Representation and generative learning

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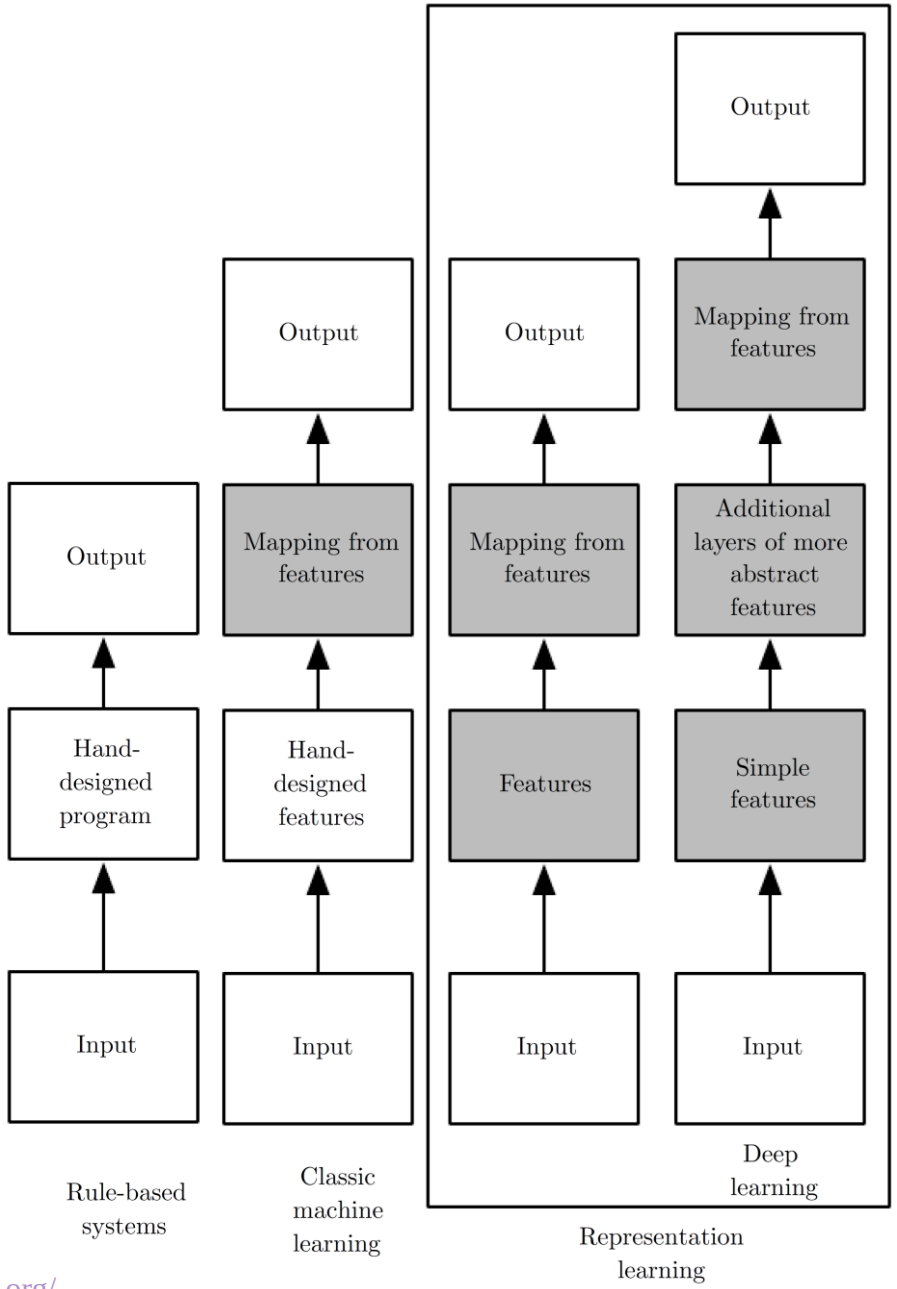
Representation matters

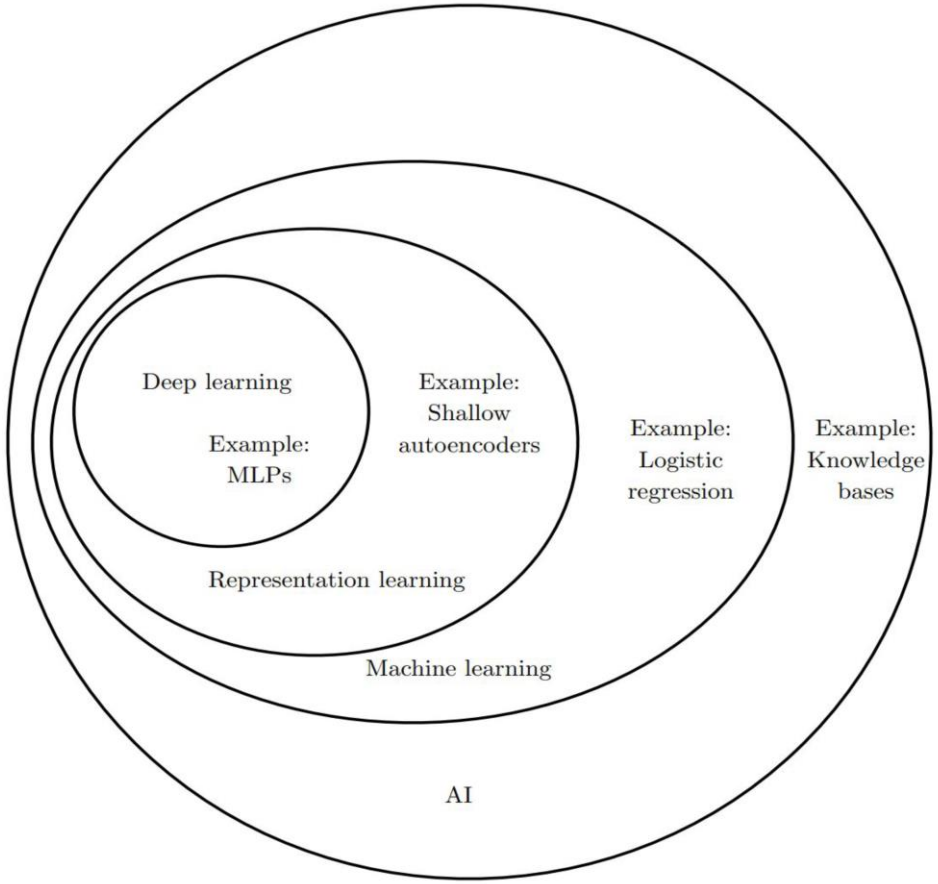
- Many tasks can be solved by designing the *right set of features* for that task, then providing these features to a simple algorithm
  - For example, a useful feature for detecting whether an e-mail is a spam or not is to count the number of occurrences of the word such as “congratulations,” “free,” and “amazing”
    - However, for many tasks, it is difficult to know what features should be extracted
    - One solution is to use machine learning to discover not only the mapping from representation to output but also the representation itself. This approach is known as *representation learning*



https://www.deeplearningbook.org/

Deep Learning is one way to learn features

- Finding all patterns = short description of raw data that can be used in downstream task
  - Recipe is clear: Collect a large labeled dataset, train a model, deploy. Good data and sufficient data are what we need
    - Unfortunately, having plenty of unlabeled data and little labeled data is common. Labeling instances is time-consuming and costly
  - Deep Unsupervised Learning - to learn representations without labels



https://www.deeplearningbook.org/

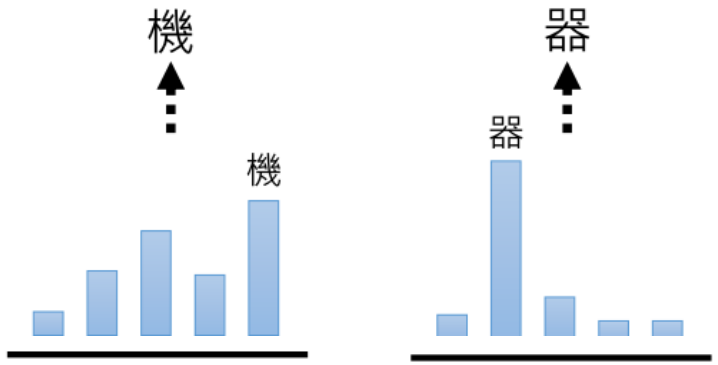
Deep Unsupervised Learning and Generative Learning

- *Autoencoders* can learn dense representations of the input data, called *latent representations*, *embedding*, *or codings*, without any supervision
  - Serve as dimension reduction and visualization tool
  - Acts as feature detectors and for unsupervised pretraining of deep neural networks
  - Some are generative models!
- *Generative adversarial networks (GANs)* can generate realistic data
  - Augmenting a dataset
  - Colorization, image editing and super-resolution
  - Generating tabular data, text, audio, time series…
- *Diffusion models*
  - Easier to train but are much slower for inference
  - Generate higher-quality images

Generative model

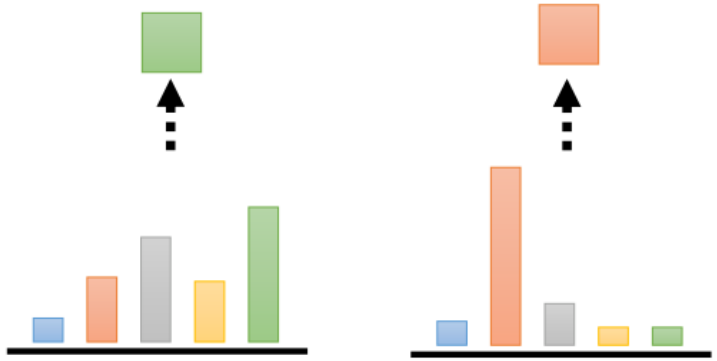
文字生成多採取 Autoregressive (各個擊破)

機器學習是甚麼？ → 文字生成 →



影像生成好像也可以
用同樣的道理

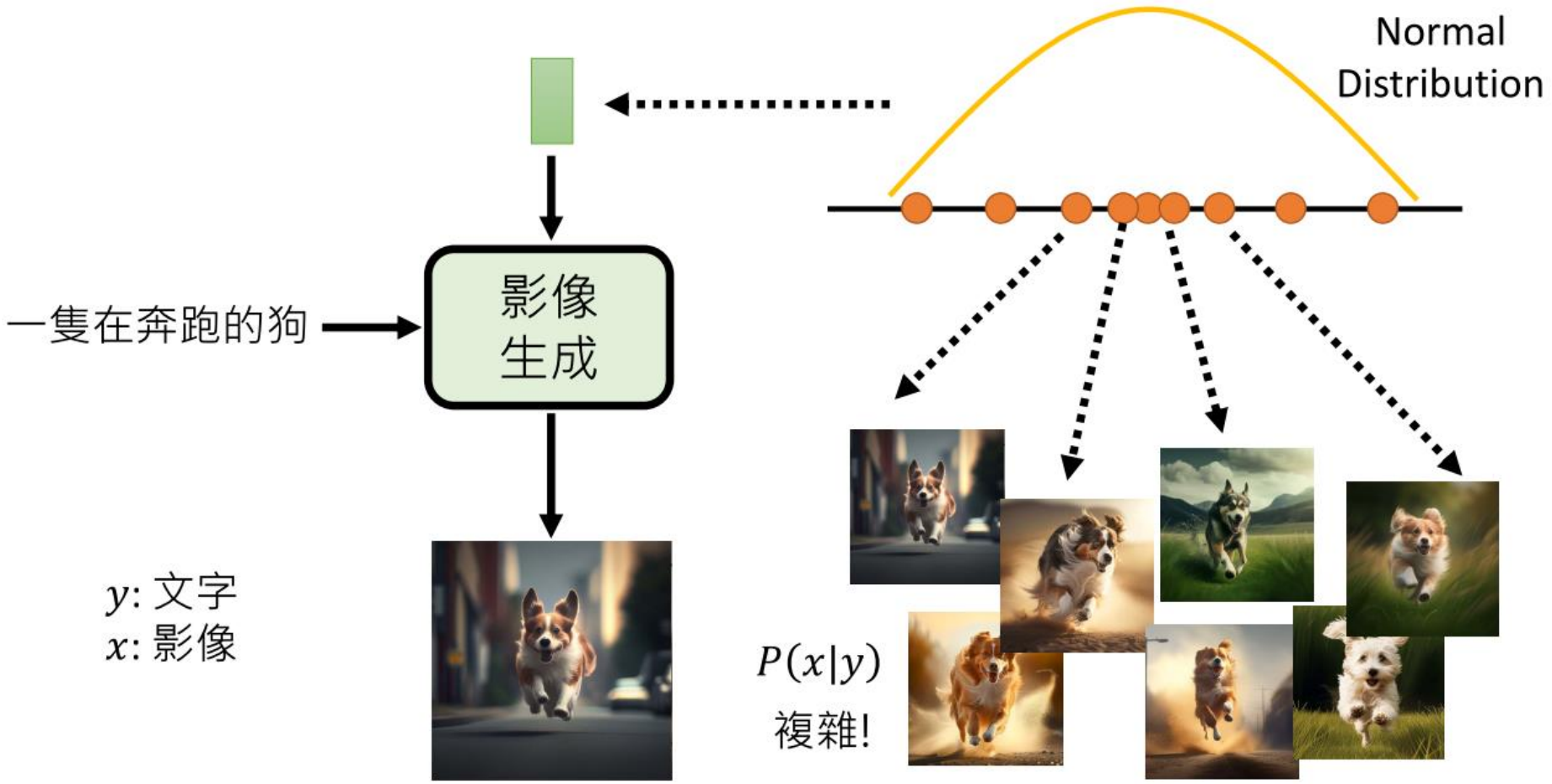
一隻在奔跑的狗 → 影像生成 →



256 x 256 images → 65536 pixels

https://speech.ee.ntu.edu.tw/~hylee/ml/2023-spring.php

Generative model

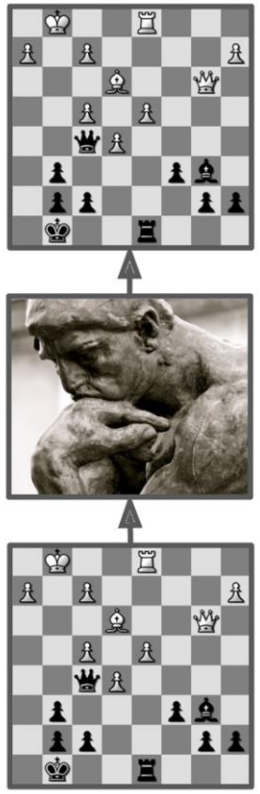


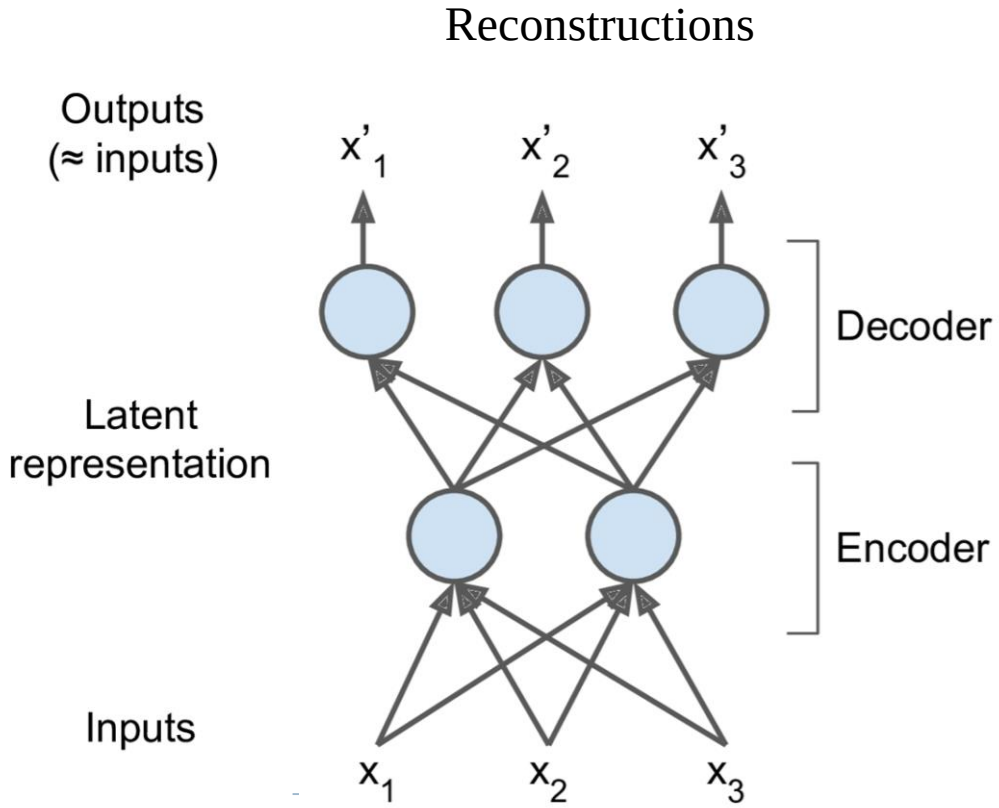
https://speech.ee.ntu.edu.tw/~hylee/ml/2023-spring.php

1. AutoEncoder - Efficient data representation

- Pattern detection
  - 40, 27, 25, 36, 81, 57, 10, 73, 19, 68
  - 50, 48, 46, 44, 42, 40, 38, 36, 34, 32, 30, 28, 26, 24, 22, 20, 18, 16, 14

- Chess players can memorize the positions of all the pieces in a game by looking at the board for just five seconds
  - This was only the case when the pieces were placed in realistic. They see chess patterns more easily





1. Linear autoencoder are similar to PCA

- Recall that if the data matrix X is column-centered. We have

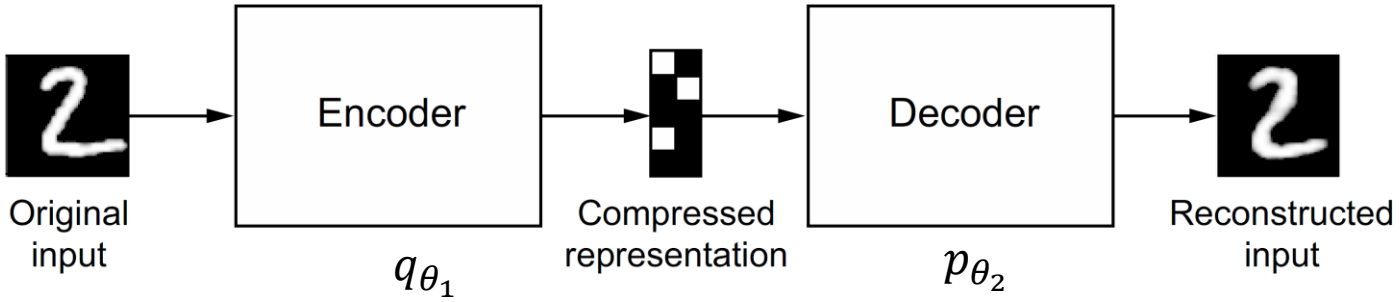
$$\min_{Z\in R^{n\times M},\Phi\in R^{p\times M}}\{\sum_{j=1}^{p}\sum_{i=1}^{n}(x_{ij}-\sum_{m=1}^{M}(\Phi_{jm}x_{ij})\Phi_{jm})^2\}$$

  - $\hat{Z}=X\hat{\Phi}$ and $\hat{\Phi}$ are the first M principal components score (coding) and loading vectors (parameters)

- The autoencoder has a similar loss

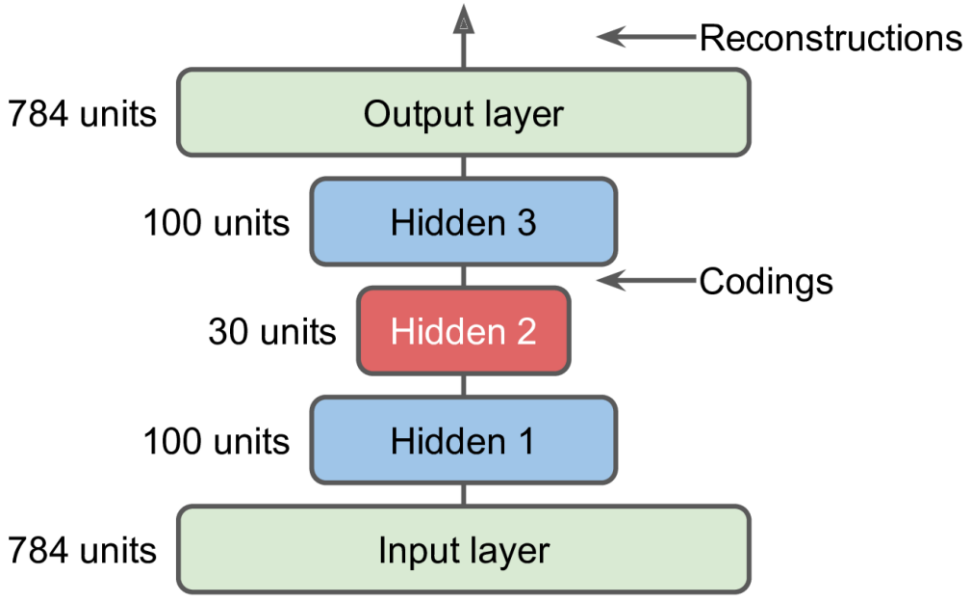
$$\min_{\theta_1\theta_2}\{\sum_{i=1}^{n}(x_i-p_{\theta_2}(q_{\theta_1}(x_i)))^2\}$$

  - Note that it can also use *nonlinear activation function*



Stacked autoencoders

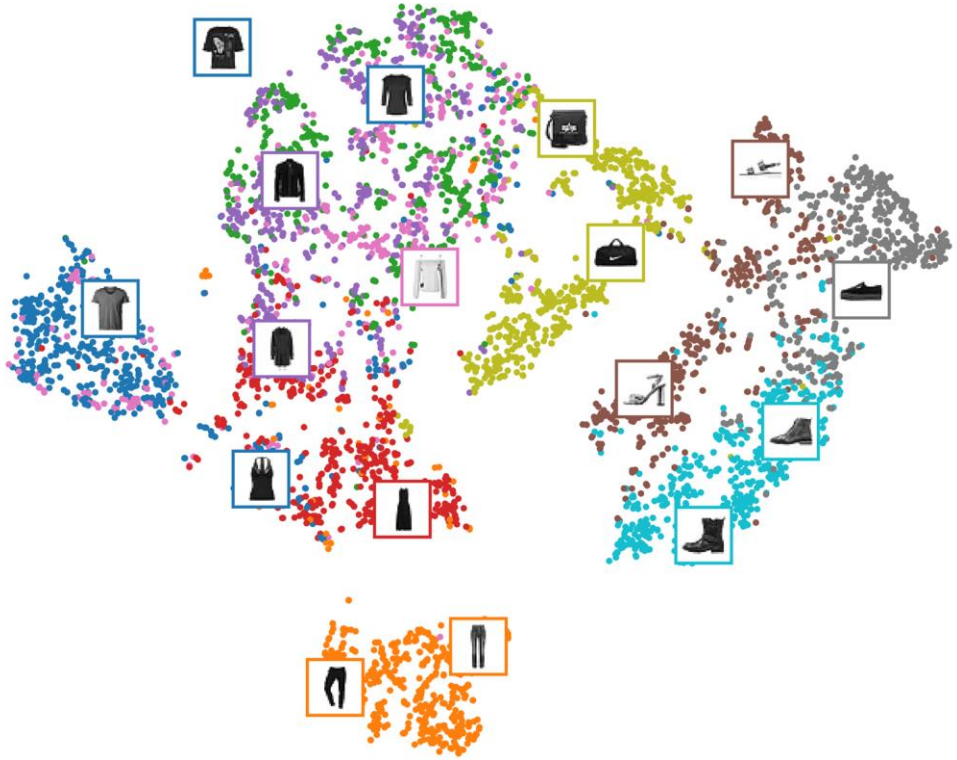
- Autoencoders can have multiple hidden layers called *Stacked Autoencoders*
  - Be careful not to make the autoencoder too powerful. An encoder that just learns to map each input to a single arbitrary number (decoder learns the reverse mapping) is not useful
  - The architecture is usually symmetric



Reconstructions of Fashion MNIST with MSE loss

Dimensional reduction and visualization

- Visualization the embedding (with the help of t-SNE)
  - One strategy is to use an autoencoder to reduce the dimensionality down to a reasonable level, then use another dimensionality reduction algorithm for visualization
  - ✓ Cope with large dataset

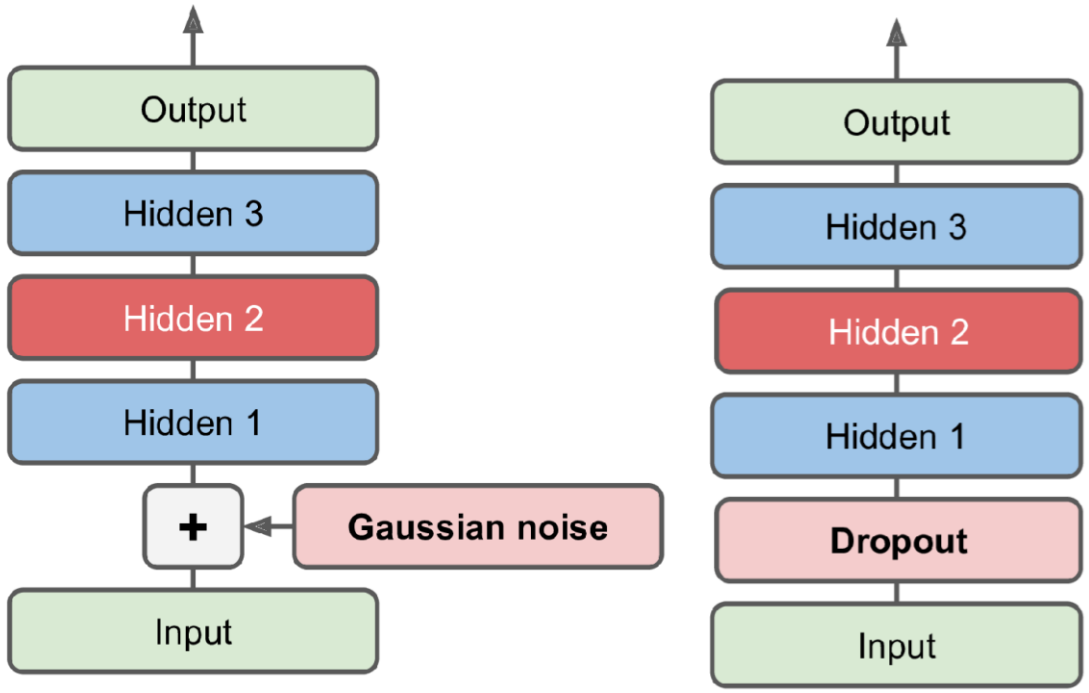


Convolutional/Recurerent Autoencoders

- If you are dealing with images, CNNs are far better suited than dense networks to work with images
  - Use the convolution layer in the encoder and transpose convolutional layers in the decoder
  - Again, try to use the pyramid design
- If you want to build an autoencoder for sequences, RNNs may be better suited
  - The encoder is typically a sequence-to-vector RNN which compresses the input sequence down to a single vector. The decoder is a vector-to-sequence RNN that does the reverse

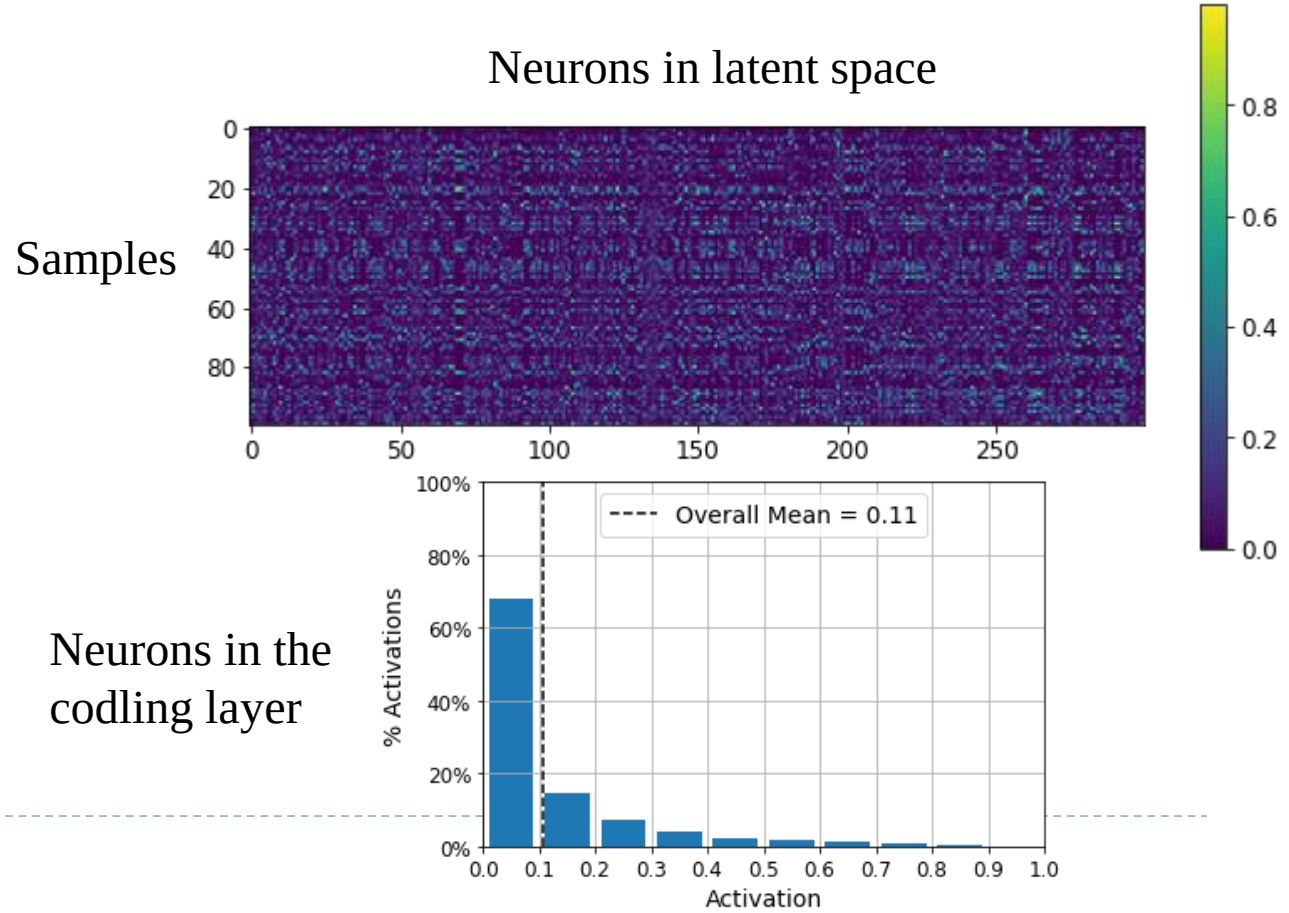
Denoising Autoencoders

- Another way to force the autoencoder to learn useful features is to *add noise* to its inputs, training it to recover the original, noise-free inputs
  - The noise can be pure Gaussian noise added to the inputs, or it can be randomly switched-off inputs, just like in dropout



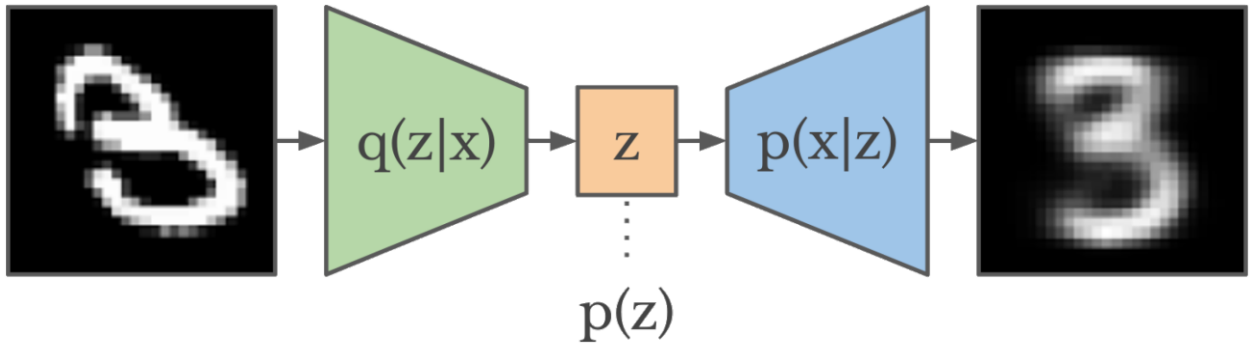
Sparse Autoencoders

- Another constraint that often leads to good *feature extraction is sparsity*
  - By adding an appropriate term to the cost function, the autoencoder is pushed to reduce the number of active neurons in the coding layer!
  - In practice, you can use sigmoid and L1 or L2 regularization in the coding layer
  - Or we can measure the actual sparsity of the coding layer at each training iteration, and penalize the model when the measured sparsity differs from a target sparsity



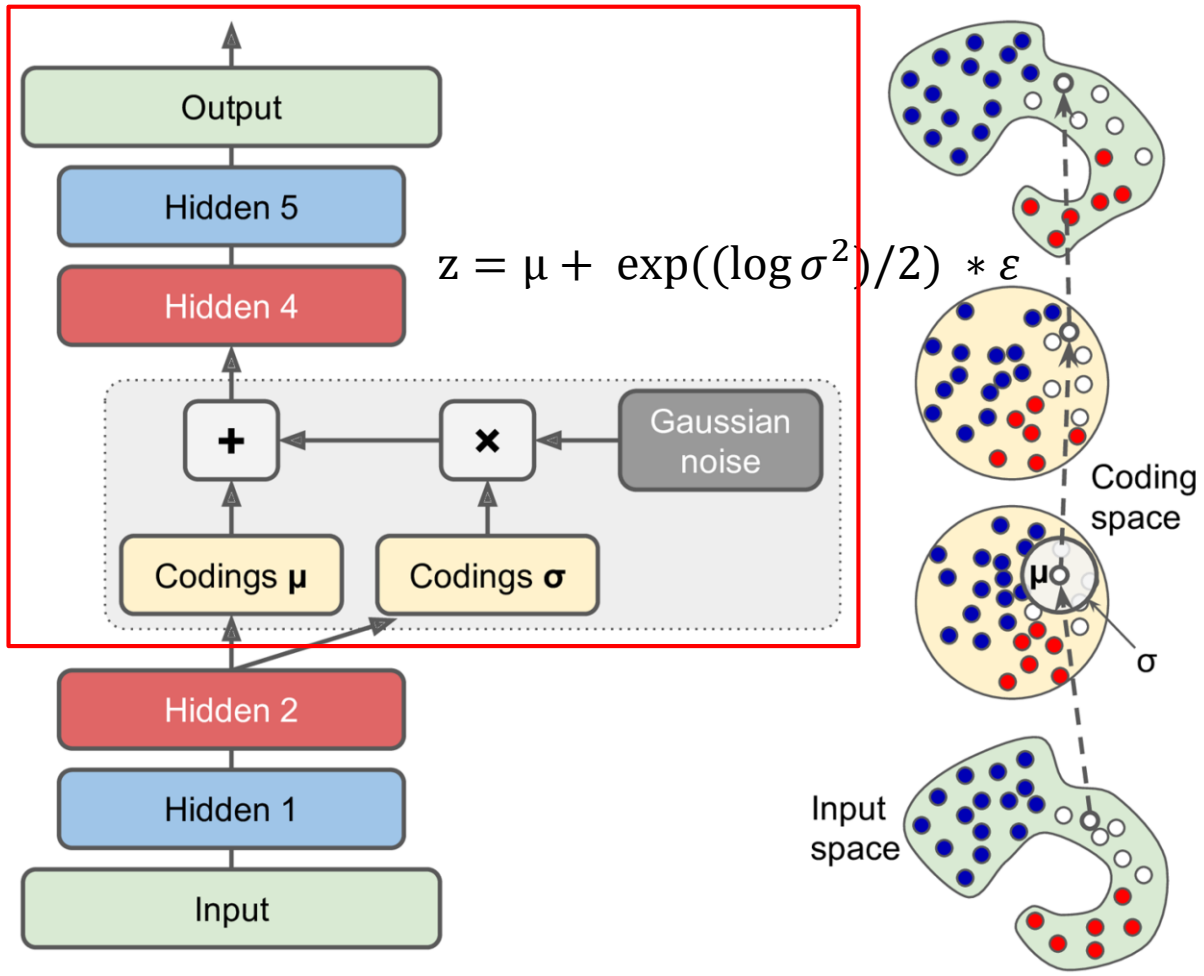
2. Variational AutoEncoders (VAE)

- Properties of VAE
  1. They are *probabilistic autoencoders*, their outputs are partly determined by chance, *even after training*. They turn the input into the parameters of a statistical distribution: a mean and a variance
  2. They can learn smooth latent spaces and they are generative autoencoders, meaning that they can generate new data that look like they were sampled from the training set
  3. It performs variational Bayesian inference, which is an efficient way to perform *approximate Bayesian inference (see appendix)*



Variational Autoencoders

- Instead of directly producing a coding for a given input, the encoder produces a mean coding μ and a variant of standard deviation $\log \sigma^2$ (for numerical stability)
  - The actual coding is then sampled randomly from a Gaussian distribution
  - One great consequence is that after training a variational autoencoder, you can very easily generate a new instance



Variational autoencoder

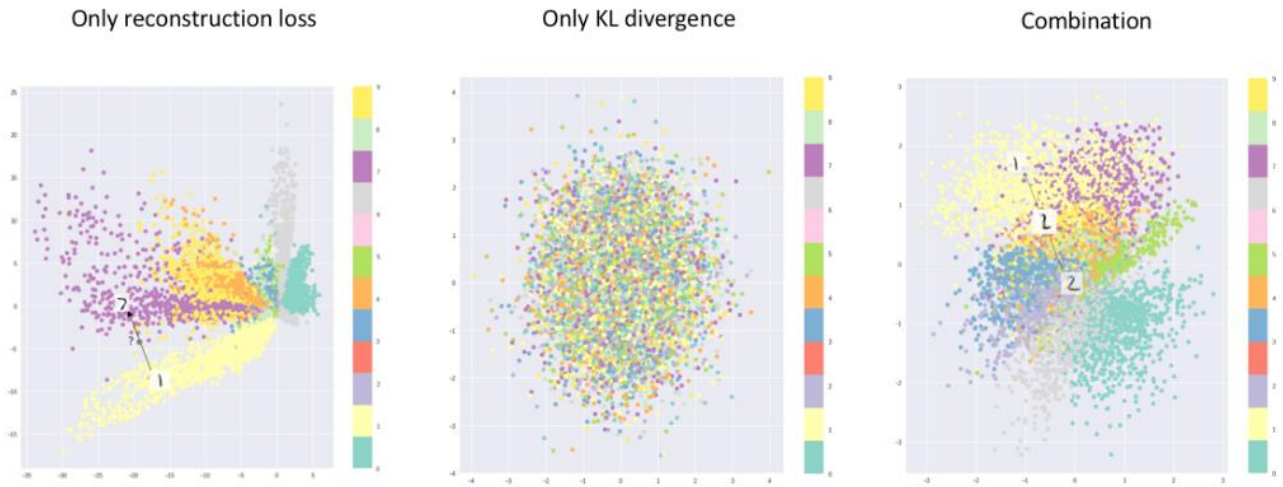
- The loss function

$$\mathcal{L}(x, \hat{x}) + \beta \sum_j KL\left(q_j(z|x) \,||N(0,1)\right)$$

Reconstruction loss

Regularization loss

MNIST Results

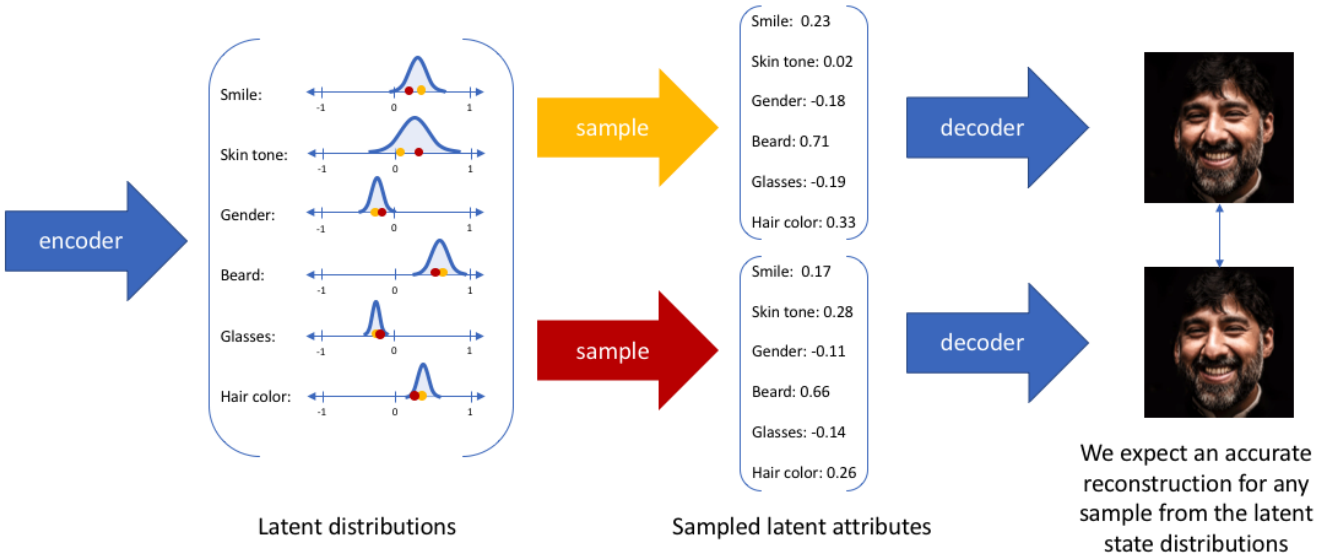


Feature disentangle through VAE

- If we observe that the latent distributions appear to be very tight, we may decide to give higher weight to the KL divergence term with a parameter $\beta > 1$, encouraging the network to learn broader distributions

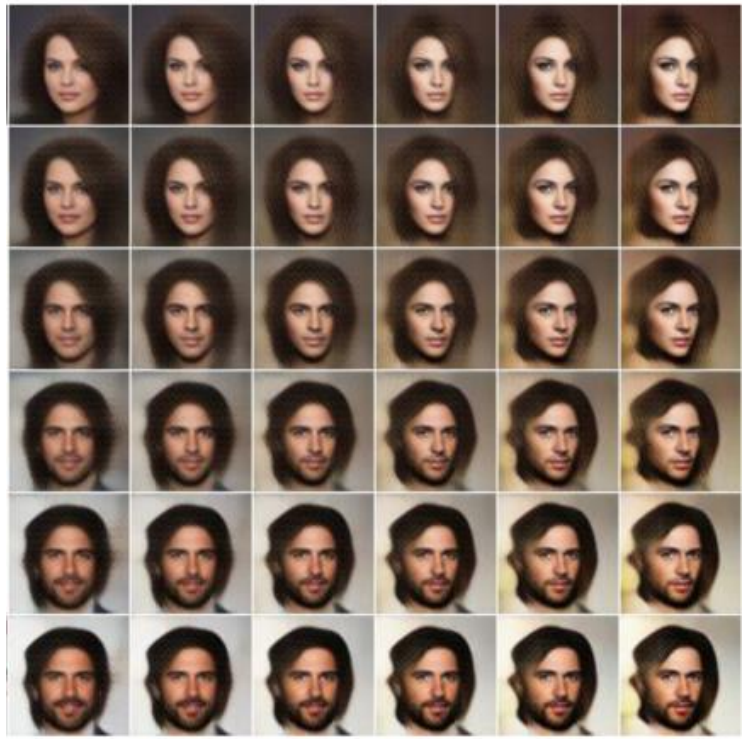
$$\mathcal{L}(x, \hat{x}) + \beta \sum_j KL(q_j(z|x) || N(0,1))$$

  - As it turns out, by placing a larger emphasis on the KL divergence term we're also implicitly enforcing that the learned latent dimensions are uncorrelated (through our simplifying assumption of a diagonal covariance matrix)
  - This is the idea of *feature disentangle*



Semantic interpolation through VAE

- Variational autoencoders make it possible to perform *semantic interpolation*: instead of interpolating two images at the pixel level (which would look as if the two images were overlaid), we can interpolate *at the codings level*
  - We first run both images through the encoder, then we interpolate the two codings we get, and finally we decode the interpolated codings to get the final image
  - Certain directions in the space may *encode interesting axes of variation* in the original data



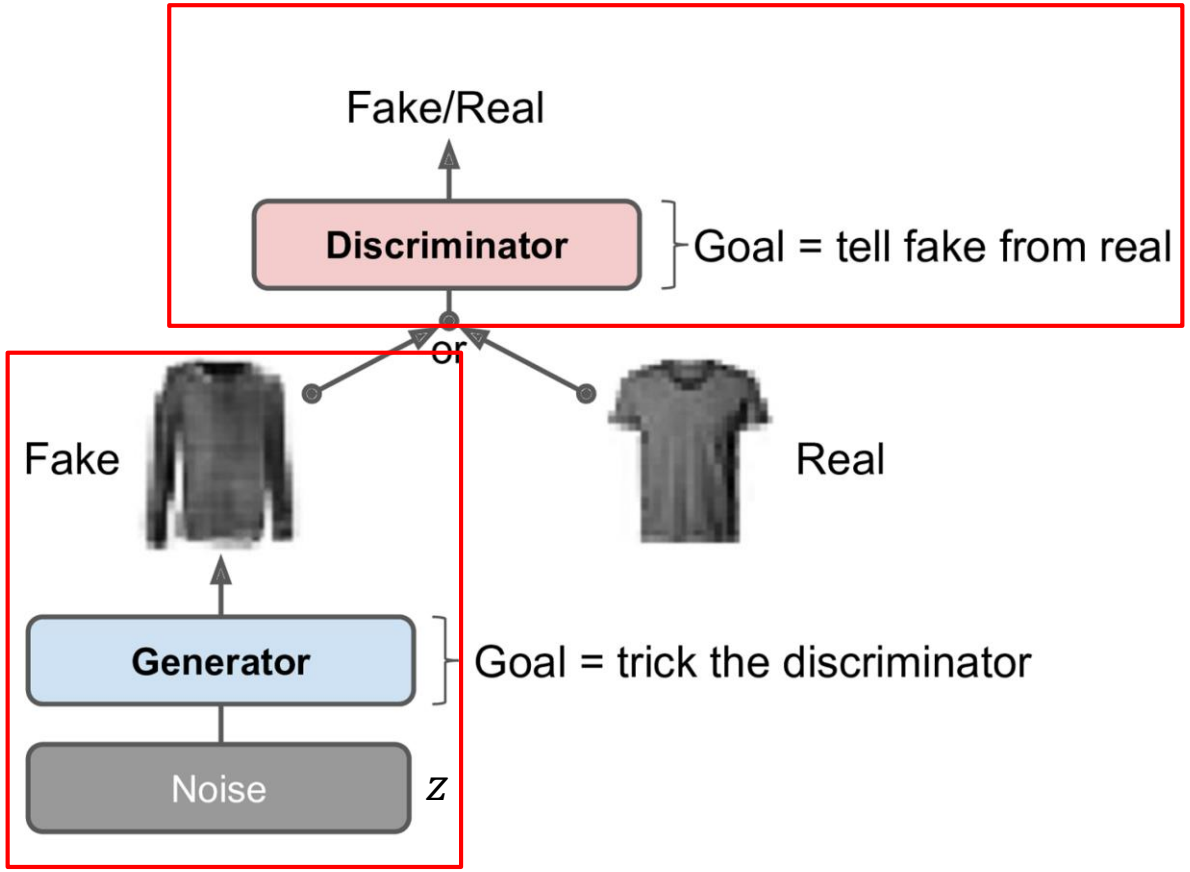
https://medium.com/vitrox-publication/generative-modeling-with-variational-auto-encoder-vae-fc449be9890e

3. Generative adversarial networks

- Generative adversarial networks (GANs), introduced in 2014 by Goodfellow et al. enable the generation of fairly realistic synthetic images by forcing the generated images to be statistically almost indistinguishable from real ones
  - An intuitive way to understand GANs is to imagine a forger trying to create a fake Picasso painting. At first, the forger is pretty bad at the task. He mixes some of his fakes with authentic Picassos and shows them all to an art dealer
  1. The art dealer makes an authenticity assessment for each painting and gives the forger feedback about what makes a Picasso look like a Picasso
  2. The forger goes back to his studio to prepare some new fake
  3. As time goes on, the forger becomes increasingly competent at imitating the style of Picasso, and the art dealer becomes increasingly expert at spotting fakes. In the end, they have on their hands some excellent fake Picassos

Generative Adversarial Networks

- Generator (forger)
  - Takes a random distribution as input (typically Gaussian) and outputs some data. You can think of the random inputs as the latent representations of the image to be generated
    - The generator offers the same functionality as a decoder in a variational autoencoder
- Discriminator (art dealer)
  - Takes either a fake image from the generator or a real image from the training set as input, and must guess whether the input image is fake or real



Training GAN

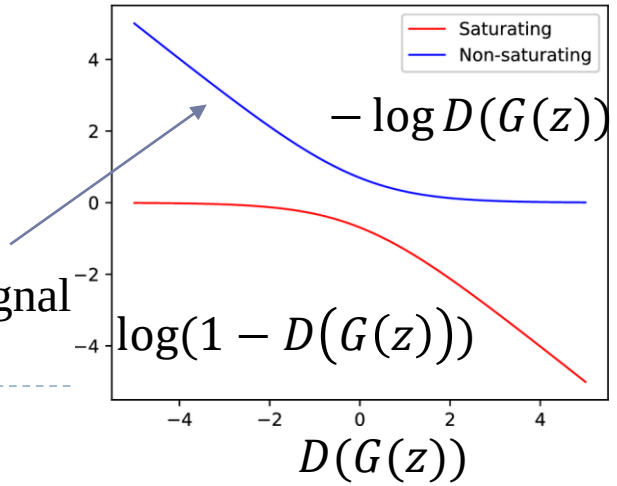
- In the first phase, we train the discriminator (generator are fixed)
  - A batch of real images is sampled from the training set and is completed with an equal number of fake images produced by the generator. *The labels are set to 0 for fake images and 1 for real images*, and the discriminator is trained using the binary cross-entropy loss

$$\max_{\theta_d} \left[\mathbb{E}_{x \sim p_{data}} \log D_{\theta_d}(x) + \mathbb{E}_{z \sim p(z)} \log(1 - D_{\theta_d}(G_{\theta_g}(z))) \right]$$

- In the second phase, we train the generator (discriminators are fixed)
  - We first use it to produce another batch of fake images, and once again the discriminator is used to tell whether the images are fake or real. The loss on the right-hand side could have a larger gradient for a bad sample

$$\min_{\theta_g} \mathbb{E}_{z \sim p(z)} \log(1 - D_{\theta_d}(G_{\theta_g}(z))) \Rightarrow \max_{\theta_g} \mathbb{E}_{z \sim p(z)} \log(D_{\theta_d}(G_{\theta_g}(z)))$$

Higher gradient signal



Training GAN

- That's what a GAN is: a forger network and an expert network, each being trained to beat the other
  - Remarkably, a GAN is a system where the optimization minimum isn't fixed. Normally, gradient descent consists of rolling down hills in a static loss landscape. But with a GAN, *every step taken down the hill changes the entire landscape a little*
  - It's a dynamic system where the optimization process is seeking not a minimum, but an equilibrium between two forces. For this reason, GANs are notoriously difficult to train

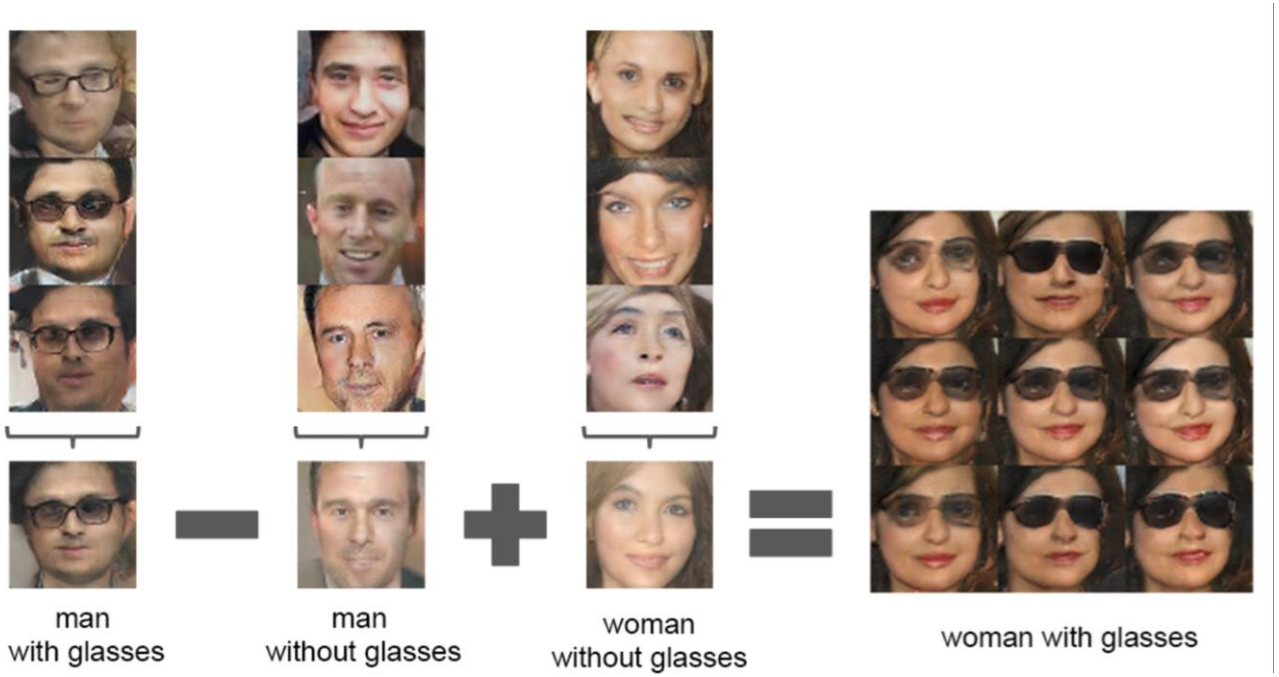
Discriminator outputs likelihood in (0,1) of real image

Minimax objective function:

$$\min_{\theta_g} \max_{\theta_d} \left[\mathbb{E}_{x \sim p_{data}} \log \underbrace{D_{\theta_d}(x)}_{\text{Discriminator output for real data x}} + \mathbb{E}_{z \sim p(z)} \log(1 - \underbrace{D_{\theta_d}(G_{\theta_g}(z))}_{\text{Discriminator output for generated fake data G(z)}}) \right]$$

Applications – Deep Convolutional GAN

- Vector arithmetic, semantic interpolation and feature disentangle - DCGANs can learn quite meaningful latent representations



4. Diffusion model

- The ideas behind diffusion models have been around for many years, but they were first formalized in their modern form in 2015
  - In 2020, Jonathan Ho et al. from UC Berkeley managed to build a diffusion model capable of generating highly realistic images, which they called a *Denoising Diffusion Probabilistic Model (DDPM)*
  - OpenAI analyzed the DDPM architecture and proposed several improvements that allowed DDPMs to become popular
  - ✓ DDPMs are much easier to train than GANs, but the generated images are more diverse and of even higher quality. The main downside of DDPMs, as you will see, is that they take a very long time to generate images compared to GANs or VAEs

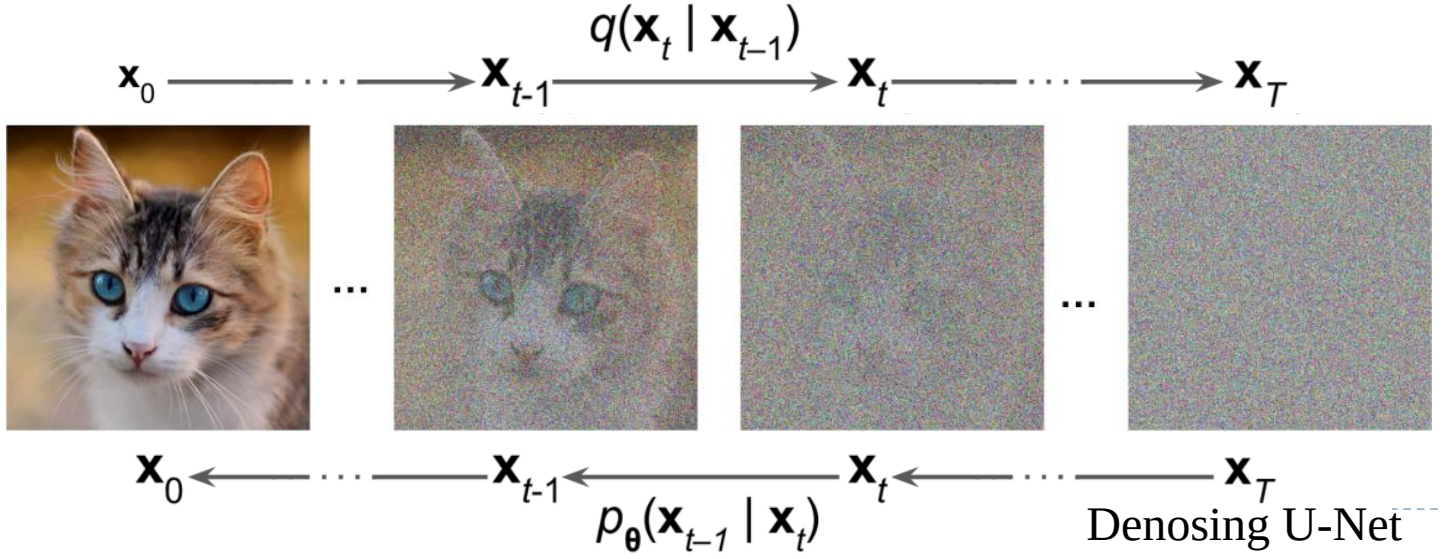
Forward/backward process

- Forward process: We add a little bit of Gaussian noise to the image, with mean 0 and variance β_t and rescale by $\sqrt{1-\beta_t}$ at each step:

$$x_t = \sqrt{1-\beta_t}x_{t-1} + \sqrt{\beta_t}\varepsilon \rightarrow q(x_t|x_{t-1}) = N(\sqrt{1-\beta_t}x_{t-1}, \beta_t I)$$

- We can train a model that can perform the reverse process: going from x_t to x_{t-1}. We can then use it to remove a tiny bit of noise from an image and repeat the operation many times until all the noise is gone

Diffusion process (no training is needed)



Denosing U-Net

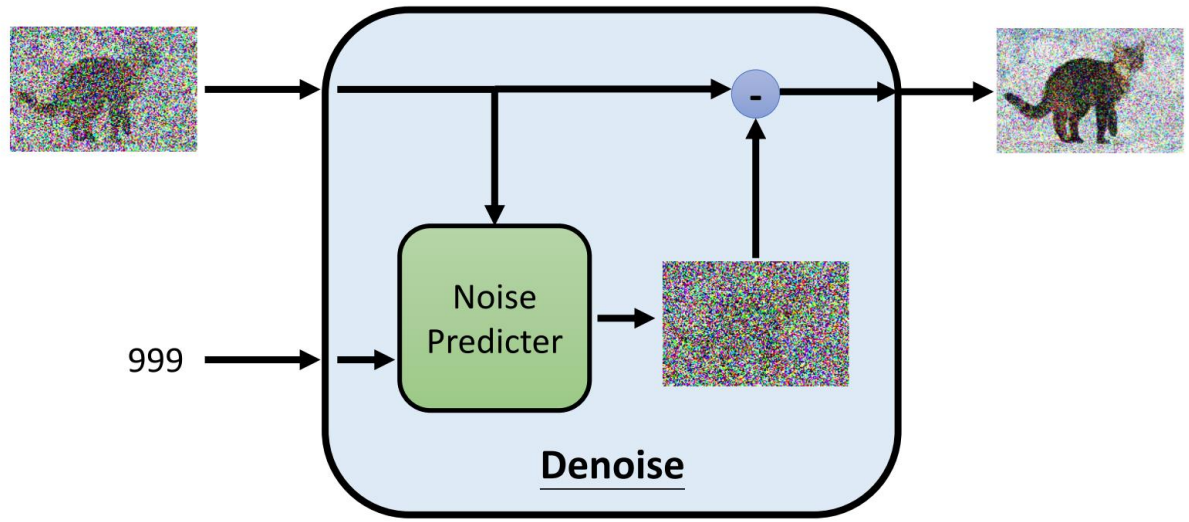
Forward/backward process

- There's a *shortcut* for the forward process:

$$q(x_t|x_0) = N\left(\sqrt{\bar{\alpha}_t}x_0, (1-\bar{\alpha}_t)I\right)$$

  - The part of the variance that comes from the initial distribution shrinks by a factor of $1-\beta_t$ at each step. If we define $\alpha_t = 1-\beta_t$, $\bar{\alpha}_t = \prod_{i=1}^{t}\alpha_t$, we want to schedule so it shrinks down from 1 to 0 gradually between time steps 0 and T (noise will become greater)

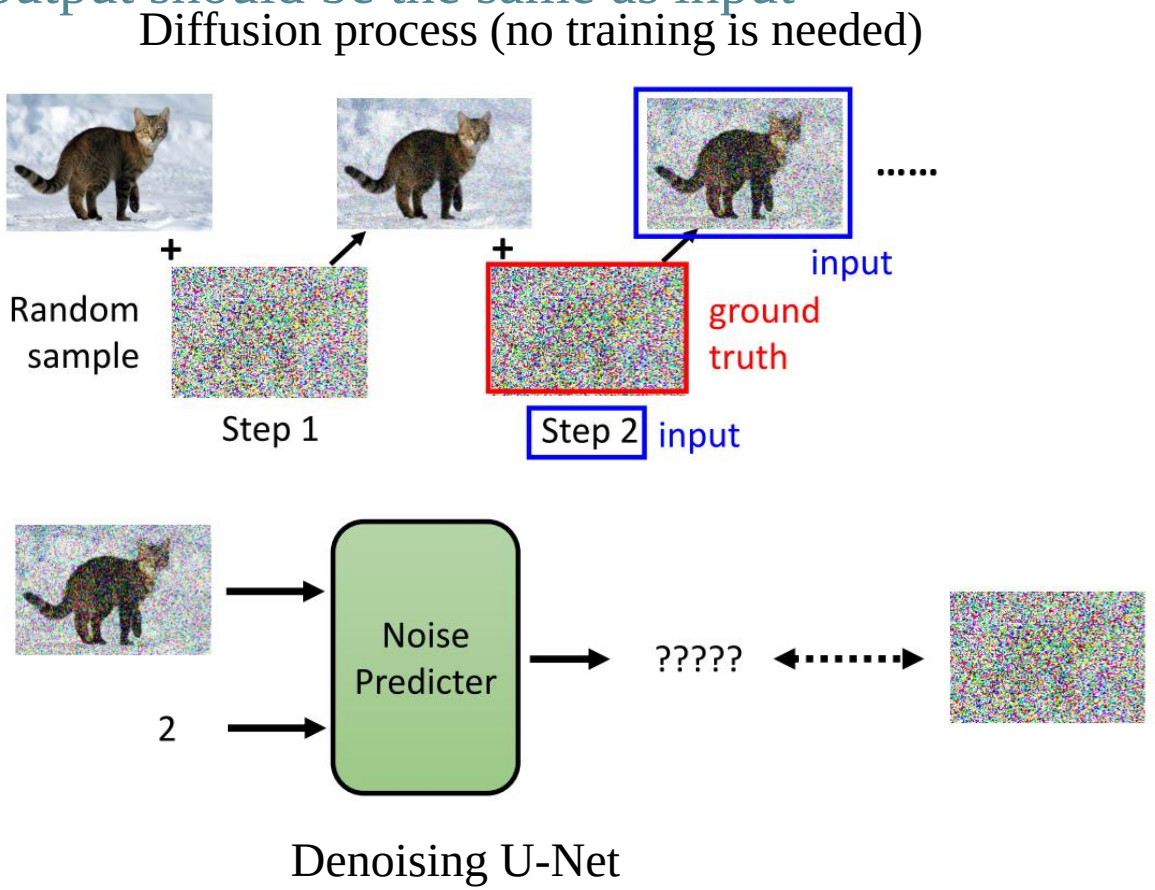
- The model is then trained to reverse that process
  - Given a noisy image produced by the forward process, and time t, the model is trained to predict the total noise that was added to the original image



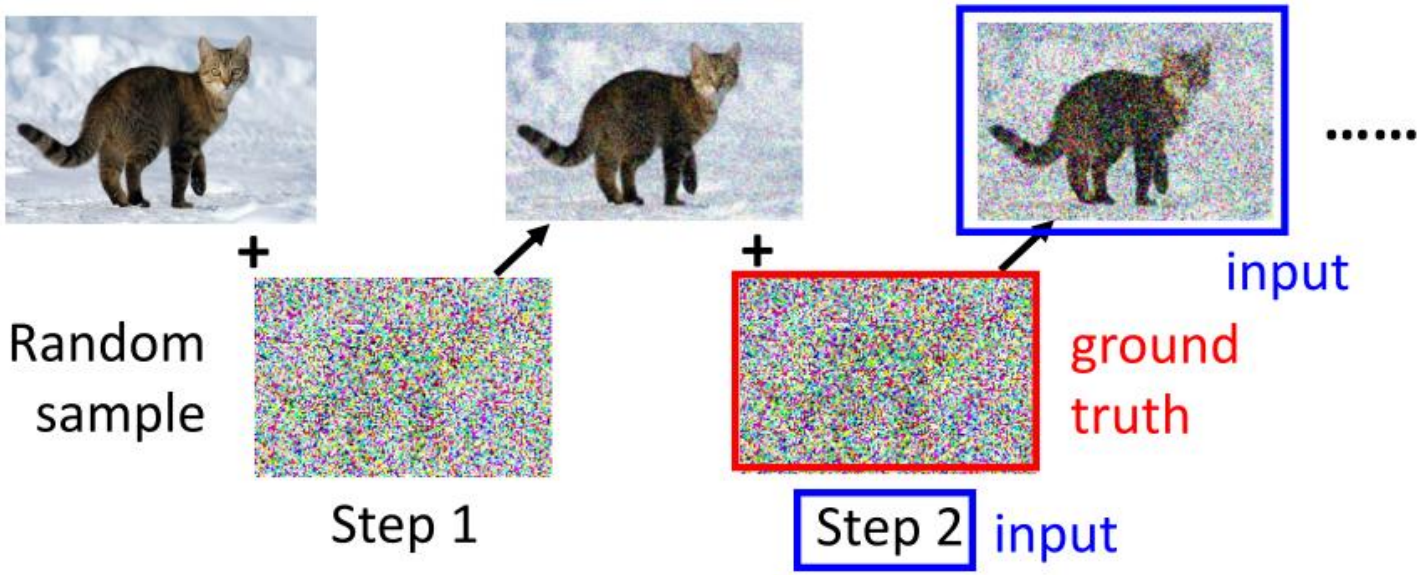
https://speech.ee.ntu.edu.tw/~hylee/ml/2023-spring.php

Forward/backward process

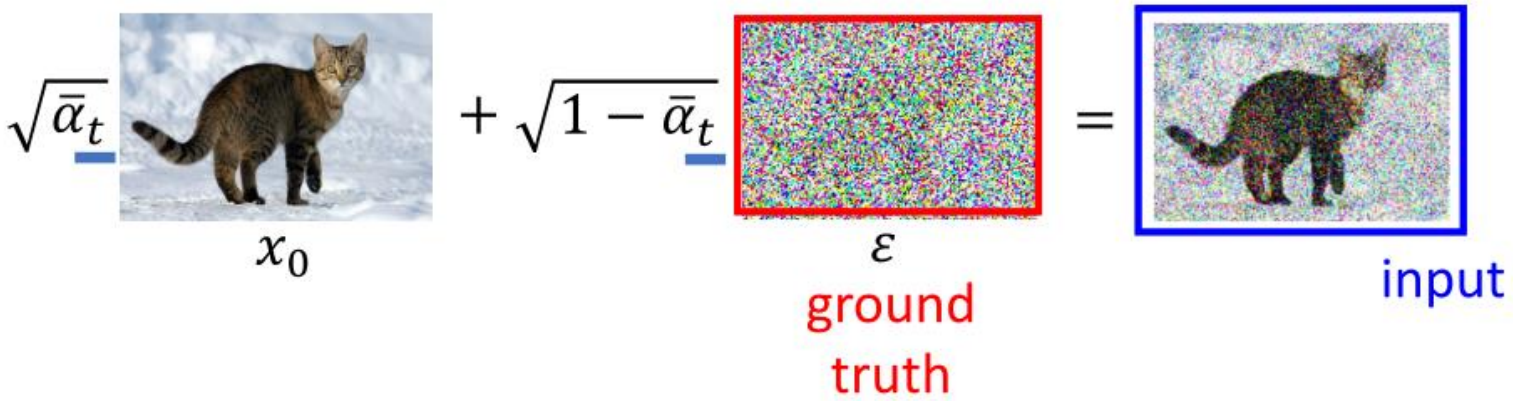
- To train the model, we will randomly draw samples *from different time steps*
  - The U-Net is employed since the size of output should be the same as input
  - The model will need to process both images and times which may be encoded using *a sinusoidal encoding*
  - There is no shortcut for the reverse process, therefore is very slow!



想像中 ...


Random sample
Step 1
Step 2
input
ground truth
input
.....

實際上 ...


$\sqrt{\bar{\alpha}_t}$
x_0
$+\sqrt{1-\bar{\alpha}_t}$
ε
ground truth
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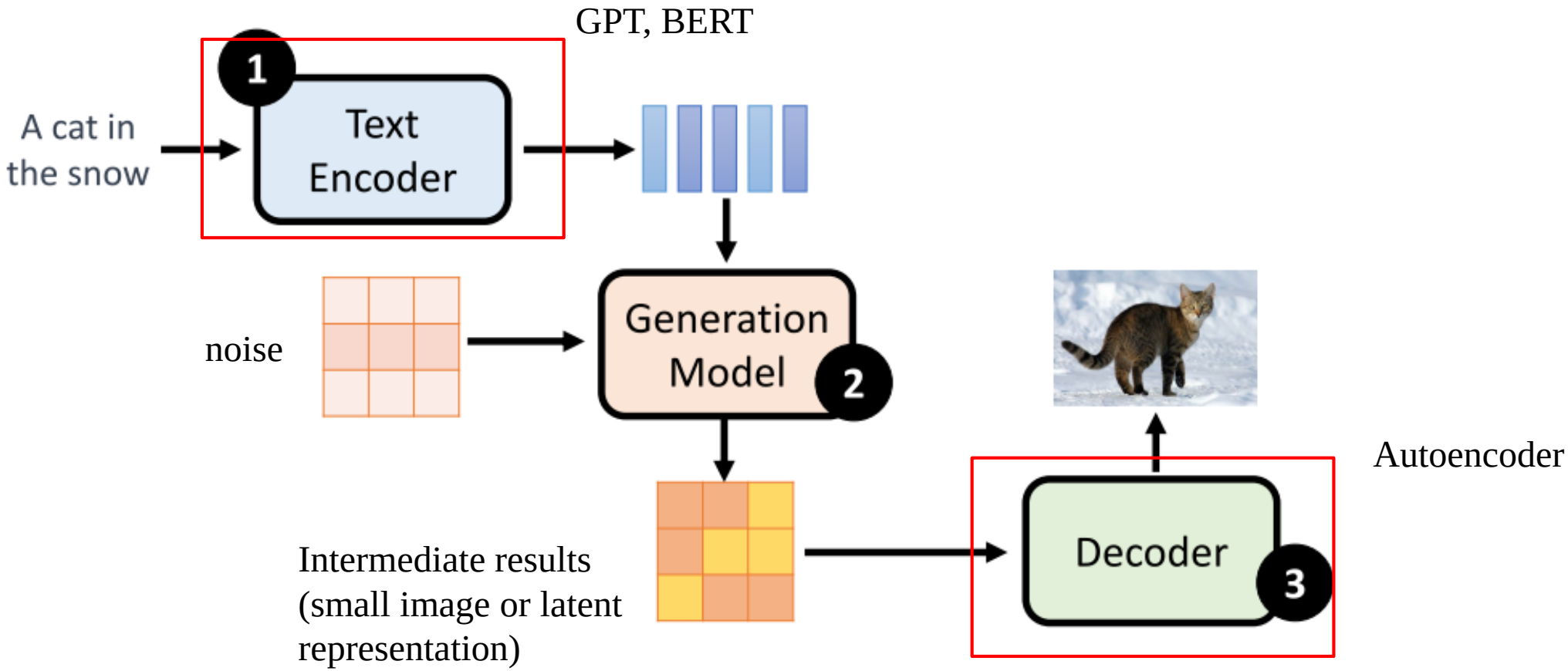
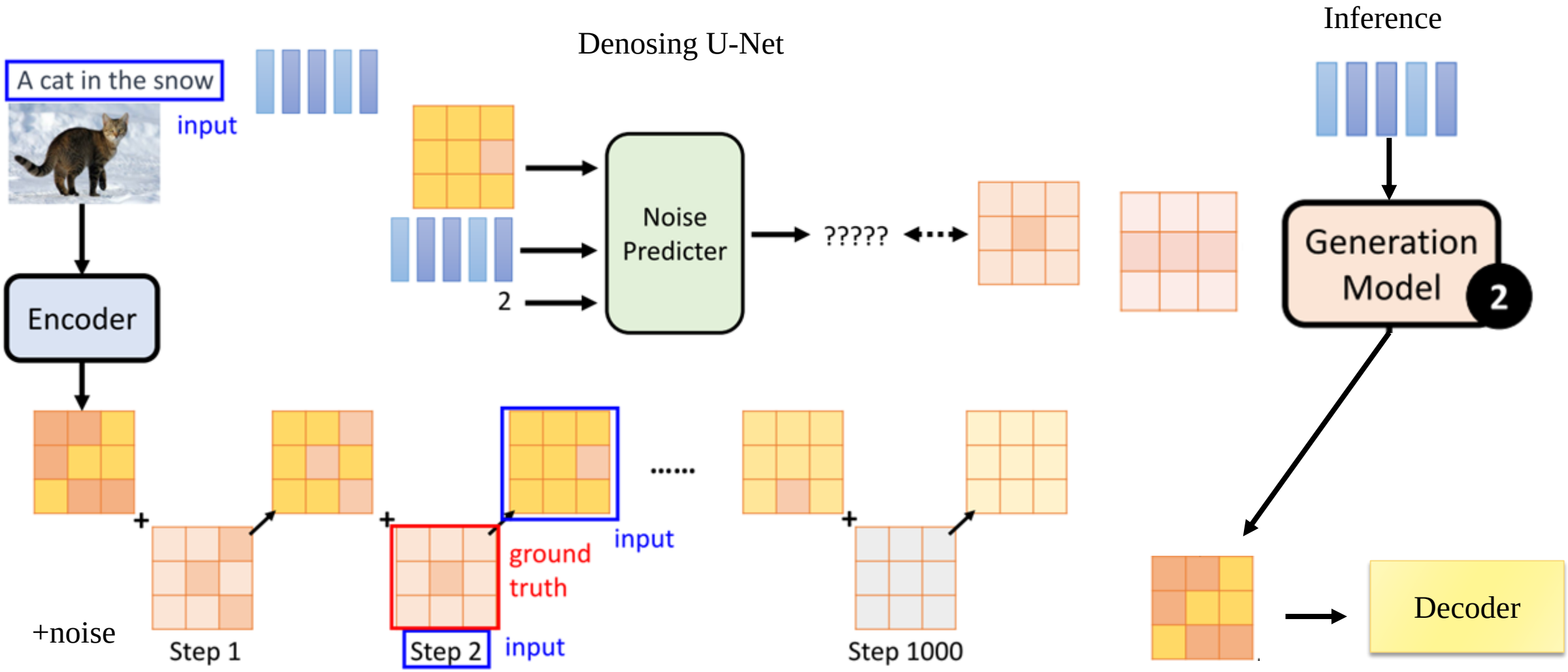
input

Text to image using diffusion model

- Diffusion models have made tremendous progress recently in generative models. The framework trains three models separately:



Forward/backward process



Diffusion process (no training is needed)

Diffusion model

- Recent advances
  - *Latent diffusion models*, where the diffusion process takes place in latent space rather than in pixel space
  - To achieve this, a powerful autoencoder is used to compress each training image into a much smaller latent space, where the diffusion process takes place, then the autoencoder is used to decompress the final latent representation, generating the output image
  - Various conditioning techniques to guide the diffusion process using text prompts, images
- Stable diffusion, DALL-E, Midjourney….

Conclusion

- Deep representation learning is an actively developed field
- There are three major tools to do this: VAEs, GANs and diffusion model
  - VAEs result in highly structured, continuous latent representations and are easier to train
  - GANs enable the generation of realistic single-frame images but may not induce latent spaces with a solid structure and high continuity
  - Diffusion models are becoming popular. However, the training/inference time is still a challenge

References

[1] Hands-On Machine Learning with Scikit-Learn, Keras, and TensorFlow, 3rd Edition Chapter 17

[2] Deep learning with Python, 2nd Edition Chapter 12

[3] https://speech.ee.ntu.edu.tw/~hylee/ml/2022-spring.php Lecture 6 and Lecture 8

[4] https://speech.ee.ntu.edu.tw/~hylee/ml/2023-spring.php

[5] https://udlbook.github.io/udlbook/ Chapter 15, 17, 18

Appendix

Resources

- Tutorials
  - https://sites.google.com/view/berkeley-cs294-158-sp24/home
- Different VAE
  - https://github.com/probml/pyprobml/tree/master/deprecated/vae
  - https://github.com/AntixK/PyTorch-VAE
  - https://mlberkeley.substack.com/p/vq-vae
- Different GAN
  - https://d2l.ai/chapter_generative-adversarial-networks/index.html
  - https://github.com/probml/pyprobml/tree/master/deprecated/gan
  - https://github.com/lucidrains/stylegan2-pytorch
  - https://github.com/junyanz/pytorch-CycleGAN-and-pix2pix
  - https://github.com/XingangPan/DragGAN

Resources

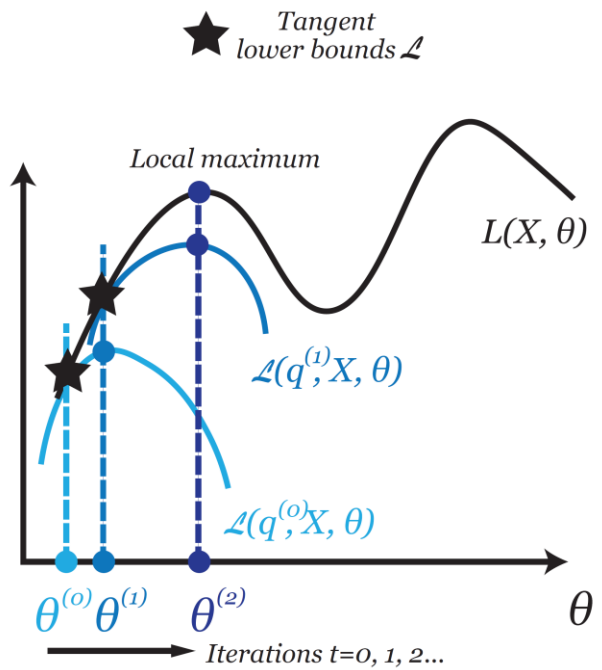
- Diffusion model
  - https://github.com/huggingface/diffusers
  - https://keras.io/guides/keras_cv/generate_images_with_stable_diffusion/
  - https://github.com/fastai/fastdiffusion
  - https://github.com/pix2pixzero/pix2pix-zero
- Playground
  - https://poloclub.github.io/ganlab/
  - https://poloclub.github.io/diffusion-explainer/

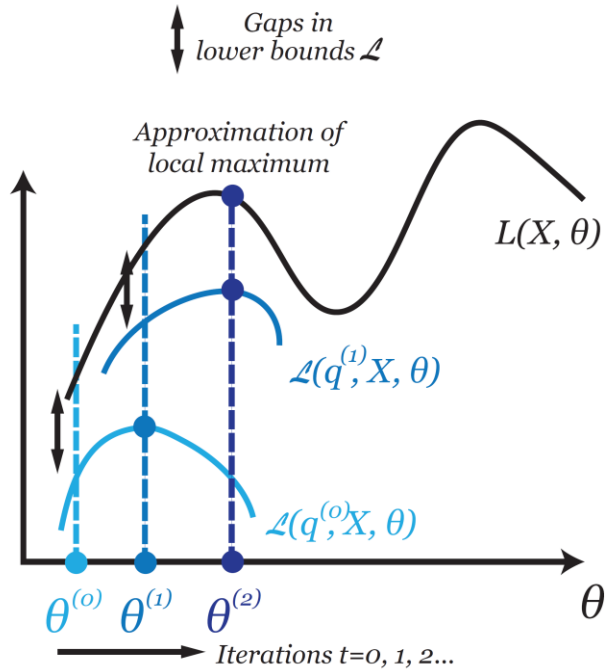
Inference

- Estimate the fix parameters θ that maximize the observed log-likelihood

$$L(X,\theta) = \log p_\theta(x_1,\ldots,x_n) = \sum_{i=1}^{n} \log p_\theta(x_i) = \sum_{i=1}^{n} \log \int_{h_i} p_\theta(x_i,h_i)dh_i$$

  - It is hard to explicitly find out $p_\theta(x_i)$. It is simpler to compute the full likelihood of each observation $p_\theta(x_i,h_i)$. However, the computation now becomes *intractable*
  - Distribution of H is inaccessible. Use a distribution q for hidden variables H, and maximizing a series of tractable lower-bounds $L(q,X,\theta)$ for $L(X,\theta)$ iteratively





https://arxiv.org/abs/2201.02867

Evidence Lower Bound (ELBO)

- $L(q, X, \theta)$ are found for the fact that for any distribution q_i on the variables h_i, the observed log-likelihood can be written as the sum of two terms

$$L(X,\theta) = \mathcal{L}(q,X,\theta) + \sum_{i=1}^{n} \mathrm{KL}(q_i(h_i) \parallel p_\theta(h_i|x_i)) = \sum_{i=1}^{n} \left[\mathcal{L}_i(q_i,x_i,\theta) + \mathrm{KL}(q_i(h_i) \parallel p_\theta(h_i|x_i))\right]$$

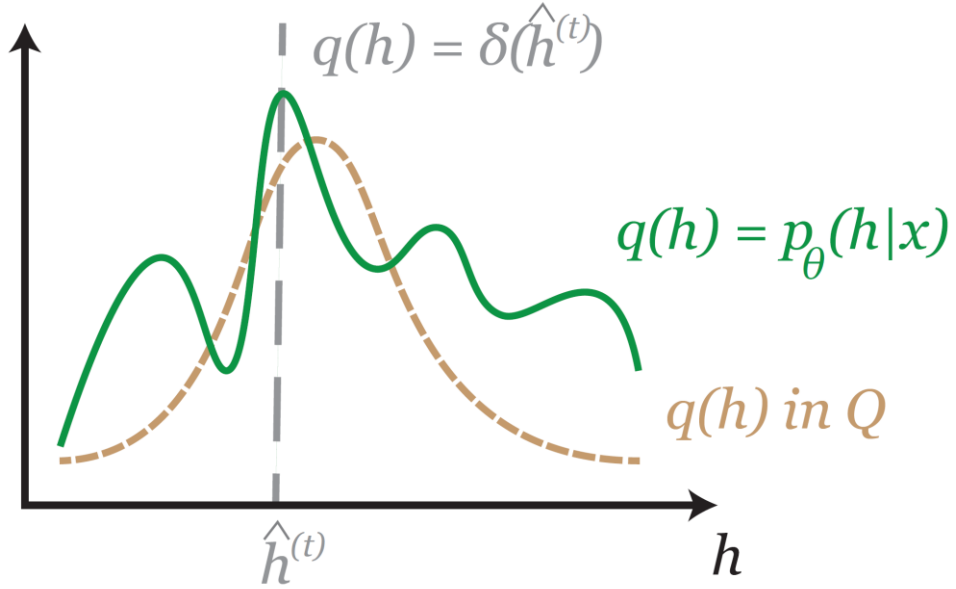
$$\mathcal{L}_i(q_i,x_i,\theta) = \int_{h_i} q_i(h_i) \log p_\theta(x_i|h_i) dh_i - \mathrm{KL}\left(q_i(h_i) \parallel p_\theta(h_i)\right)$$

- The Evidence Lower Bound is the function $\theta \to \mathcal{L}(q,X,\theta)$

$$\forall q \in \mathcal{Q}, \forall \theta, \quad \mathcal{L}(q,X,\theta) \leq L(X,\theta)$$

Inference Methods Based on an ELBO

- The choices of the distributions $q_i^{(t)}$ at each iteration t
  - Choosing $q_i^{(t)}(z) = p_{\theta^{(t)}}(h_i|x_i)$ makes the lower bound tangent to $L(X,\theta)$ at $\theta = \theta^{(t)}$ which is the Expectation-Maximization (EM) algorithm
  - For computational reasons, we can choose to have $q_i^{(t)}(z)$ approximate the posteriors $p_{\theta^{(t)}}(h_i|x_i)$ by
    - Their "mode", i.e. the value $\hat{h}_i$ of h_i that maximizes them: q_i effectively becomes a Dirac distribution at $\hat{h}_i$
    - A general distribution q_i within a family Q



Inference Methods Based on an ELBO

- The modal approximation in the EM algorithm speeds up the E-step; yet it has the drawback of summarizing the posterior distribution by a single estimate
  - Variational Inference (VI) (Variational Bayes) has appeared as a compromise between EM and modal EM during the computation of the E-step.
  - VI replaces the evaluation of the posterior of the latent variables, by an optimization converging to an approximation q of this posterior distribution. VI selects q from some parametric family of distributions Q, called the "variational family"
  - The variational family $Q = \{q_\eta = N(\mu, \sigma) | \eta = (\mu, \sigma)\}$ with parameters η is typically chosen to be a family of Gaussian distributions

Inference Methods Based on an ELBO

- The previous methods have the drawback of learning one (approximate) posterior for each hidden variable in h_i, and for each image i. This is computationally expensive and increase with n
  - Amortized inference (AI) collapses the n optimizations problems of the E-step into one
  - Instead of solving an optimization problem for each i and finding $\hat{h}_i$ or η_i defining q_{η_i}, AI optimizes the parameters ξ of a function Enc_ξ that predicts $\hat{h}_i$ or η_i when given x_i
  - The function Enc is traditionally called an encoder. Inference is made more tractable, but adds an additional error, called the amortization error

Inference Methods Based on an ELBO

- The main realizations of amortized variational EM in come through variational autoencoders

Amortized Variational EM as Dual Optimization Algorithm AI in the E-step of variational EM leads to an amortized variational EM:

(a) Inference on hidden variables h_i The parameters ξ parametrizing the encoder is computed via:

$$\xi^{(t)} = \arg\max_{\xi} \sum_{i=1}^{n} KL(q_{\text{Enc}_\xi(x_i)} || p_{\theta^{(t-1)}}(h_i|x_i) \tag{16}$$

which generates n distributions $q_{\eta_i}^{(t)}$ parametrized by $\eta_i = \text{Enc}_{\xi^{(t)}}(x_i)$.

(b) Maximization on model's parameter θ The parameter θ is updated via:

$$\theta^{(t)} = \arg\max_{\theta} \mathcal{L}(q_{\eta_i}^{(t)}, \theta). \tag{17}$$

Training GAN

Some find k=1 more stable, others use k > 1, no best rule.

Recent work (e.g. Wasserstein GAN) alleviates this problem, better stability!

for number of training iterations **do**

  for k steps **do**

- Sample minibatch of m noise samples $\{\boldsymbol{z}^{(1)}, \ldots, \boldsymbol{z}^{(m)}\}$ from noise prior $p_g(\boldsymbol{z})$.
- Sample minibatch of m examples $\{\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(m)}\}$ from data generating distribution $p_{\text{data}}(\boldsymbol{x})$.
- Update the discriminator by ascending its stochastic gradient:

$$\nabla_{\theta_d} \frac{1}{m} \sum_{i=1}^{m} \left[\log D_{\theta_d}(x^{(i)}) + \log(1 - D_{\theta_d}(G_{\theta_g}(z^{(i)}))) \right]$$

  end for

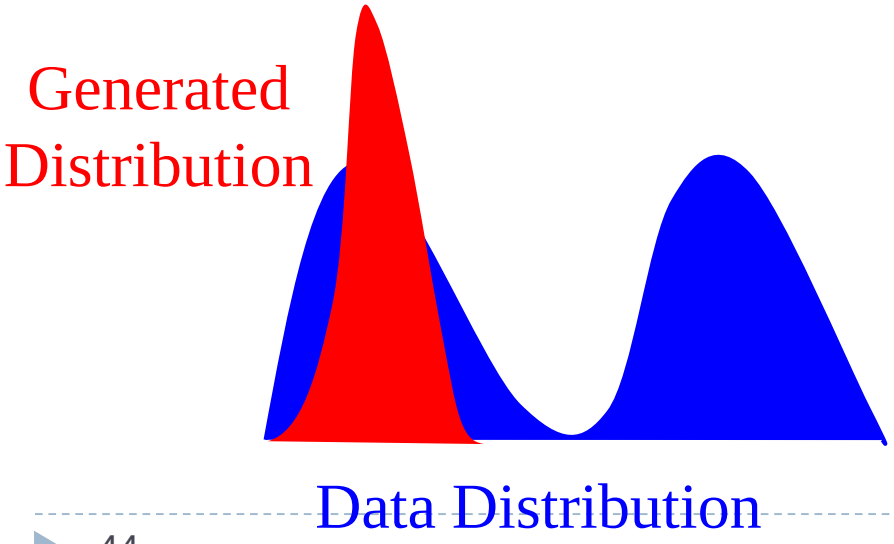
- Sample minibatch of m noise samples $\{\boldsymbol{z}^{(1)}, \ldots, \boldsymbol{z}^{(m)}\}$ from noise prior $p_g(\boldsymbol{z})$.
- Update the generator by ascending its stochastic gradient (improved objective):

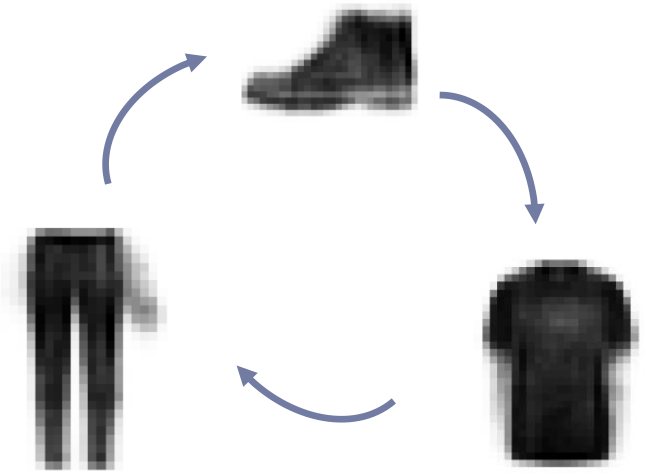
$$\nabla_{\theta_g} \frac{1}{m} \sum_{i=1}^{m} \log(D_{\theta_d}(G_{\theta_g}(z^{(i)})))$$

end for

Training GAN

- The biggest difficulty is called *mode collapse*: this is when the generator's outputs gradually become less diverse
  - Suppose that the generator gets better at producing convincing shoes than any other class. It will fool the discriminator a bit more with shoes, and this will encourage it to produce even more images of shoes. Gradually, it will forget how to produce anything else. Meanwhile, the only fake images that the discriminator will see will be shoes, so it will also forget how to discriminate fake images of other classes





Training GAN

- Tricks
  1. Replace any pooling layers with strided convolutions (in the discriminator) and use transposed convolutions (in the generator)
  2. Use batch normalization in both the generator and the discriminator, except in the generator's output layer and the discriminator's input layer
  3. Use LeakyReLU/ELU instead of a ReLU since sparse gradients can hinder GAN training
  4. Adding random noise to the labels for the discriminator since stochasticity is good for inducing robustness. Because GAN training results in a dynamic equilibrium, GANs are likely to get stuck in all sorts of ways
  5. In generated images, it's common to see artifacts. To fix this, use a kernel size that's divisible by the stride size whenever we use a strided convolution
  6. Proposing new cost functions and architecture or https://github.com/soumith/ganhacks

Applications – Conditional GAN (Text to Image)

It is a distribution

c: train

Prior distribution z

G

Image

$x = G(c, z)$

dropout

x is realistic or not

x

D (type 1)

scalar

Positive example:

Negative example: Image

x is realistic or not + c and x are matched or not

c

x

D (type 2)

scalar

Positive example: (train , )

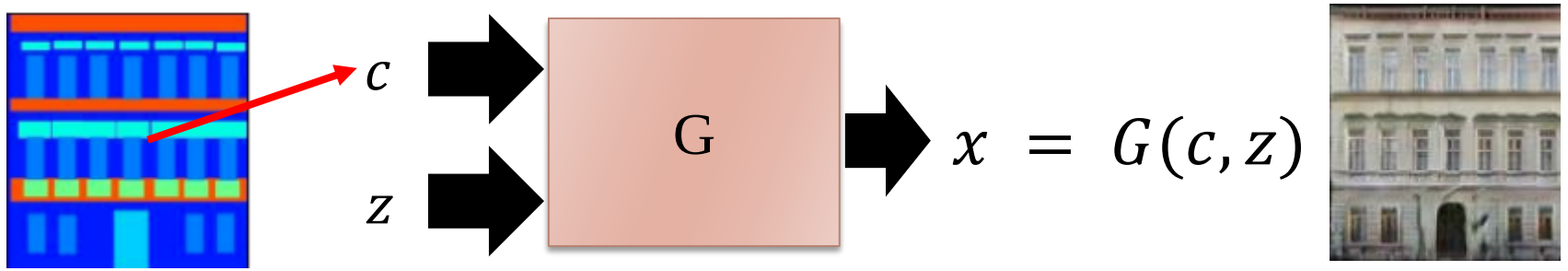
Negative example:

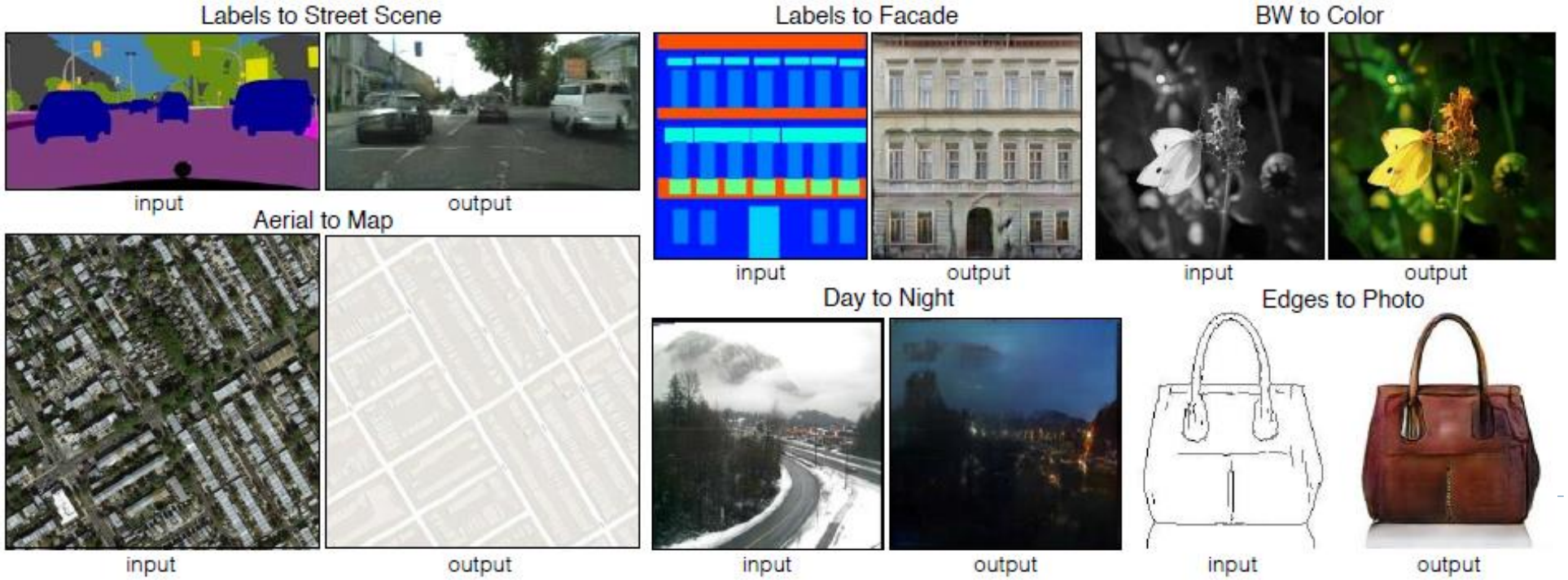
(train , Image) (cat , )

https://speech.ee.ntu.edu.tw/~hylee/ml/ml2021-course-data/gan_v10.pptx

Applications – Conditional GAN with Paired Data (pix2pix)

- Application with unsupervised learning is also possible





VAE

Encoder

Decoder

Flow-based

Encoder

Invertible

Encoder

Diffusion

Add noise

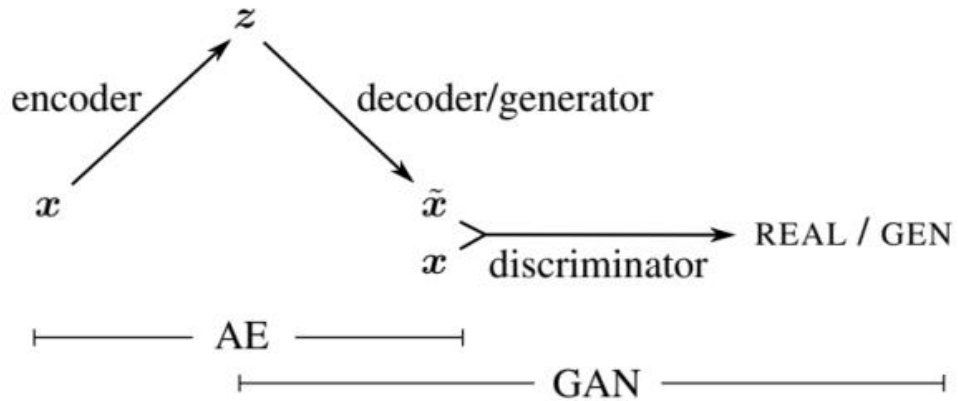
X N

X N

Denoise

VAE + GAN

https://arxiv.org/abs/1512.09300

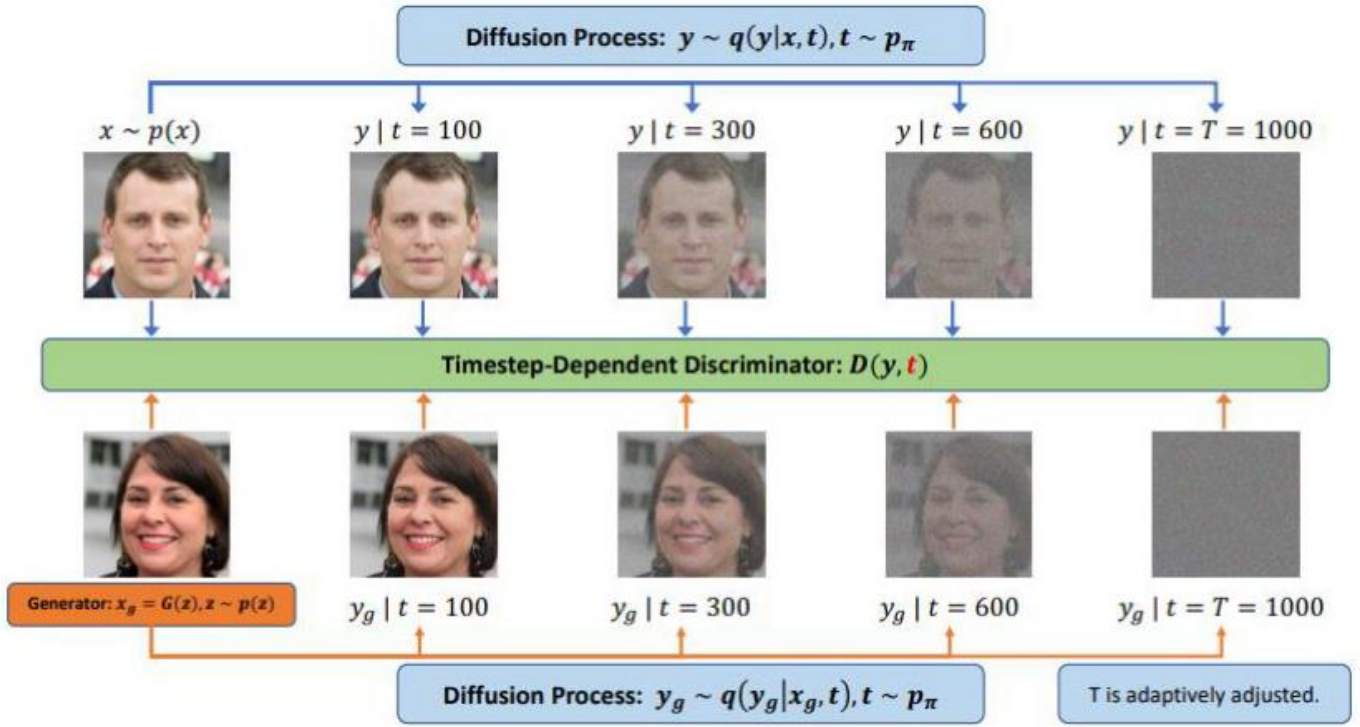


Flow + GAN

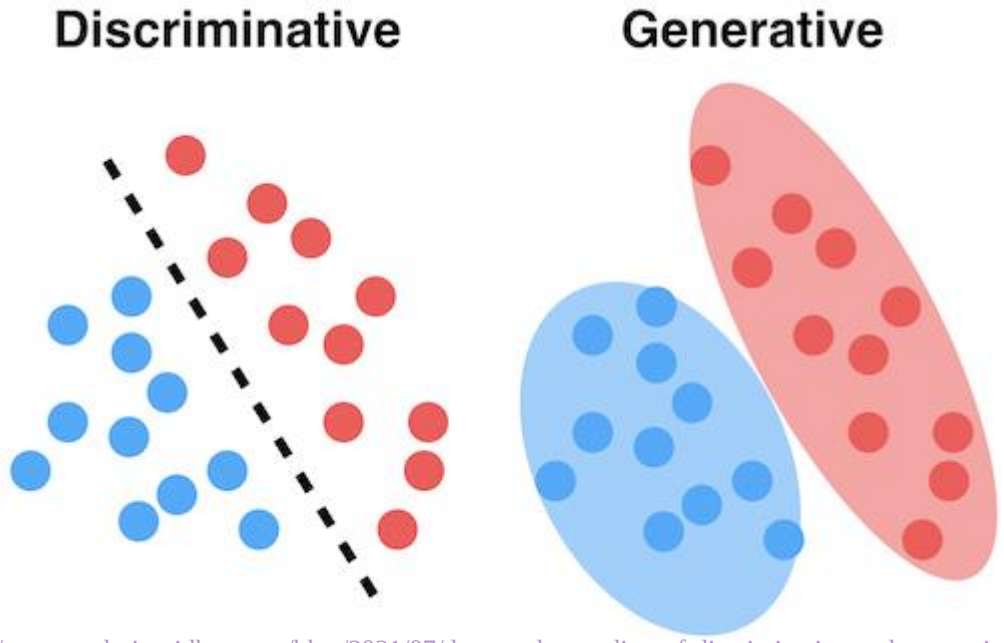
https://arxiv.org/abs/1705.08868

Diffusion + GAN

https://arxiv.org/abs/2206.02262



Generative Models for Classification



https://www.analyticsvidhya.com/blog/2021/07/deep-understanding-of-discriminative-and-generative-models-in-machine-learning/

- Model the distribution of X in each of the classes separately, and then use Bayes theorem to flip things around and obtain $\Pr(Y \mid X)$
  - When we use normal (Gaussian) distributions for each class, this leads to linear or quadratic discriminant analysis. Other distributions can be used as well!

Bayes theorem for classification

- According to the Bayes' theorem:

$$\Pr(Y = k|X = x) = \frac{\Pr(Y = k) \times \Pr(X = x|Y = k)}{\Pr(X = x)}$$

One writes this for discriminant analysis:

$$p_k(x) = \Pr(Y = k|X = x) = \frac{\pi_k f_k(x)}{\sum_{l=1}^{K} \pi_l f_l(x)}$$

  - $f_k(x) = Pr(X = x|Y = k)$ is the density for X in class k. Here we will use normal densities for these, separately in each class
  - $\pi_k = Pr(Y = k)$ is the marginal or prior probability for class k

- Common classifiers that use different estimates of $f_k(x)$ to approximate the Bayes classifier: *linear discriminant analysis, quadratic discriminant analysis, and naive Bayes*