# Moving Beyond Linearity

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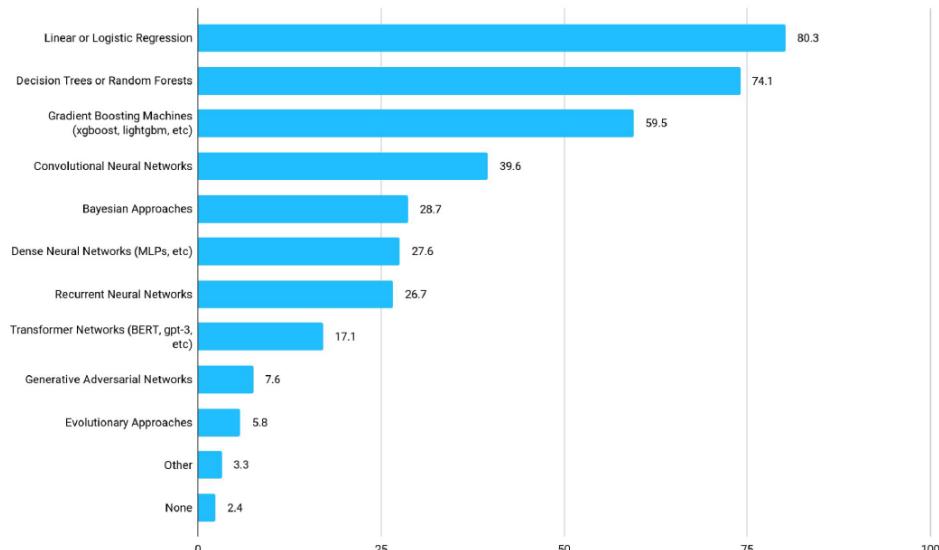
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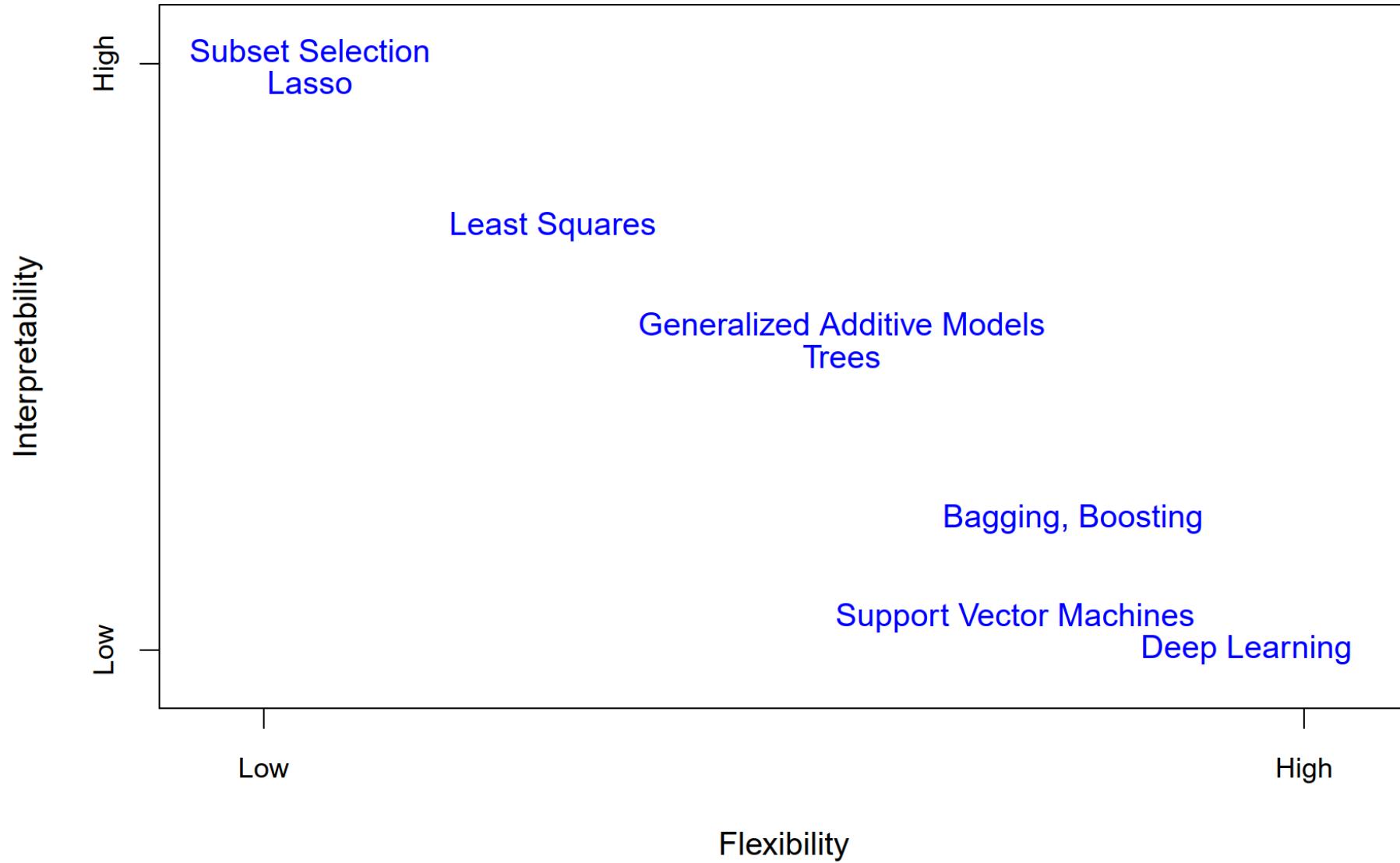
This course – focus on modeling and interpretation

1. Introduction
2. Statistical learning
3. Regression
4. Classification
5. Resampling methods
6. Linear model selection and regularization
7. Moving Beyond Linearity
8. Tree-Based Methods
9. Support Vector Machines
10. Unsupervised Learning

Algorithms Used by: Data Scientists

Methods and Algorithms Usage

Flexibility

Interpretability

High — Subset Selection, Lasso

Least Squares

Generalized Additive Models, Trees

Bagging, Boosting

Support Vector Machines

Deep Learning

Low — High

Moving Beyond Linearity
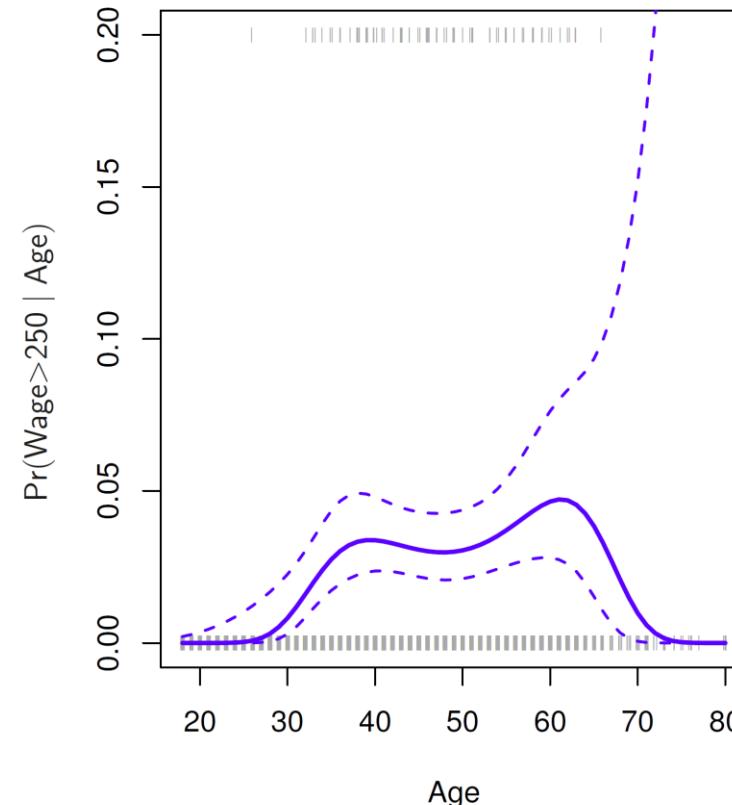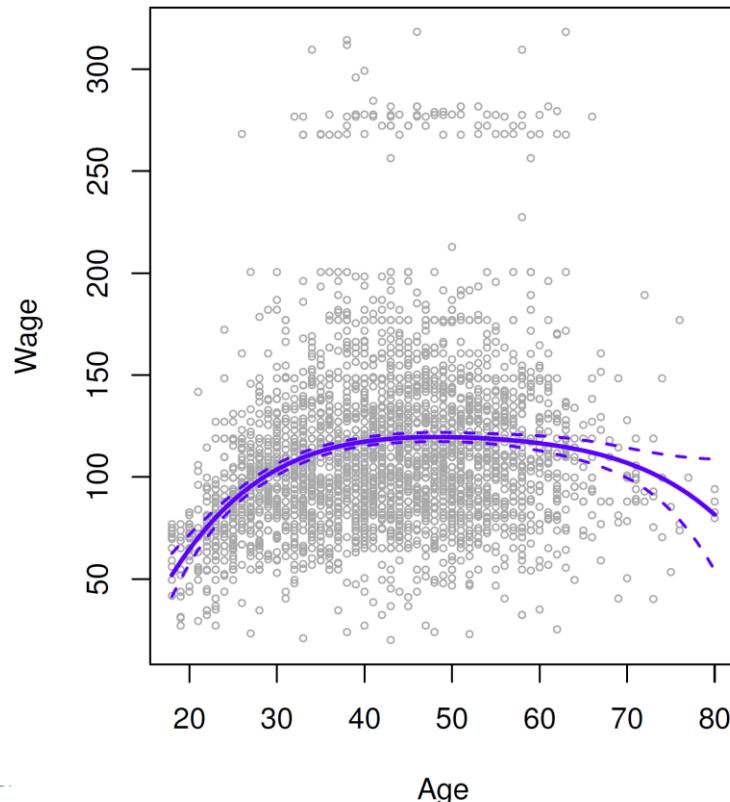
▸ The truth is never linear! Or almost never!

▸ But often the linearity assumption is good enough

▸ When it is not . . .

    ▸ polynomials,

    ▸ step functions,

    ▸ splines,

    ▸ local regression, and

    ▸ generalized additive models

offer a lot of flexibility, without losing the ease and interpretability of linear models

1. Polynomial Regression

$$y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \cdots + \beta_p x_p^p + \epsilon_i$$

Degree−4 Polynomial

Details

▸ Create new variables $X_1 = X$, $X_2 = X^2$, etc and then treat as multiple linear regression ([confidence interval](#))

$$C.I. = \hat{y}_i \pm t_{\alpha/2}\,\hat{\sigma}\,\sqrt{\left(\frac{1}{n} + \frac{(x_i - \bar{x})^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right)}$$

▸ Not really interested in the coefficients; more interested in the fitted function values at any value x_0 (pointwise):

$$\hat{f}(x_0) = \hat{\beta}_0 + \hat{\beta}_1 x_0 + \hat{\beta}_2 x_0^2 + \hat{\beta}_3 x_0^3 + \hat{\beta}_4 x_0^4$$

▸ In the figure we have computed the fit and pointwise standard errors on a grid of values for x_0. We show $\hat{f}(x_0) \pm 2 \cdot se\big[\hat{f}(x_0)\big]$

▸ We either fix the degree p at some reasonably low value, else use cross-validation to choose p

Details continued

▸ Logistic regression follows naturally. For example, in figure we model

$$\Pr(y_i > 250 | x_i) = \frac{\exp(\beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \cdots + \beta_p x_i^p)}{1 + \exp(\beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \cdots + \beta_p x_i^p)}$$
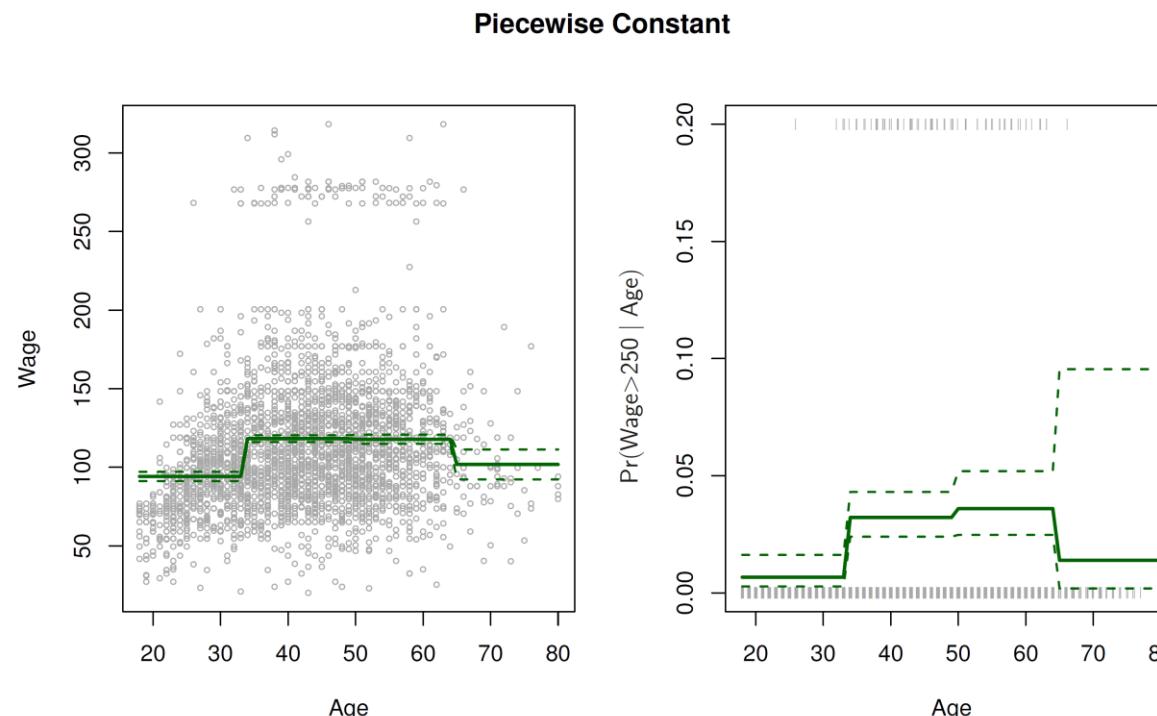
  ▸ To get confidence intervals, compute upper and lower bounds on the logit scale, and then invert to get on probability scale

▸ Can do separately on several variables—just stack the variables into one matrix, and separate out the pieces afterwards (see GAMs later)

▸ Can fit using poly = PolynomialFeatures(4) in formula

▸ Caveat: polynomials have notorious tail behavior — very bad for extrapolation

2. Step Functions (Piecewise-constant)

▶ Another way of creating transformations of a variable — cut the variable into distinct regions with K cutpoints $c_j, 1 \leq j \leq K$ and $K + 1$ new variables

$$C_0(X) = I(X < 35), C_1(X) = I(35 \leq X < 50), C_2(X) = I(50 \leq X < 65), C_3(X) = I(X \geq 65)$$

$$y_i = \beta_0 + \beta_1 C_1(x_i) + \beta_2 C_2(x_i) + \cdots + \beta_K C_K(x_i) + \epsilon_i$$

Piecewise Constant

Step functions continued

- Easy to work with. Creates a series of dummy variables representing each group

- Useful way of creating interactions that are easy to interpret. For example, interaction effect of Year and Age:

$$I(\text{Year} < 2005) \cdot \text{Age}, \quad I(\text{Year} \geq 2005) \cdot \text{Age}$$

would allow for different linear functions in each age category

- In Python: pd.cut(age,4) or pd.cut(age, [0, 25, 40, 65, 90])

- Choice of cutpoints or knots can be problematic. For creating nonlinearities, smoother alternatives such as splines are available

Basis function

▸ Instead of fitting the model in X, we fit
$$y_i = \beta_0 + \beta_1 b_1(x_i) + \beta_2 b_2(x_i) + \cdots + \beta_K b_K(x_i) + \epsilon_i$$

▸ For polynomial regression, $b_j(x_i) = x_i^j$

▸ For step function, $b_j(x_i) = I(c_j \leq x_i < c_{j+1})$

▸ We can think of the model as a standard linear model with predictors $b_1(x_i), b_2(x_i), \ldots, b_K(x_i)$

  ▸ We can use least squares to estimate the unknown regression coefficients

  ▸ All of the inference tools for linear models can be used, such as standard errors for the coefficient estimates and F-statistics for the model's overall significance…
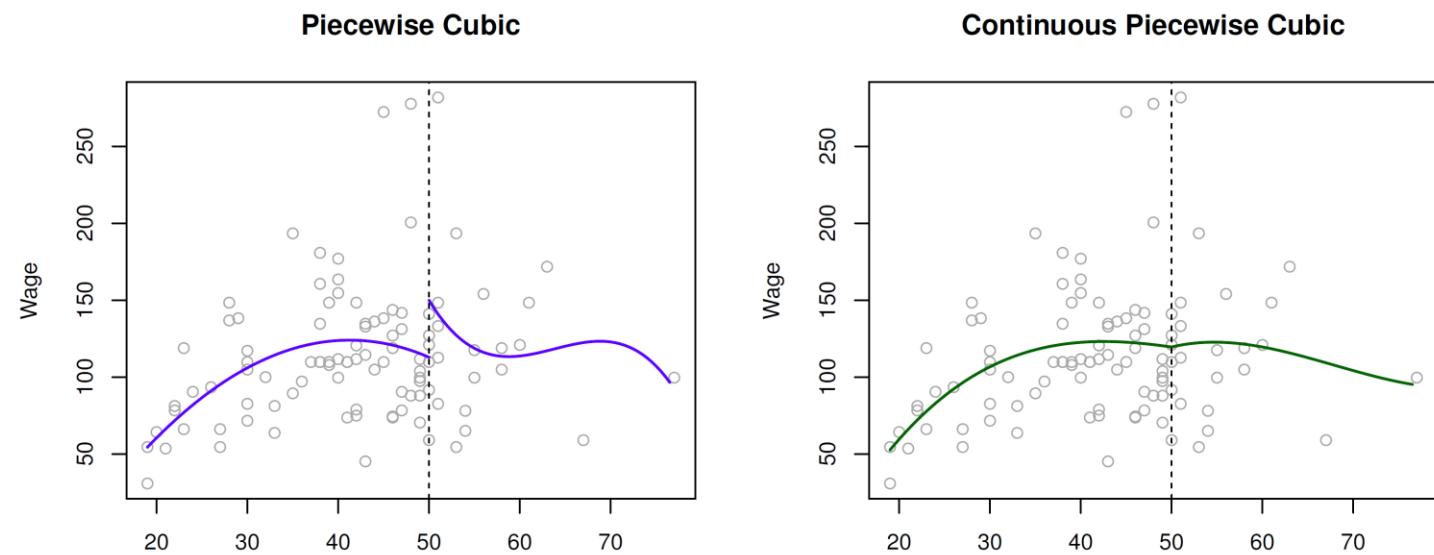
3. Regression Splines - Piecewise Polynomials

▸ Instead of a single polynomial in X over its whole domain, we can rather use different polynomials in regions defined by *knots*

$$y_i = \begin{cases} \beta_{01} + \beta_{11}x_i + \beta_{21}x_i^2 + \beta_{31}x_i^3 + \epsilon_i & if \ x_i < c; \\ \beta_{02} + \beta_{12}x_i + \beta_{22}x_i^2 + \beta_{32}x_i^3 + \epsilon_i & if \ x_i \geq c. \end{cases}$$
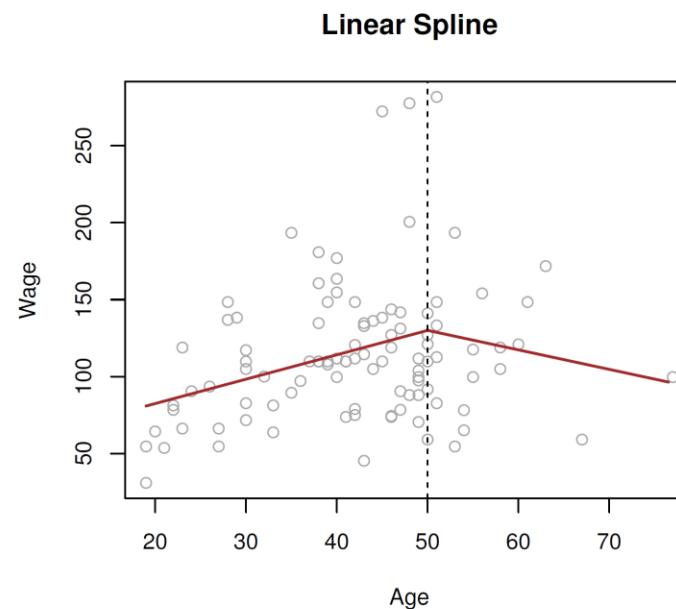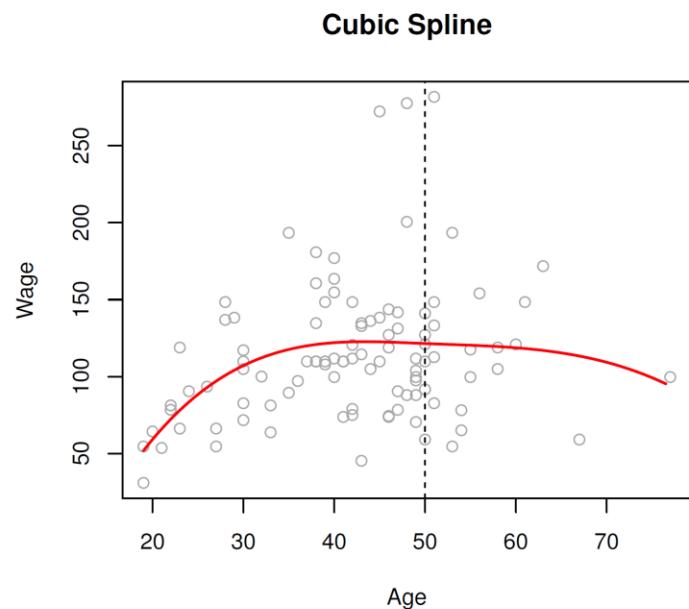
▸ If we place K different knots throughout the range of X, then we will end up fitting $K + 1$ different cubic polynomials

▸ Better to add constraints to the polynomials, e.g. continuity

3. Regression Splines - Piecewise Polynomials

▸ The general definition of a degree-d spline is that it is a piecewise degree-d polynomial, with continuity in derivatives up to degree $d - 1$ at each knot

  ▸ Note that we do not need to use a cubic polynomial. For example, we can instead fit piecewise linear functions!

3.1 Regression Splines - Linear Splines

▸ How can we fit a piecewise degree-d polynomial under the constraint that it (and possibly its first $d-1$ derivatives) be continuous?

  ▸ It turns out that we can use the basis function to represent the regression spline (ESL)

▸ A linear spline with knots at ξ_k, $k = 1, \ldots, K$ is a piecewise linear polynomial continuous at each knot

  ▸ We can represent this model as

 $$y_i = \beta_0 + \beta_1 b_1(x_i) + \beta_2 b_2(x_i) + \cdots + \beta_{K+1} b_{K+1}(x_i) + \epsilon_i$$
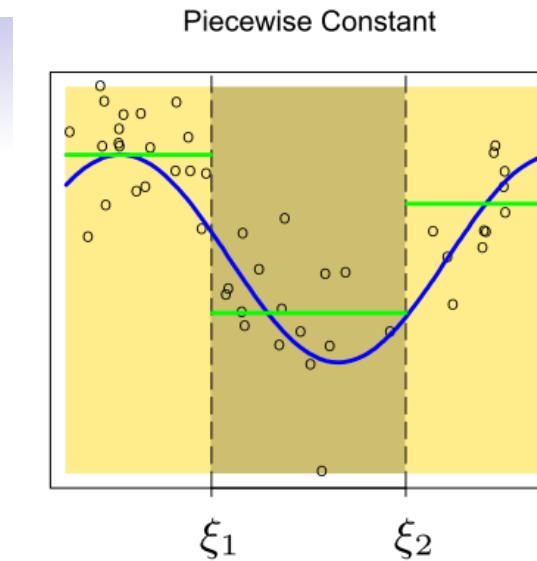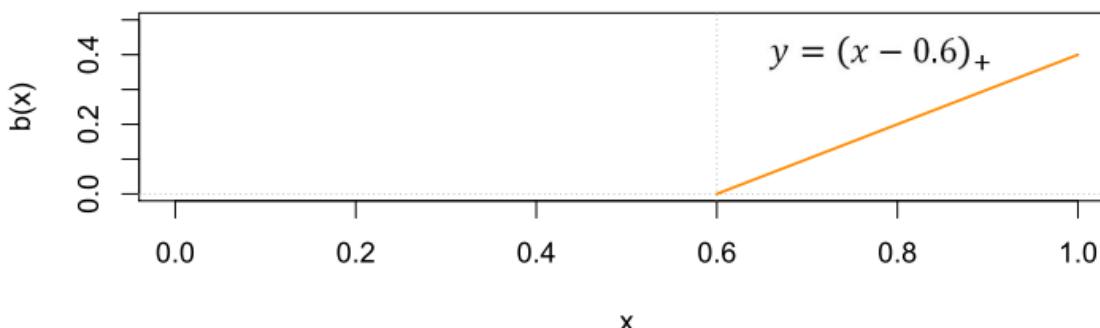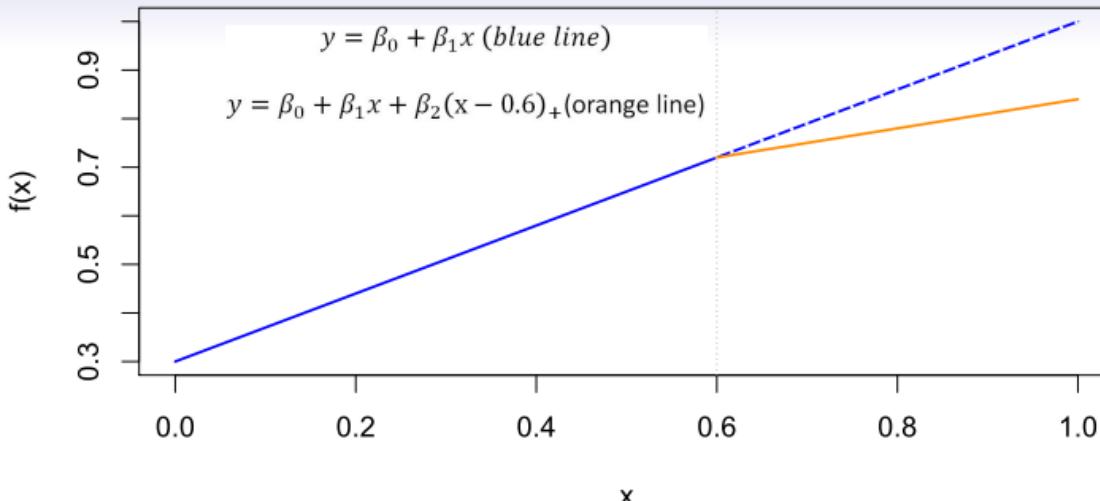
  Where the b_k are basis functions, here we use *truncated power basis* per not

  $$b_1(x_i) = x_i$$
  $$b_{k+1}(x_i) = (x_i - \xi_k)_+, \qquad k = 1, \ldots, K$$
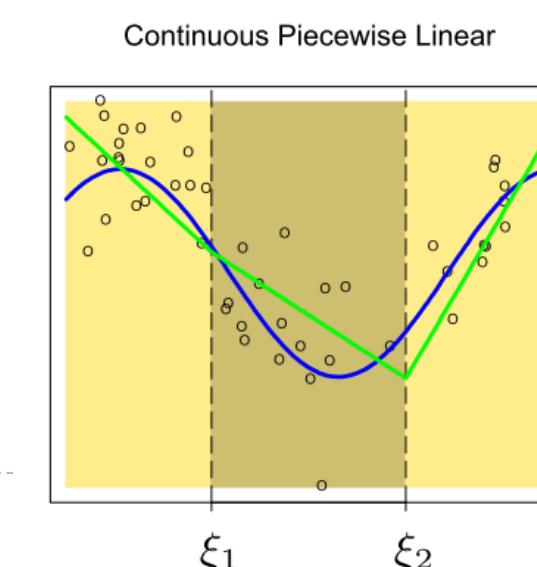
  Here the $()_+$ means positive part; i.e.

  $$(x_i - \xi_k)_+ = \begin{cases} x_i - \xi_k & if\ x_i > \xi_k \\ 0 & otherwise \end{cases}$$

3.1 Regression Splines - Linear Splines

$$y = \beta_0 + \beta_1 x \; (blue\; line)$$

$$y = \beta_0 + \beta_1 x + \beta_2 (x - 0.6)_+ \;(orange\; line)$$

$$y = (x - 0.6)_+$$

Piecewise Constant

ξ_1 ξ_2

Piecewise Linear

ξ_1 ξ_2
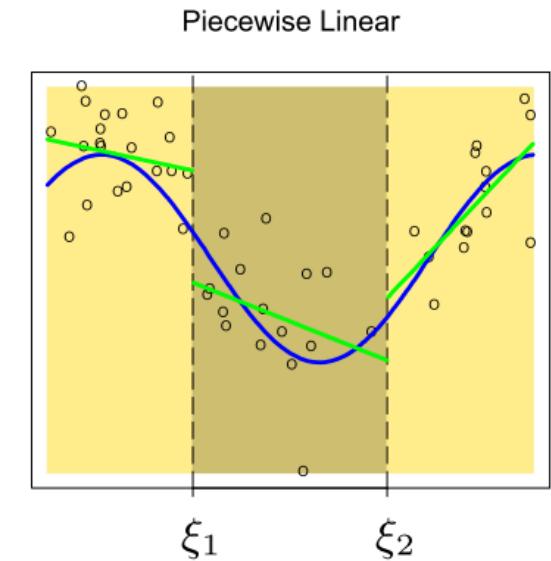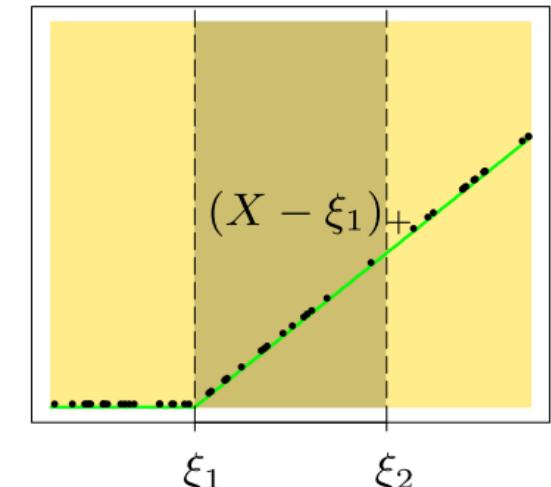
Continuous Piecewise Linear

ξ_1 ξ_2

Piecewise-linear Basis Function

$$(X - \xi_1)_+$$

ξ_1 ξ_2

3.2 Regression Splines - Cubic Splines
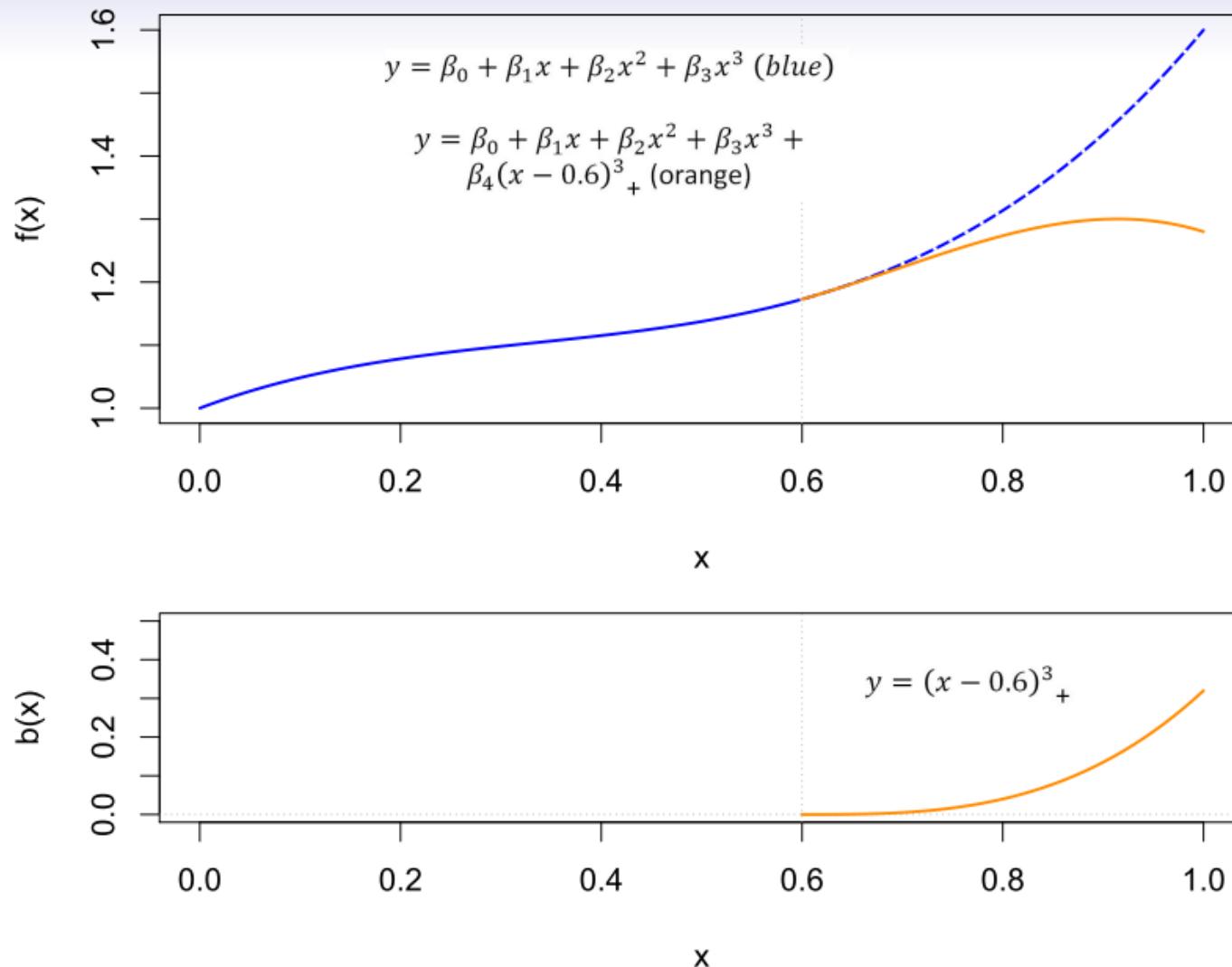
▸ A cubic spline with knots at $\xi_k, k = 1, \ldots, K$ is a piecewise cubic polynomial with continuous derivatives up to order 2 at each knot

▸ Again we can represent this model with truncated power basis functions ($K + 4$ degrees of freedom)

$$y_i = \beta_0 + \beta_1 b_1(x_i) + \beta_2 b_2(x_i) + \cdots + \beta_{K+3} b_{K+3}(x_i) + \epsilon_i$$

$$b_1(x_i) = x_i$$
$$b_2(x_i) = x_i^2$$
$$b_3(x_i) = x_i^3$$
$$b_{k+3}(x_i) = (x_i - \xi_k)_+^3, k = 1, \ldots, K$$
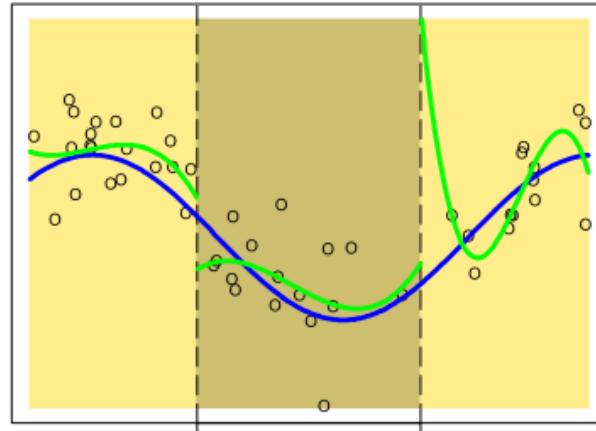
Where

$$(x_i - \xi_k)_+^3 = \begin{cases} (x_i - \xi_k)^3 & if\, x_i > \xi_k \\ 0 & otherwise \end{cases}$$

$$y = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 \ (blue)$$

$$y = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 (x - 0.6)^3_+ \ (orange)$$

$$y = (x - 0.6)^3_+$$
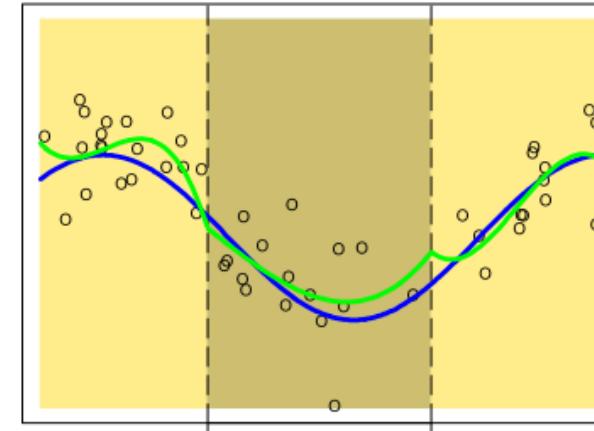
Piecewise Cubic Polynomials

Discontinuous

ξ_1 ξ_2
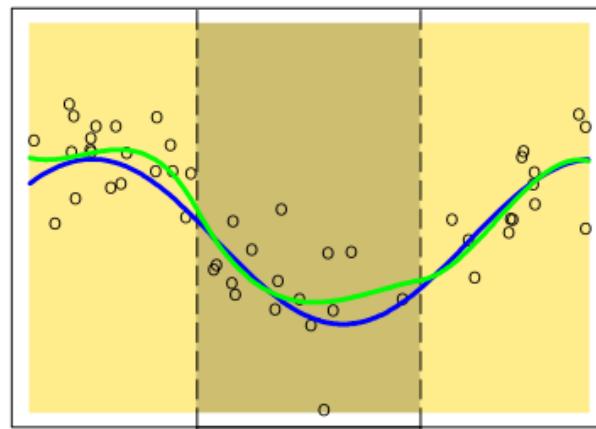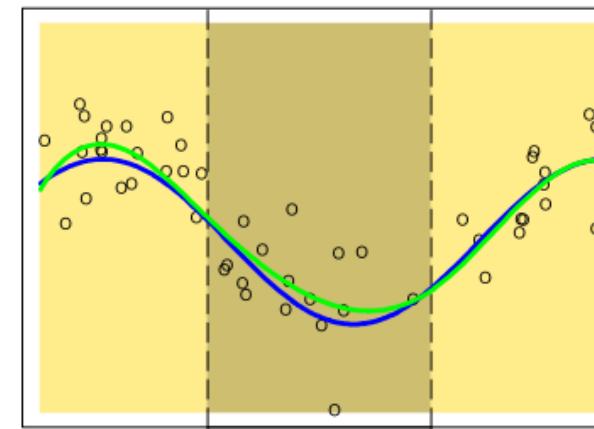
Continuous

ξ_1 ξ_2

Continuous First Derivative

ξ_1 ξ_2
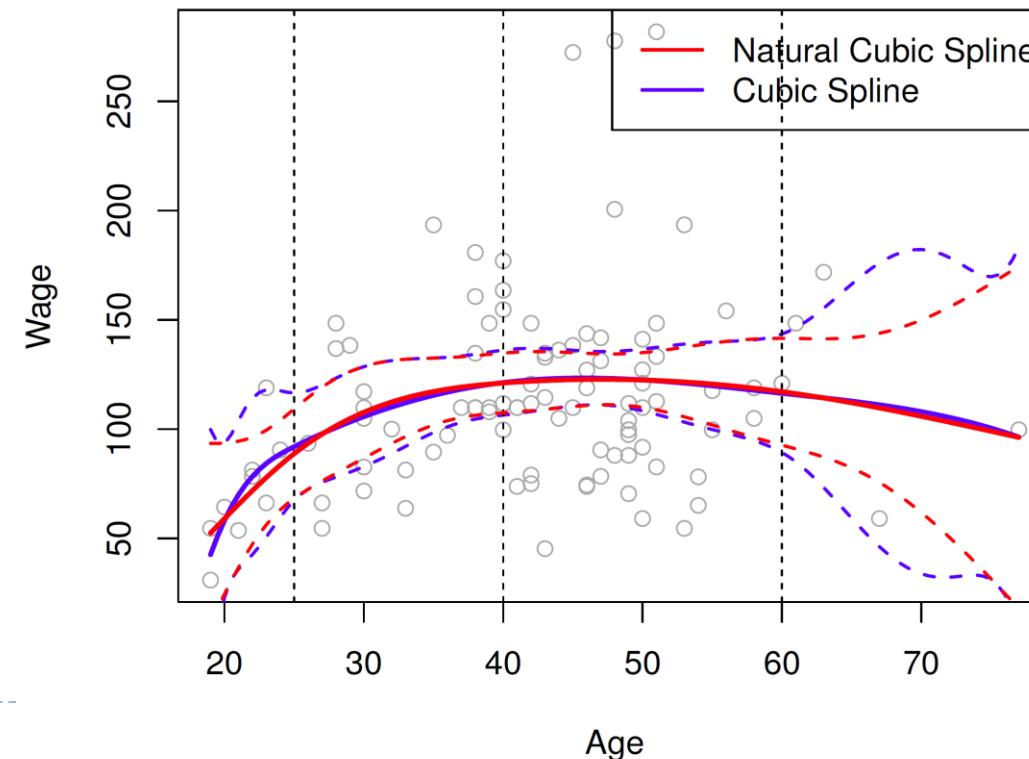
Continuous Second Derivative

ξ_1 ξ_2
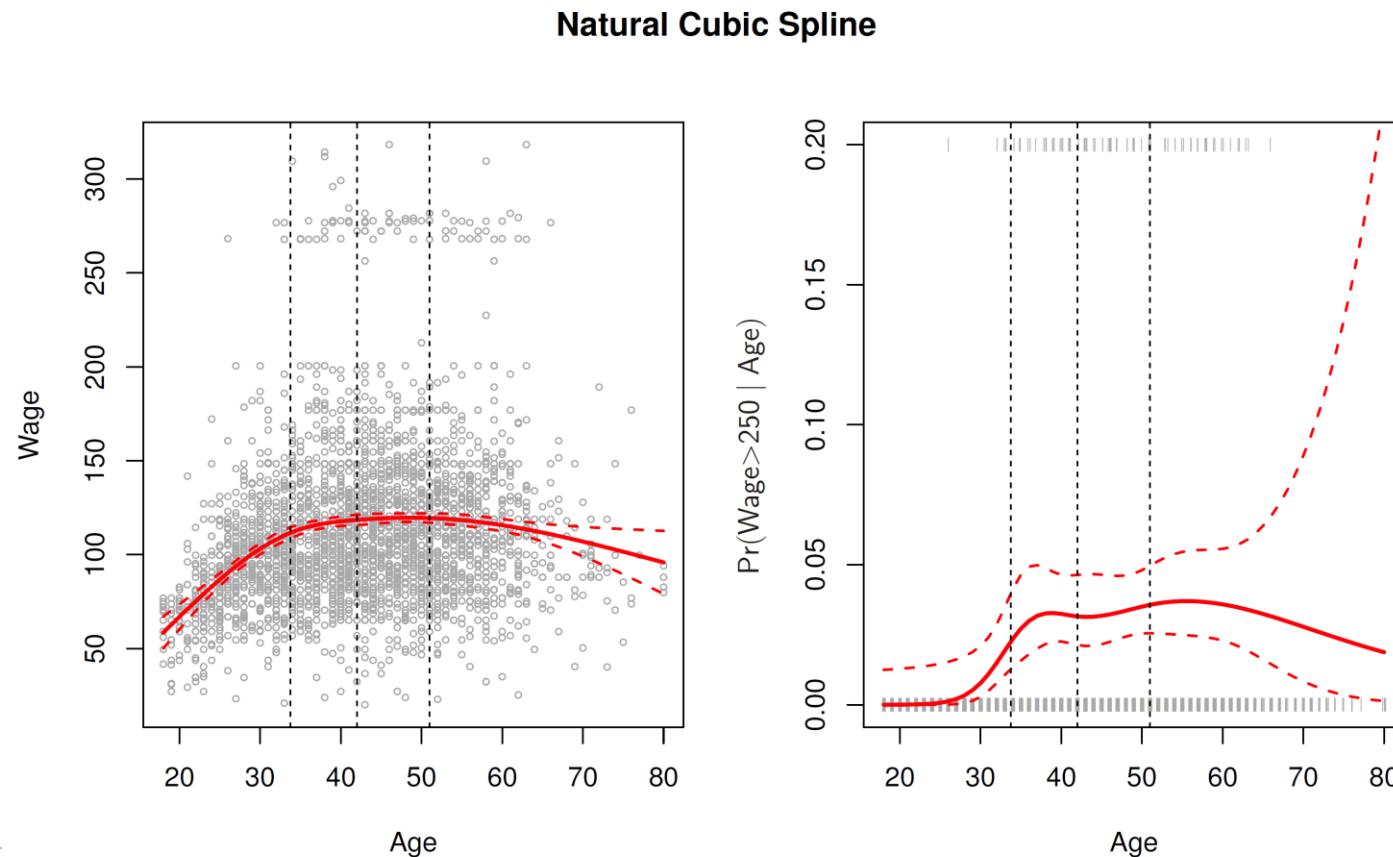
- A natural cubic spline extrapolates *linearly* beyond the boundary knots. This adds $4 = 2 \times 2$ extra constraints, and allows us to put more internal knots for the same degrees of freedom as a regular cubic spline

  - It has zero 2nd and 3rd derivative outside the boundary knots

Regression Splines

▶ Fitting splines in Python is easy: $bs(x, ...)$ for any degree splines, and $cr(x, ...)$ for natural cubic splines with any degree of freedom

Natural Cubic Spline

Regression Splines

▶ The regression spline is most flexible in regions that contain a lot of knots, because in those regions the polynomial coefficients can change rapidly

  ▶ Hence, one option is to place more knots in places where we feel the function might vary most rapidly, and to place fewer knots where it seems more stable

  ▶ In specific fields like biostatistics and epidemiology. For example, 5-year age groups are often used to define the bins

  ▶ While these option can work well, in practice it is common to place knots in a <u>uniform fashion</u>

    ▸ We have fit a natural cubic spline with three knots, the knot locations were chosen as the 25th, 50th, and 75th percentiles of age

    ▸ You can also specify the desired degrees of freedom, and then have the software automatically place the corresponding number of knots at uniform quantiles of the data
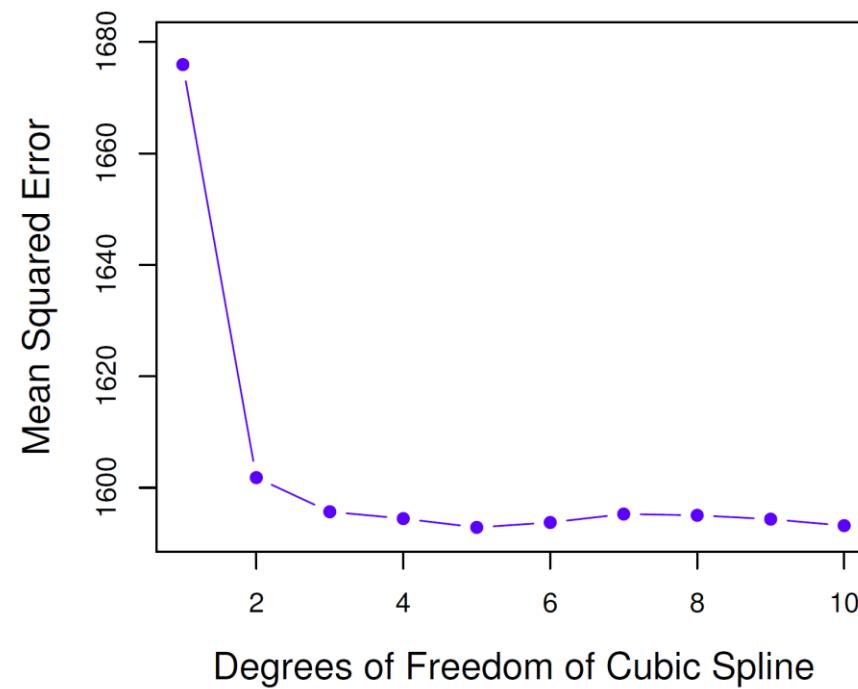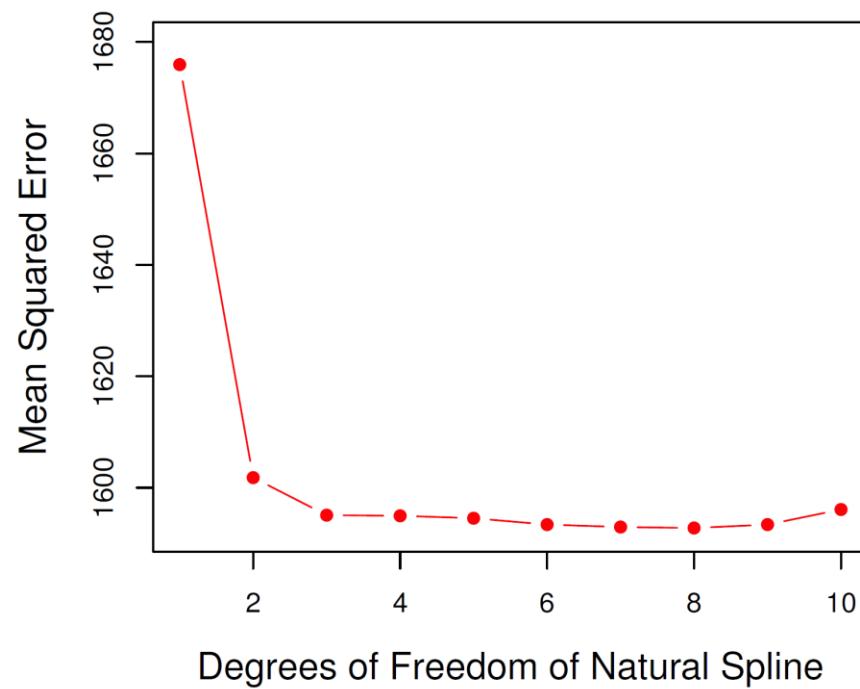
Regression Splines

‣ How many knots should we use, or equivalently how many degrees of freedom should our spline contain?

> ‣ One option is to try out different numbers of knots and see which produces the best looking curve

‣ A more objective approach is to use cross-validation

1. With this method, we remove a portion of the data (say 10%), fit a spline with a certain number of knots to the remaining data, and then use the spline to make predictions for the held-out portion

2. We repeat this process multiple times until each observation has been left out once, and then compute the overall cross-validated RSS

3. This procedure can be repeated for different numbers of knots K

4. Then the value of K giving the smallest test RSS is chosen
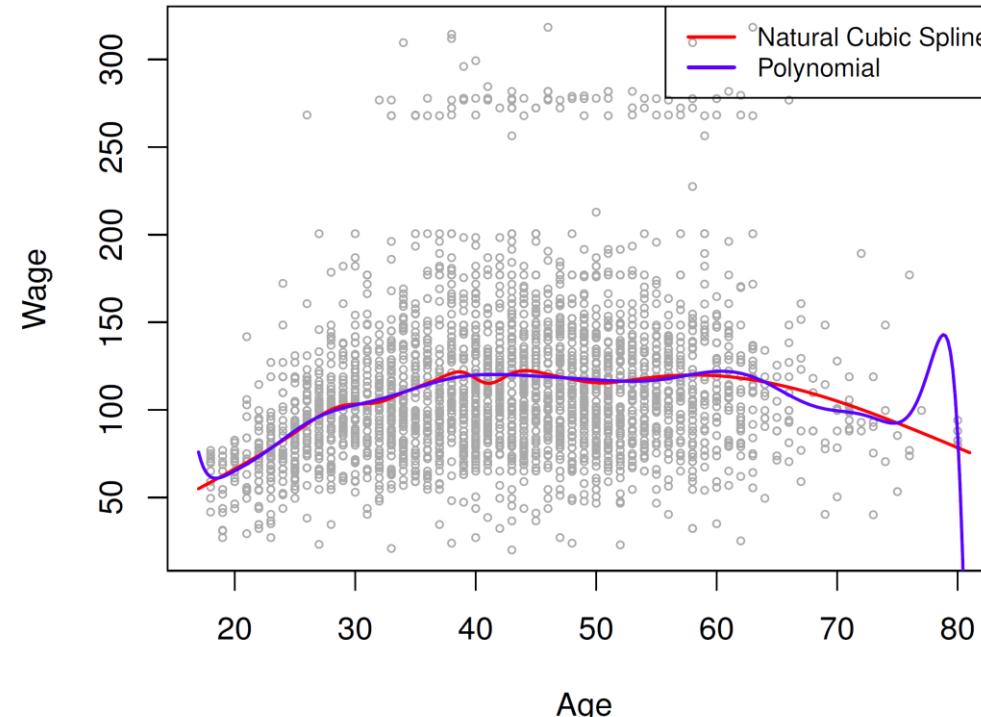
Regression Splines

▸ Ten-fold cross-validated mean squared errors for selecting the degrees of freedom when fitting splines to the Wage data. The response is wage and the predictor age. Left: A natural cubic spline. Right: A cubic spline

Knot placement

▸ One strategy is to decide K, the number of knots, and then place them at appropriate quantiles of the observed X

▸ A cubic spline with K knots has $K + 4$ parameters or degrees of freedom

▸ A natural cubic spline with K knots has K degrees of freedom

Comparison of a degree-14 polynomial and a natural cubic spline, each with 15df.

4. Smoothing Splines

▶ Consider this criterion for fitting a smooth function $g(x)$ to some data:

$$\min_{g \in S} \sum_{i=1}^{n} (y_i - g(x_i))^2 + \lambda \int g''(t)^2 dt$$

▶ The first term is RSS, and tries to make $g(x)$ match the data at each x_i

▶ The second term is a *roughness penalty* and controls how wiggly $g(x)$ is. It is modulated by the tuning parameter $\lambda \geq 0$

  ▶ The smaller λ, the more wiggly the function, eventually interpolating y_i when $\lambda = 0$

  ▶ As $\lambda \rightarrow \infty$, the function $g(x)$ becomes linear - it will just be a straight line that passes as closely as possible to the training points

Smoothing Splines

▸ The unique minimizer of this penalized RSS is *a shrunken version of natural cubic spline* with knots at the unique values of $x_i, i = 1, \ldots, n$

▸ Seems like there will be n features and presumably overfitting of the data. But, the smoothing term shrinks the model towards the linear fit

  ▸ Effective degrees of freedom decrease from n to 2 as we increase λ

  ▸ Smoothing splines avoid the knot-selection issue, leaving a single λ to be chosen

  ▸ The vector of n fitted values can be written as $\hat{g}_\lambda = S_\lambda y$, where S_λ is a $n \times n$ matrix (determined by the x_i and λ)

  ▸ The *effective degrees of freedom* are given by

$$df_\lambda = \sum_{i=1}^{n} \{S_\lambda\}_{ii}$$

Smoothing Splines – choosing λ
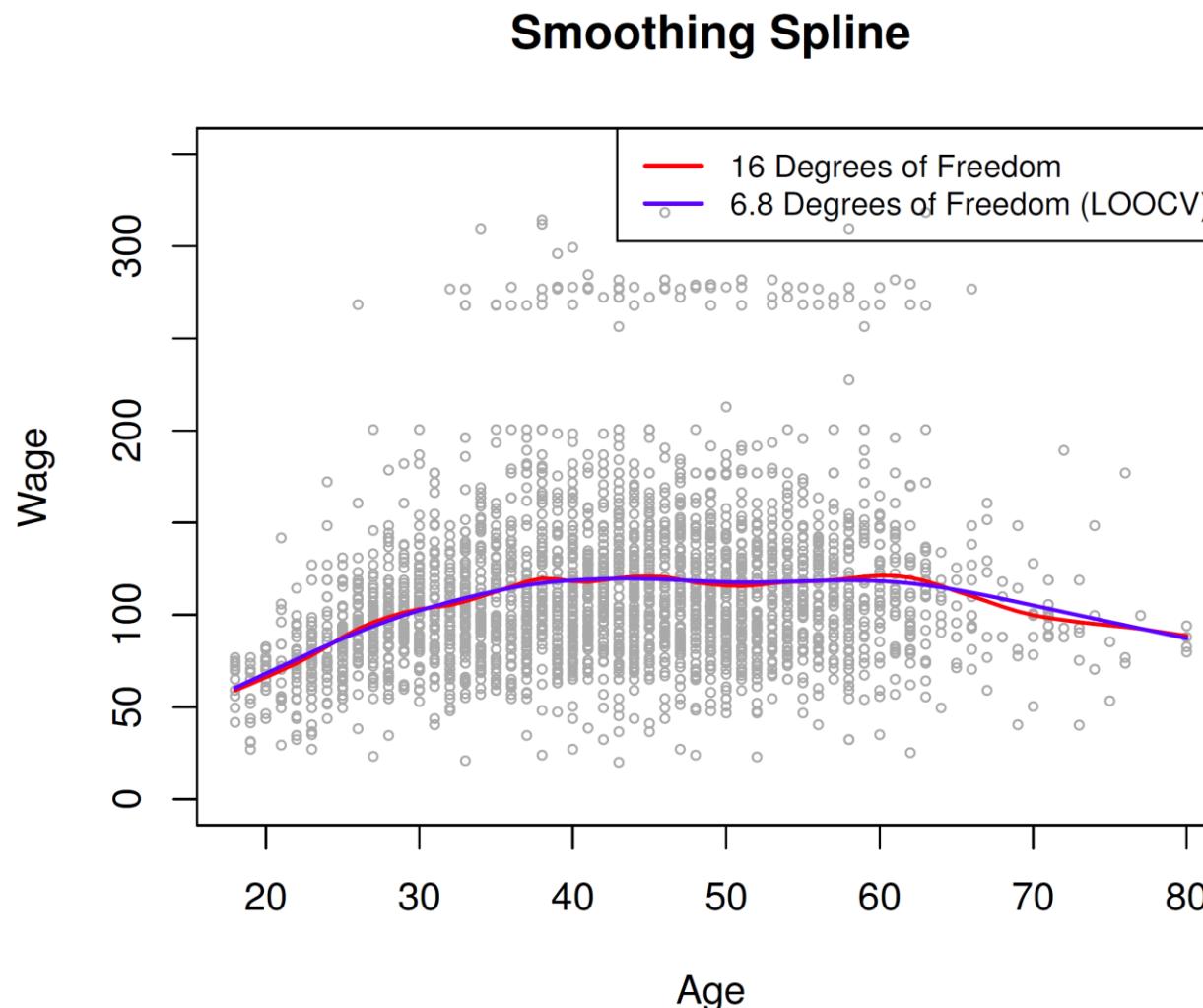
‣ We can also specify df rather than λ!

‣ The leave-one-out (LOO) cross-validated error is given by

$$RSS_{cv}(\lambda) = \sum_{i=1}^{n} (y_i - \hat{g}_\lambda^{(-i)}(x_i))^2 = \sum_{i=1}^{n} \left[\frac{y_i - \hat{g}_\lambda(x_i)}{1 - \{S_\lambda\}_{ii}} \right]^2$$

‣ The notation $\hat{g}_\lambda^{(-i)}(x_i)$ indicates the fitted value for this smoothing spline evaluated at x_i, where the fit uses all of the training observations except for the ith observation (x_i, y_i)

Smoothing Splines

Smoothing Spline

5. Local Regression

▸ Local regression computes the fit at a target point x_0 using only the regression nearby training observations

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Algorithm 7.1 *Local Regression At $X = x_0$*

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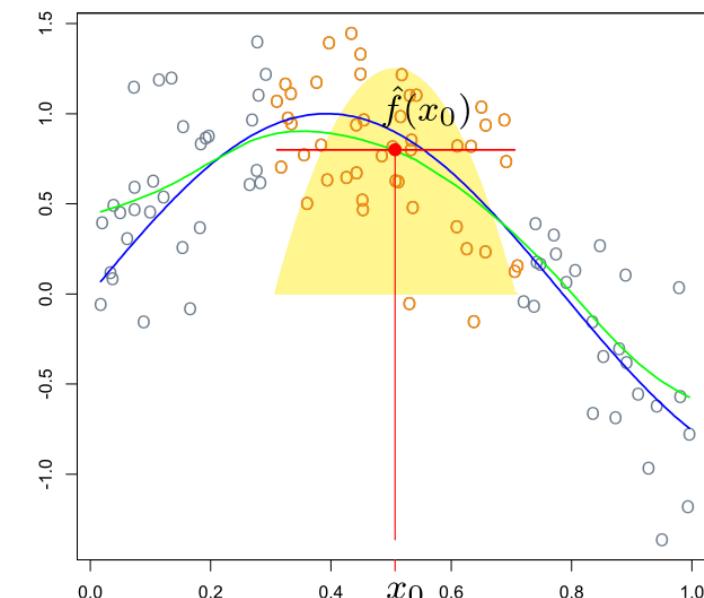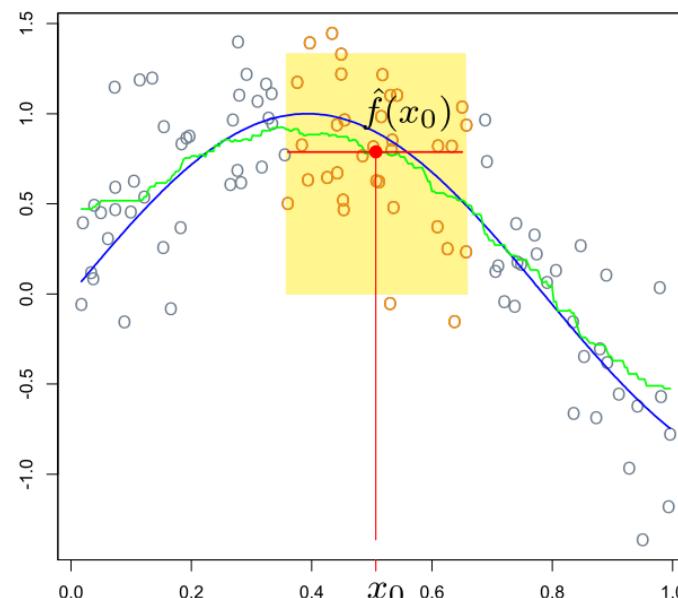
1. Gather the fraction $s = k/n$ of training points whose x_i are closest to x_0.

2. Assign a weight $K_{i0} = K(x_i, x_0)$ to each point in this neighborhood, so that the point furthest from x_0 has weight zero, and the closest has the highest weight. All but these k nearest neighbors get weight zero.

3. Fit a *weighted least squares regression* of the y_i on the x_i using the aforementioned weights, by finding $\hat{\beta}_0$ and $\hat{\beta}_1$ that minimize

$$\sum_{i=1}^{n} K_{i0}(y_i - \beta_0 - \beta_1 x_i)^2. \qquad (7.14)$$

4. The fitted value at x_0 is given by $\hat{f}(x_0) = \hat{\beta}_0 + \hat{\beta}_1 x_0$.
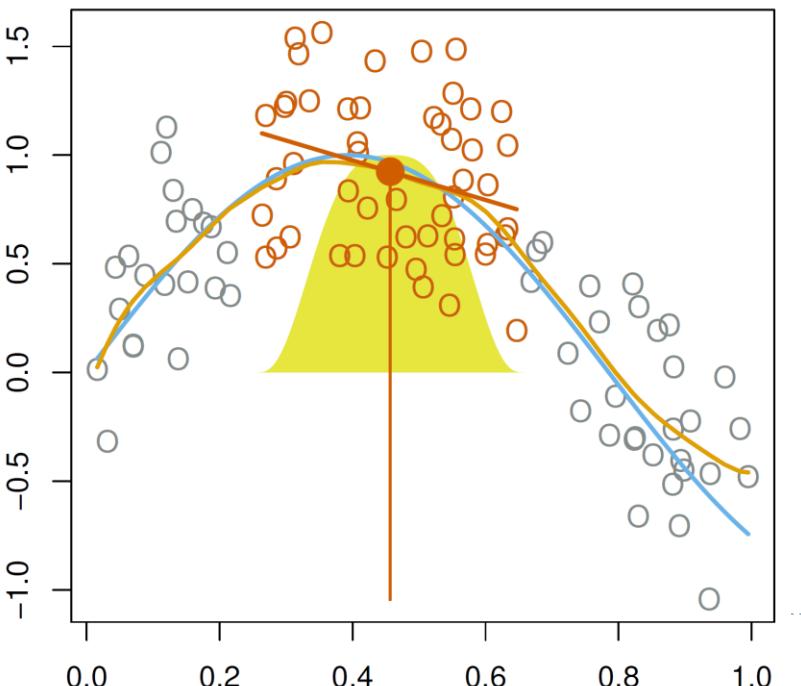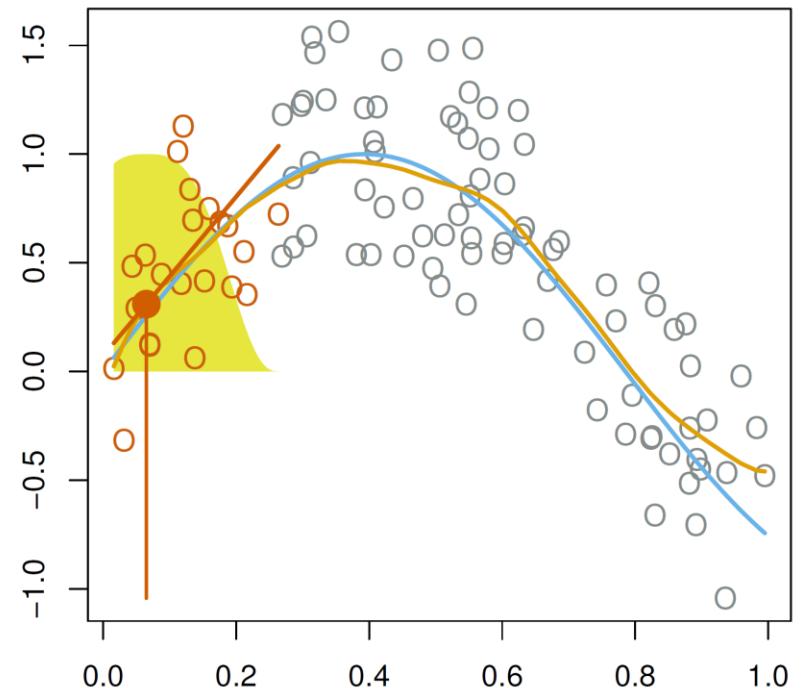
Local Regression

▸ In Step 3 of Algorithm 7.1, the weights K_{i0} will differ for each value of x_0.

　　▸ At a new point, we need to fit a new weighted least squares regression model by minimizing (7.14) for a new set of weights

　　▸ Like KNN, local regression is referred to as a *memory-based* procedure, because we need all the training data each time we wish to compute a prediction

▸ With a sliding weight function, we can fit separate linear fits over the range of X by weighted least squares

　　▸ See text for more details, and *lowess()* function in Python

Local Regression

▸ A simulated data, with one target point near 0.4, and another near the boundary at 0.05

  ▸ The blue line represents the function $f(x)$ from which the data were generated, and the light orange line corresponds to the local regression estimate $\hat{f}(x)$

  ▸ The yellow bell-shape indicates weights assigned to each point, decreasing to zero with distance from the target point. The fitted value $\hat{f}(x_0)$ at x_0 is obtained by fitting a weighted linear regression (orange line segment)
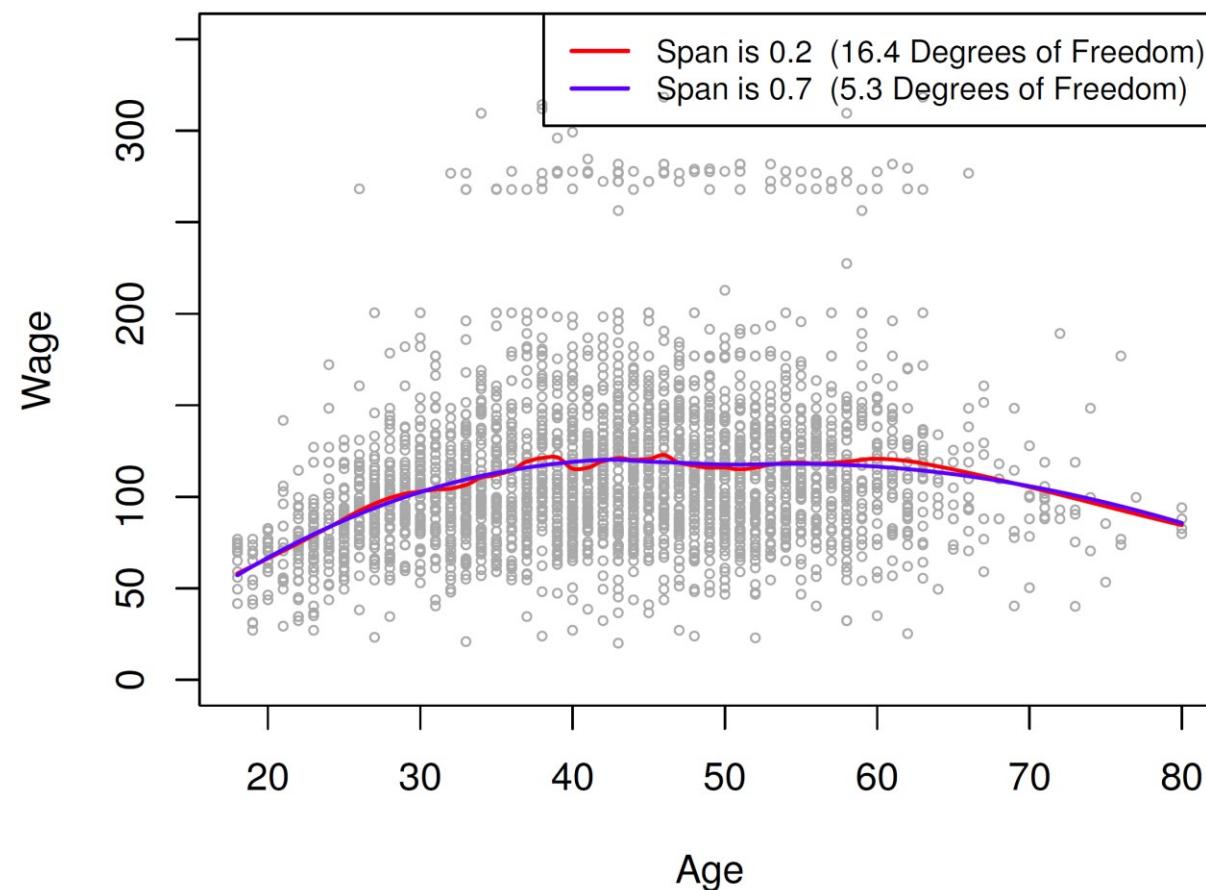
Local Regression - considerations

▸ Choices: span s (Step 1), the weighting function K (Step 2), and whether to fit a linear, constant, or quadratic regression (Step 3)

▸ The most important choice is the span s, similar to the tuning parameter λ in smoothing splines: it controls the flexibility of the non-linear fit

  ▸ The smaller the value of s, the more local and wiggly will be our fit; alternatively, a very large value of s will lead to a global fit to the data using all of the training observations

  ▸ We can use cross-validation to choose s, or we can specify it directly

▸ Figure 7.10 displays local linear regression fits on the Wage data, using two values of s: 0.7 and 0.2

  ▸ The fit obtained using $s = 0.7$ is smoother than that obtained using $s = 0.2$

Local Regression

Local Regression

▸ The idea of local regression can be generalized in many different ways

  ▸ In a setting with multiple features $X_1, X_2, \ldots, X_p$, one very useful generalization involves fitting a multiple linear regression model that is global in some variables, but local in another, such as time. Such varying coefficient models are a useful way of adapting a model to the most recently gathered data

  ▸ Local regression also generalizes very naturally when we want to fit models that are local in a pair of variables X_1 and X_2, rather than one. We can simply use two-dimensional neighborhoods, and fit bivariate linear regression models using the observations that are near each target point in two-dimensional space.

GAMs for Regression Problems

▶ A natural way to extend the multiple linear regression model

$$y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \cdots + \beta_p X_{ip} + \epsilon_i$$

in order to allow for non-linear relationships between each feature and the response

▶ The generalized additive model (GAM) can be written as

$$y_i = \beta_0 + \sum_{j=1}^{p} f_j(x_{ij}) + \epsilon_i$$

▶ It is called an additive model because we calculate a separate f_j for each X_j, and then add together all of their contributions

GAMs for Regression Problems

▸ The beauty of GAMs is that we can use these methods as building blocks for fitting an additive model. Consider the task of fitting the model on the Wage data:
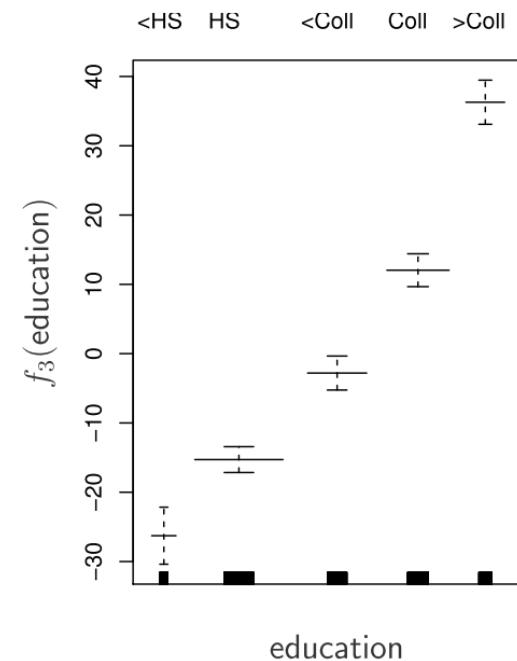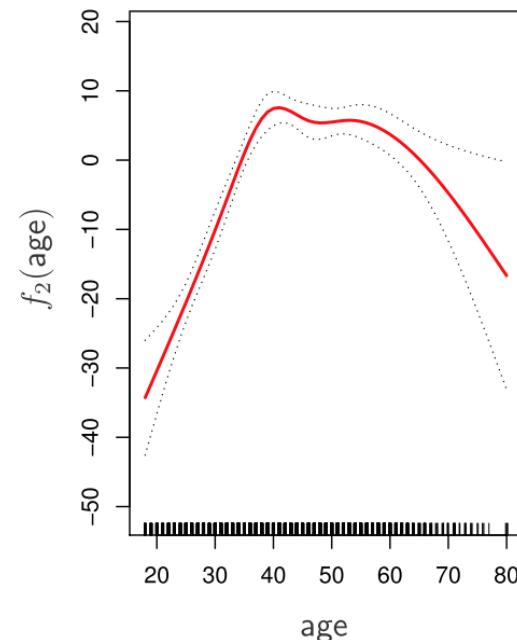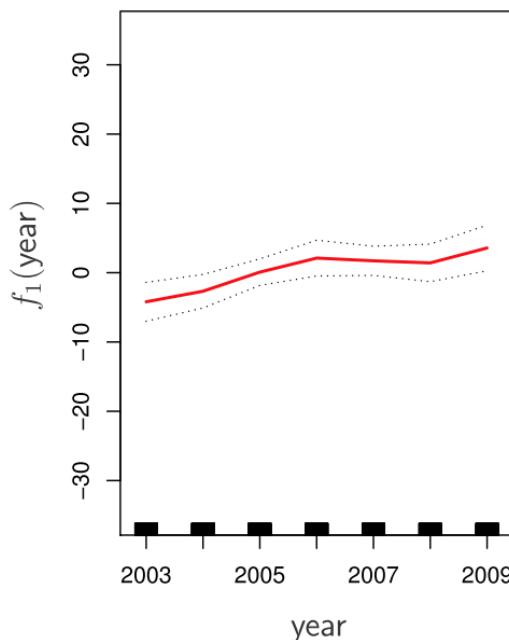
$$wage = \beta_0 + f_1(year) + f_2(age) + f_3(education) + \epsilon$$

  ▸ *Year* and *age* are quantitative variables

  ▸ *Education* is a qualitative variable with five levels: <HS, HS, <Coll, Coll, >Coll, referring to the amount of high school or college education that an individual has completed

  ▸ f_1 and f_2: natural splines

  ▸ f_3: separate constant for each level, via the dummy variable approach

▸ Figure next page shows the fitted model using least squares

Generalized Additive Models

▸ Allows for flexible nonlinearities in several variables, but retains the additive structure of linear models ([Partial residual plot](#))

$$y_i = \beta_0 + f_1(x_{i1}) + f_2(x_{i2}) + \cdots + f_p(x_{ip}) + \epsilon_i$$
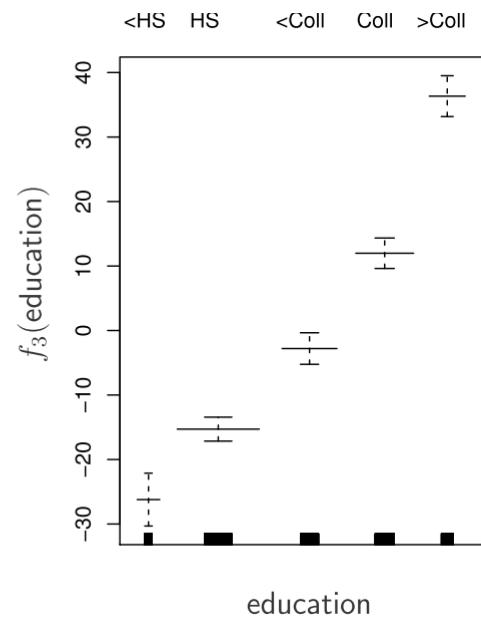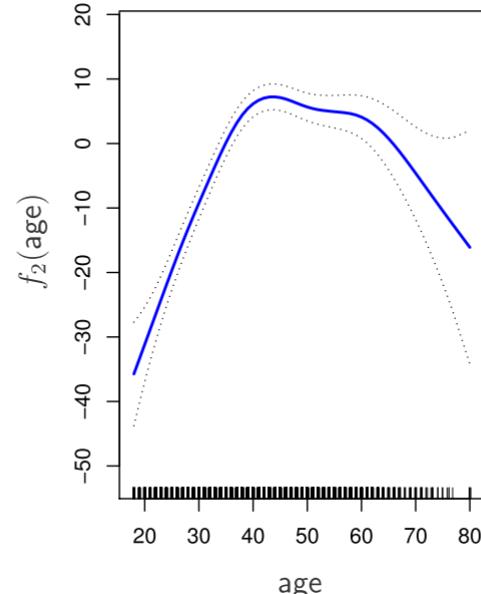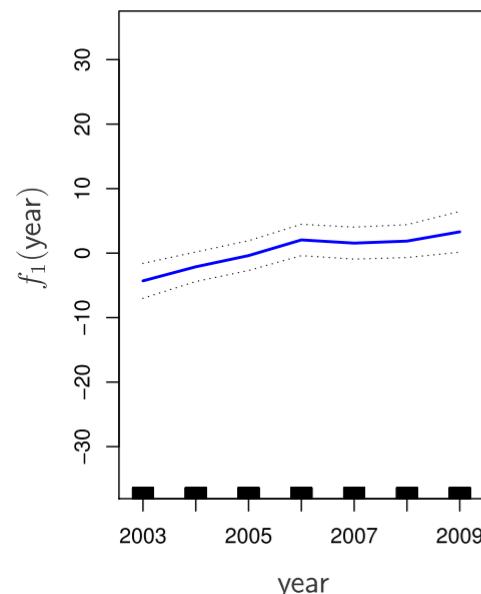
Generalized Additive Models

▸ The above figure can be easily interpreted

▸ The left-hand panel indicates that holding age and education fixed, wage tends to increase slightly with year; this may be due to inflation (通貨膨脹)

▸ The center panel indicates that holding education and year fixed, wage tends to be highest for intermediate values of age, and lowest for the very young and very old

▸ The right-hand panel indicates that holding year and age fixed, wage tends to increase with education: the more educated a person is, the higher their salary, on average

Generalized Additive Models

‣ Figure 7.12 shows a similar triple of plots, but this time f_1, f_2: smoothing splines with 4 and 5 d.f.s, respectively

  ‣ It can be fit via *backfitting* which repeatedly updating the fit for each predictor in turn, holding the others fixed

  ‣ As demonstrated, in most situations, the differences in the GAMs obtained using smoothing splines versus natural splines are small

Pros and Cons of GAMs

▶ The advantages and limitations of a GAM

▸ GAMs allow us to fit a non-linear f_j to each X_j, so that we can automatically model non-linear relationships

▸ The non-linear fits can potentially make more accurate predictions for the response Y

▸ Because the model is additive, we can still examine the effect of each X_j on Y individually while holding all of the other variables fixed

▸ The smoothness of the function f_j for the variable X_j can be summarized via degrees of freedom

Pros and Cons of GAMs

▸ The advantages and limitations of a GAM

  ▸ The main limitation of GAMs is that the model is restricted to be additive: important *interactions* can be missed

    ▸ We can manually add interaction terms to the GAM model by including additional predictors of the form $X_j \times X_k$

    ▸ In addition, we can add low-dimensional interaction functions of the form $f_{jk}(X_j, X_k)$ into the model; such terms can be fit using two-dimensional smoothers such as local regression, or two-dimensional splines

▸ For fully general models, we have to look for even more flexible approaches such as random forests and boosting, described in Chapter 8. GAMs provide a useful compromise between linear and fully nonparametric models

GAMs for Classification Problems

▸ GAMs can also be used when Y is qualitative

▸ Assume Y takes on values zero or one, and let
$$p(X) = \Pr(Y = 1|X)$$

▸ Extend the logistic regression model to the logistic regression GAM:
$$\log(\frac{p(X)}{1 - p(X)}) = \beta_0 + f_1(X_1) + f_2(X_2) + \cdots + f_p(X_p)$$

▸ The general form of GAM
$$\eta\left(E(Y|X_1, \ldots, X_p)\right) = \beta_0 + f_1(X_1) + f_2(X_2) + \cdots + f_p(X_p)$$

GAMs for Classification Problems

▶ We fit a GAM to the Wage data to predict the probability that an individual's income exceeds $250,000 per year:
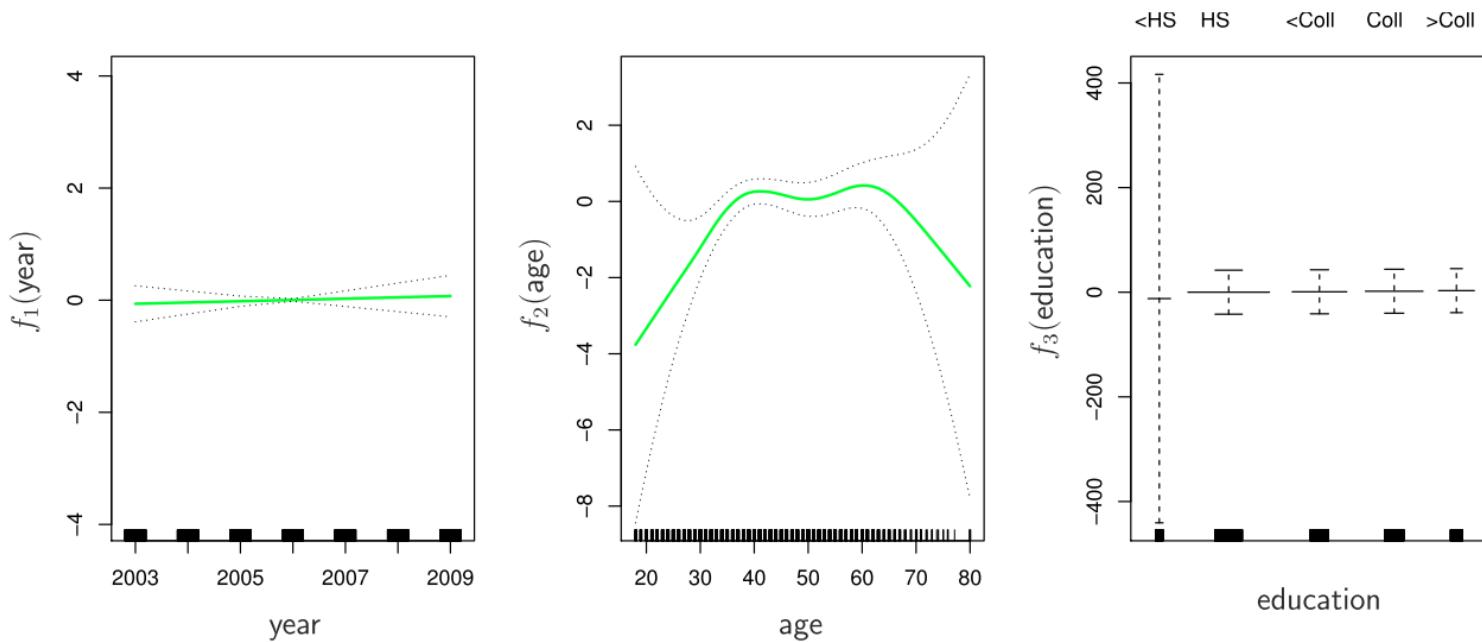
$$p(X) = \Pr(wage > 250 | year, age, education)$$

▶ Consider the GAM:

$$\log\left(\frac{p(X)}{1 - p(X)}\right) = \beta_0 + \beta_1 \times year + f_2(age) + f_3(education)$$

▶ f_2: smoothing spline with five d.f.

▶ f_3: a step function, with dummy variables for each level of education

▶ The resulting fit is shown in Figure 7.13

GAMs for Classification Problems

FIGURE 7.13. *For the* `Wage` *data, the logistic regression GAM given in (7.19) is fit to the binary response* `I(wage>250)`. *Each plot displays the fitted function and pointwise standard errors. The first function is linear in* `year`, *the second function a smoothing spline with five degrees of freedom in* `age`, *and the third a step function for* `education`. *There are very wide standard errors for the first level* `<HS` *of* `education`.
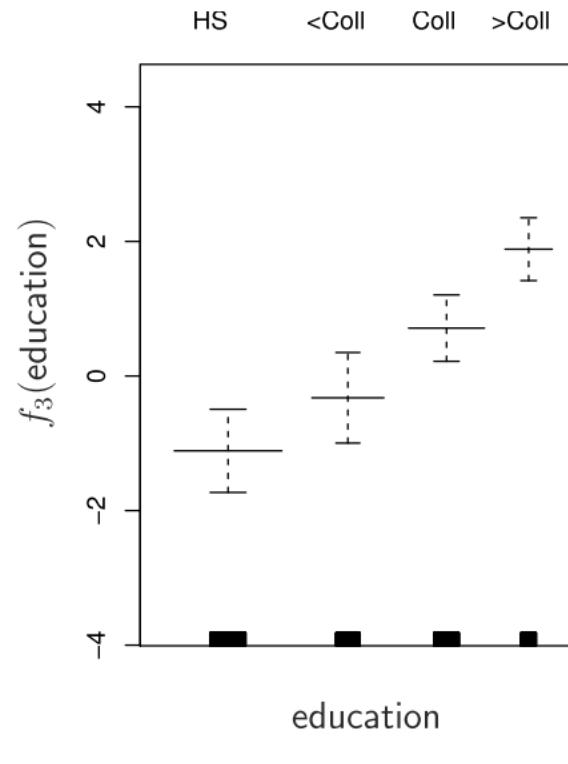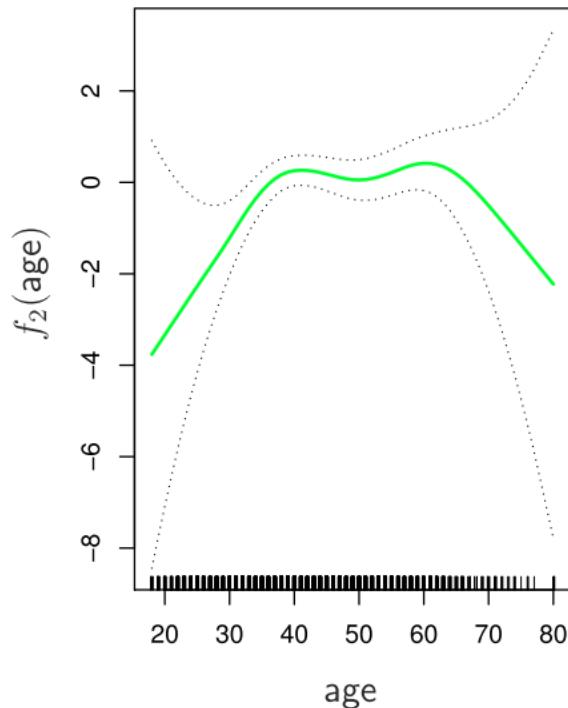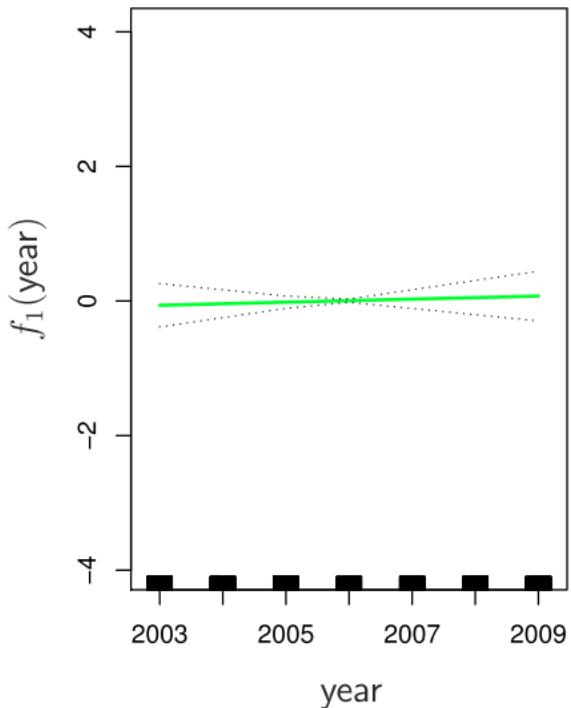
GAMs for Classification Problems

▸ The last panel looks suspicious, with very wide confidence intervals for level <HS. In fact, there are no ones for that category: <u>no individuals with less than a high school education make more than $250,000 per year</u>

▸ Hence we refit the GAM, excluding the individuals with less than a high school education

▸ The resulting model is shown in Figure 7.14

GAMs for Classification Problems

We observe that age and education have a much larger effect than year on the probability of being a high earner

FIGURE 7.14. *The same model is fit as in Figure 7.13, this time excluding the observations for which* `education` *is* `<HS`. *Now we see that increased education tends to be associated with higher salaries.*

Appendix

Natural Cubic Spline Basis (ESL 5.2)

- A natural cubic spline model with K knots is represented by K basis functions:

$$f(x_i) = y_i = \beta_0 b_0(x_i) + \beta_1 b_1(x_i) + \beta_2 b_2(x_i) + \cdots + \beta_{K-1} b_{K-1}(x_i) + \epsilon_i$$

$$b_0(x_i) = 1$$
$$b_1(x_i) = x_i$$
$$b_k(x_i) = d_{k-2}(x_i) - d_{K-1}(x_i)$$

where

$$d_k(x_i) = \frac{(x_i - \xi_{k+1})^3 - (x_i - \xi_K)^3}{\xi_K - \xi_{k+1}}, \qquad k = 0, \ldots, K - 2$$

Smoothing Splines (ESL 5.4)

▸ The objective function is the penalized RSS

$$RSS(\beta, \lambda) = (y - H\beta)^T (y - H\beta) + \lambda \beta^T \Omega_H \beta$$

Where $y = (y_1, \ldots, y_n)', \beta = (\beta_1, \ldots, \beta_n)'$ and

$(H)_{ij} = h_j(x_i)$, and $\Omega_{jk} = \int h_j''(t) h_k''(t) \, dt$

▸ The estimate

$$\hat{\beta} = (H^T H + \lambda \Omega_H)^{-1} H^T y = S_\lambda y$$

▸ This is a generalized ridge regression (when $\Omega_H = I$, it's a ridge regression)

▸ Can show that $S_\lambda = (I + \lambda M)^{-1}$ where the matrix M does not depend on λ

Nonparametric Logistic Regression (ESL 5.6)

▸ Consider logistic regression with a single x:

$$\log(\frac{p(X)}{1-p(X)}) = g(x)$$

And a penalized log-likelihood criterion

$$l(g,\lambda) = \sum_{i=1}^{N}\{y_i \log(p(x_i)) + (1-y_i)\log(1-p(x_i))\} - \frac{1}{2}\lambda \int g''(t)^2 dt$$

$$= \sum_{i=1}^{N}\{y_i g(x_i) - \log(1+e^{g(x_i)})\} - \frac{1}{2}\lambda \int g''(t)^2 dt$$

▸ Again can show that the optimal g is a natural spline with knots at the datapoint

▸ Can use Newton-Raphson to do the fitting

Thin-Plate Splines (Multidimensional Splines) (ESL 5.7)

▸ The discussion up to this point has been one-dimensional. The higher-dimensional analogue of smoothing splines are "thin-plate splines." In 2-D, instead of minimizing

$$\min_{g \in S} \sum_{i=1}^{n} (y_i - g(x_i))^2 + \lambda \int g''(t)^2 dt$$

Minimized

$$\min_{g \in S} \sum_{i=1}^{n} (y_i - g(x_i))^2 + \lambda J(f)$$

Where

$$J(f) = \iint \left(\frac{\partial^2 g(x)}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 g(x)}{\partial x_1 x_2}\right)^2 + \left(\frac{\partial^2 g(x)}{\partial x_2^2}\right)^2 dx_1 dx_2$$

Thin-Plate Splines

▸ The solution has the form:

$$g(x) = \beta_0 + \beta^T x + \sum_{j=1}^{N} \alpha_j h_j(x)$$

Where

$$h_j(x) = \eta(|x - x_j|) \text{ and } \eta(z) = z^2 \log z^2$$

B-spline (ESL 5-Appendix)

- Since the space of spline functions of a particular order and knot sequence is a vector space, there are many equivalent bases for representing them (just as there are for ordinary polynomials.)

- While the truncated power basis is conceptually simple, it is not too attractive numerically: powers of large numbers can lead to severe rounding problems

- In practice, we often use another basis: the B-spline basis, which allows for efficient computations even when the number of knots K is large (each basis function has a local support)

- See here for more information

Some notes about plotting

- Partial regression
  - https://en.wikipedia.org/wiki/Partial_regression_plot
- Partial residual plot
  - https://en.wikipedia.org/wiki/Partial_residual_plot
  - https://stats.stackexchange.com/questions/279576/partial-residuals-plot
  - https://stats.stackexchange.com/questions/423375/why-is-there-residual-dots-in-plot-of-gam
- Partial dependence
  - More general method!
- Implementation
  - https://www.statsmodels.org/dev/generated/statsmodels.gam.generalized_additive_model.GLMGamResults.html

Partial dependence plots (PDP)

▸ Partial dependence plots show *how a feature affects predictions*

▸ This is useful to answer questions like:

　▸ Controlling for all other house features, what impact do longitude and latitude have on home prices? Are predicted health differences between two groups due to differences in their diets, or due to some other factor?

▸ If you are familiar with linear or logistic regression models, partial dependence plots can be interpreted similarly to the coefficients in those models

　▸ Though, partial dependence plots on **sophisticated models** can capture more complex patterns than coefficients from simple models

　▸ Partial dependence plots are calculated after a *model has been fitted*

Partial dependence plots (PDP)

▶ The partial dependence function for regression is defined as

$$\hat{f}_S(x_S) = E_{X_C}[\hat{f}(x_S, X_C)] = \int \hat{f}(x_S, X_C)p(x_C)dx_C = \frac{1}{n}\sum_{i=1}^{n}\hat{f}\left(x_S, X_C^{(i)}\right)$$

The x_S are the features for which the partial dependence function should be plotted and X_C are the other features used in the model $\hat{f}$, which are here treated as random variables

▶ It works by marginalizing the model output over the distribution of the features in set C, so that the function shows the relationship between the S we are interested in and the predicted outcome. Computing this integral for various values of x_S produces a PDP plot

▶ It is assumed that the feature(s) for which the partial dependence is computed are not correlated with other features