Number Systems

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Introduction

- A number system defines how a number can be represented using distinct symbols
  - A number can be represented differently in different systems. For example, the two numbers $(2A)_{16}$ and $(52)_8$ both refer to the same quantity, $(42)_{10}$, but their representations are different

- Several number systems have been used in the past and can be categorized into two groups: *positional* (位置) and *non-positional* (非位置) systems. Our main goal is to discuss the positional number systems, but we also give examples of non-positional systems

Positional Number Systems

- In a positional number system, the position a symbol occupies in the number determines the value it represents
  - In this system, a number is represented as:

$$\pm(S_{k-1} \ldots S_2 S_1 S_0 . S_{-1} S_{-2} \ldots S_{-L})_b$$

has the value of

$$n = \pm S_{k-1} \times b^{k-1} + \cdots + S_1 \times b^1 + S_0 \times b^0 + S_{-1} \times b^{-1} + \cdots + S_{-L} \times b^{-L}$$

in which S is the set of symbols, b is the *base (基底)* (or *radix*) which is equal to the total number of the symbols in the set S

  - Notice the radix point (decimal point)

The decimal system (十進位系統) (base 10)

- In this system, the base $b = 10$ and we use ten symbols

$$S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

  The symbols in this system are often referred to as *decimal digits* or just *digits*

- A number is written as

$$\pm(S_{k-1} \ldots S_2 S_1 S_0 . S_{-1} S_{-2} \ldots S_{-L})_{10}$$

- For simplicity, we often drop the parentheses, the base, and the plus sign

$$+(552.23)_{10} \rightarrow 552.23$$

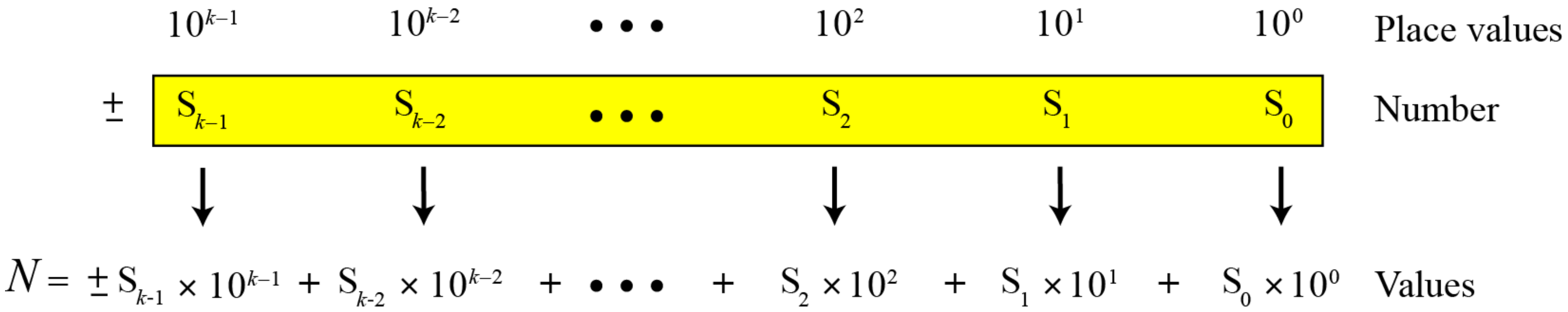
Integers

- We represent an integer as

$$\pm S_{k-1} \ldots S_2 S_1 S_0$$

in which S_i is a digit, $b = 10$ is the base, and K is the number of digits

- The *place values (位值)* is the power of the base $(10^0, 10^1, \ldots, 10^{K-1})$



Maximum value and reals

- Sometimes we need to know the maximum value of a decimal integer that can be represented by K digit

$$N_{max} = 10^K - 1$$

- A real (a number with a fractional part) in the decimal system is also familiar. We can represent a real as $\pm S_{k-1} \dots S_2 S_1 S_0 . S_{-1} S_{-2} \dots S_{-L}$ and the value is

Integral part | Fractional part

$$R = \pm \underbrace{S_{K-1} \times 10^{K-1} + \dots + S_1 \times 10^1 + S_0 \times 10^0}_{\text{Integral part}} + \underbrace{S_{-1} \times 10^{-1} + \dots + S_{-L} \times 10^{-L}}_{\text{Fractional part}}$$

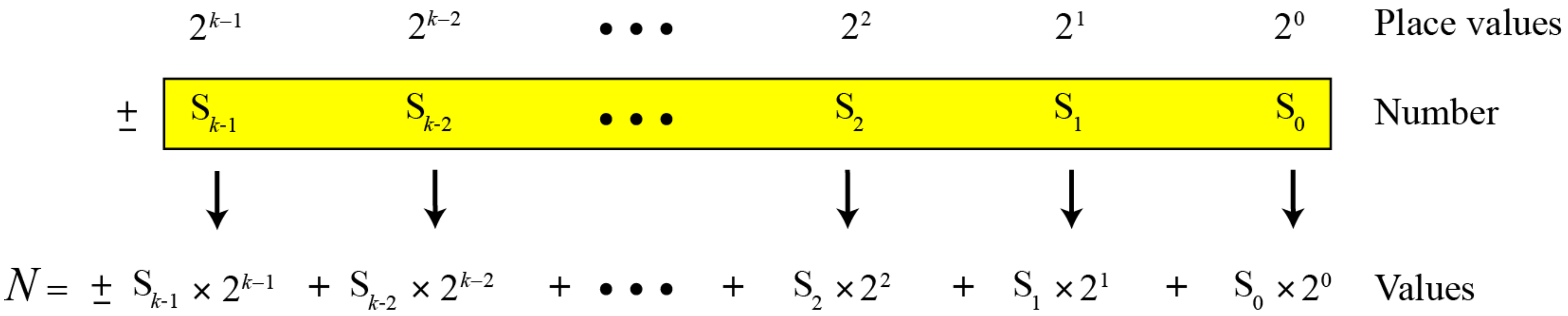
The binary system (二進位系統) (base 2)

- In this system, the base $b = 2$ and we use only two symbols, $S = \{0, 1\}$. The symbols in this system are often referred to as *binary digits* or *bits*
- We can represent an integer as

$$\pm(S_{k-1} \dots S_2 S_1 S_0)_2$$

in which S_i is a binary digit, $b = 2$ is the base, and K is the number of bits

- What is the corresponding decimal of $(11001)_2$?



Maximum value and reals

- The maximum value of a binary integer with K digits is

$$N_{max} = 2^K - 1$$

- A real (a number with a fractional part) in the binary system is represented as $\pm\,(S_{k-1} \dots S_2 S_1 S_0 . S_{-1} S_{-2} \dots S_{-L})_2$ and the value is

$$\underbrace{R = \pm\ S_{K-1} \times 2^{K-1} \times \dots \times S_1 \times 2^1 \times S_0 \times 2^0}_{\text{Integral part}} \ \bullet\ + \underbrace{S_{-1} \times 2^{-1} + \dots + S_{-L} \times 2^{-L}}_{\text{Fractional part}}$$

- What is the corresponding decimal of $(101.11)_2$?

The hexadecimal system (十六進位系統) (base 16)

- Base $b = 16$ and we use sixteen symbols to represent a number

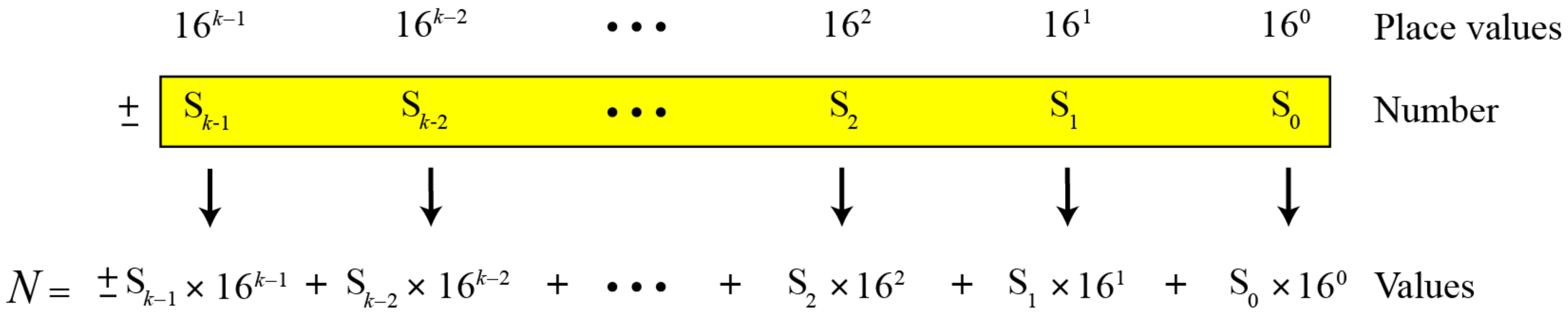
$$S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F\}$$

  - The symbols in this system are often referred to as *hexadecimal digits*
- We can represent an integer as

$$\pm(S_{k-1} \dots S_2 S_1 S_0)_{16}$$

S_i is a hexadecimal digit, $b = 16$ is the base, and K is the number of hexadecimal digits

- What is the corresponding decimal of $(2AE)_{16}$?



Maximum value and reals

- The maximum value of a hexadecimal integer with K digits is

$$N_{max} = 16^K - 1$$

- Although a real number can be also represented in the hexadecimal system, it is not very common

The octal system (八進位系統) (base 8)

- In this system, the base $b = 8$ and we use eight symbols to represent a number. The set of symbols is

$$S = \{0, 1, 2, 3, 4, 5, 6, 7\}$$

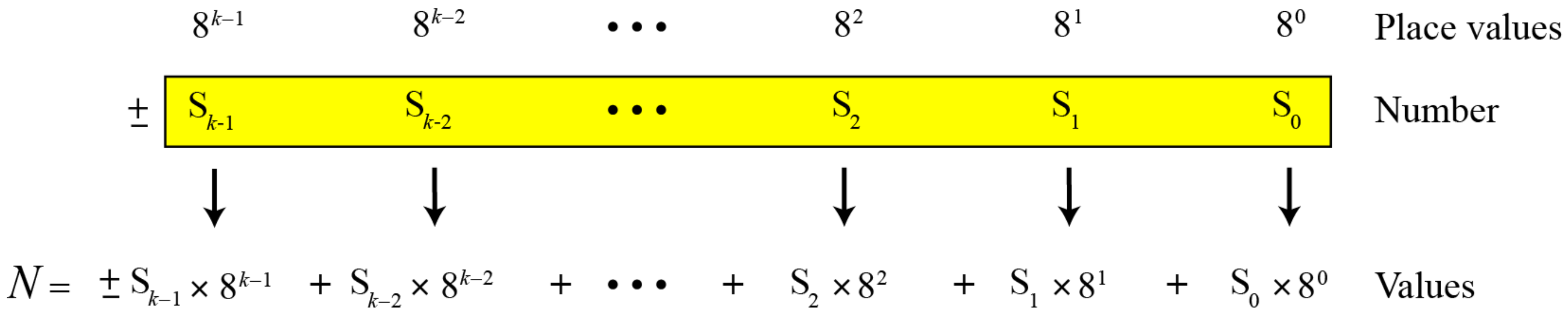
  - The symbols in this system are often referred to as *octal digits*

- We can represent an integer as

$$\pm(S_{k-1} \ldots S_2 S_1 S_0)_8$$

  S_i is a octal digit, $b = 8$ is the base, and K is the number of octal digits

- What is the corresponding decimal of $(1256)_8$?



Maximum value and reals

- The maximum value of an octal integer with K digits is
$$N_{max} = 2^8 - 1$$
- Although a real number can be also represented in the octal system, it is not very common

Summary of the four positional systems

Table 2.1 Summary of the four positional number systems

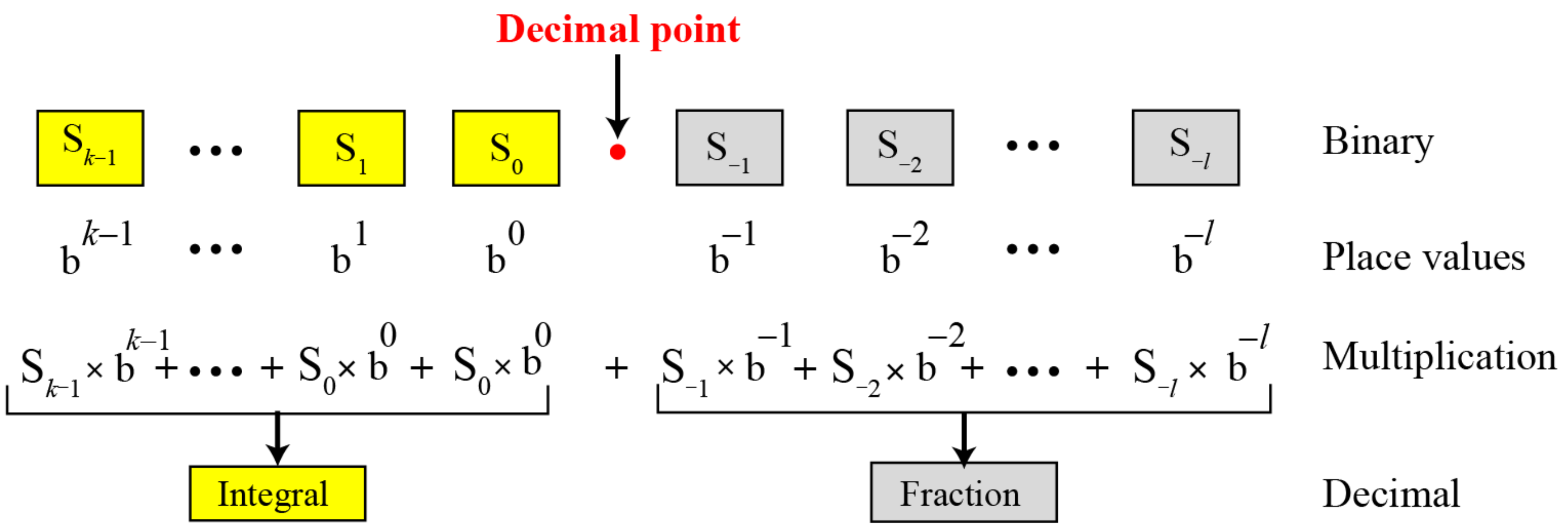
System	*Base*	*Symbols*	*Examples*
Decimal	10	0, 1, 2, 3, 4, 5, 6, 7, 8, 9	2345.56
Binary	2	0, 1	$(1001.11)_2$
Octal	8	0, 1, 2, 3, 4, 5, 6, 7	$(156.23)_8$
Hexadecimal	16	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F	$(A2C.A1)_{16}$

Table 2.2 Comparison of numbers in the four systems

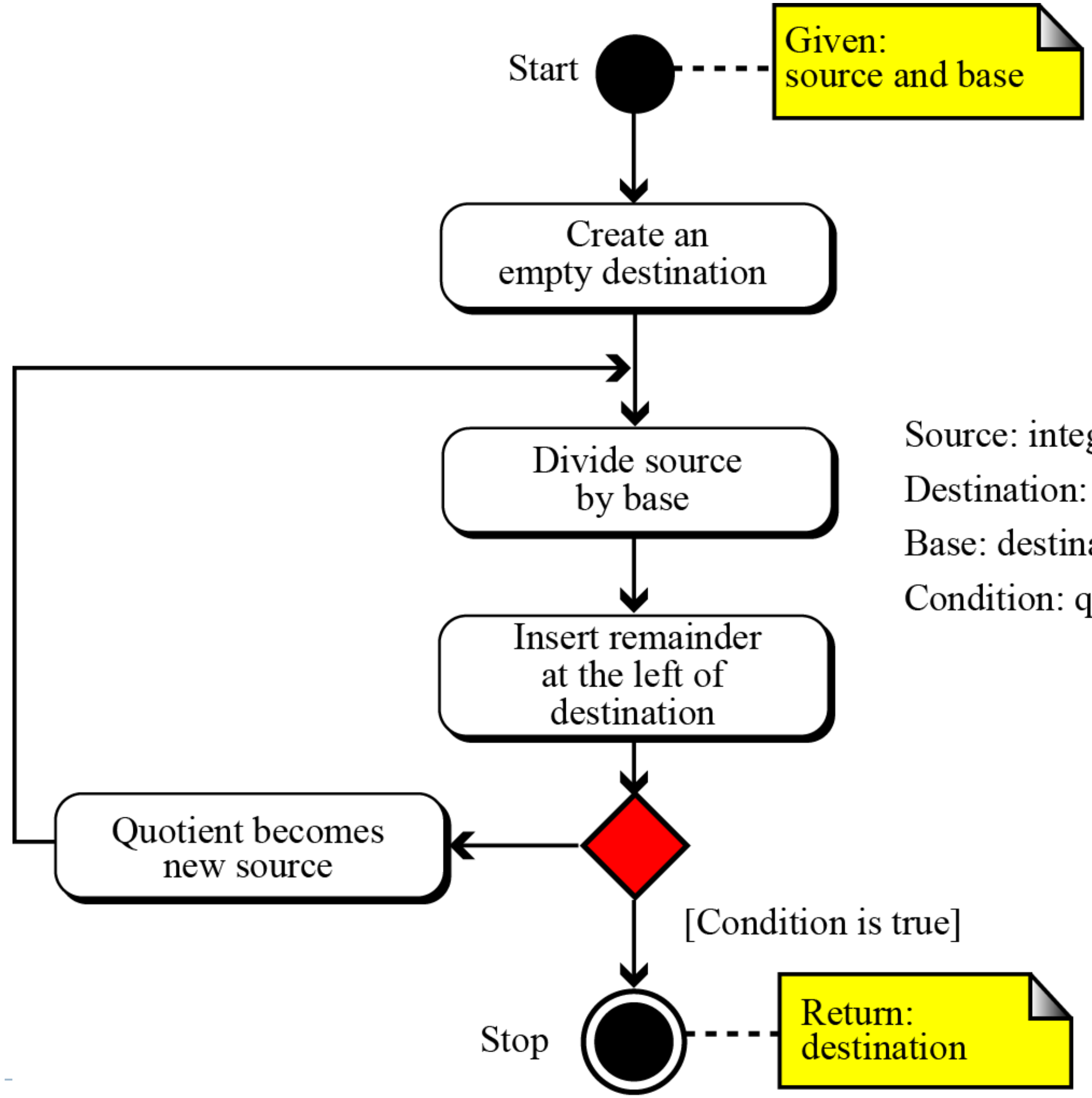
Decimal	*Binary*	*Octal*	*Hexadecimal*
0	0	0	0
1	1	1	1
2	10	2	2
3	11	3	3
4	100	4	4
5	101	5	5
6	110	6	6
7	111	7	7
8	1000	10	8
9	1001	11	9
10	1010	12	A
11	1011	13	B
12	1100	14	C
13	1101	15	D
14	1110	16	E
15	1111	17	F

Conversion - Any base to decimal conversion

- We need to know how to convert a number in one system to the equivalent number in another system
  - What is the corresponding decimal of $(110.11)_2$, $(1A.23)_{16}$, $(23.17)_8$?



Decimal to any base - integral part



Source: integral part of decimal number

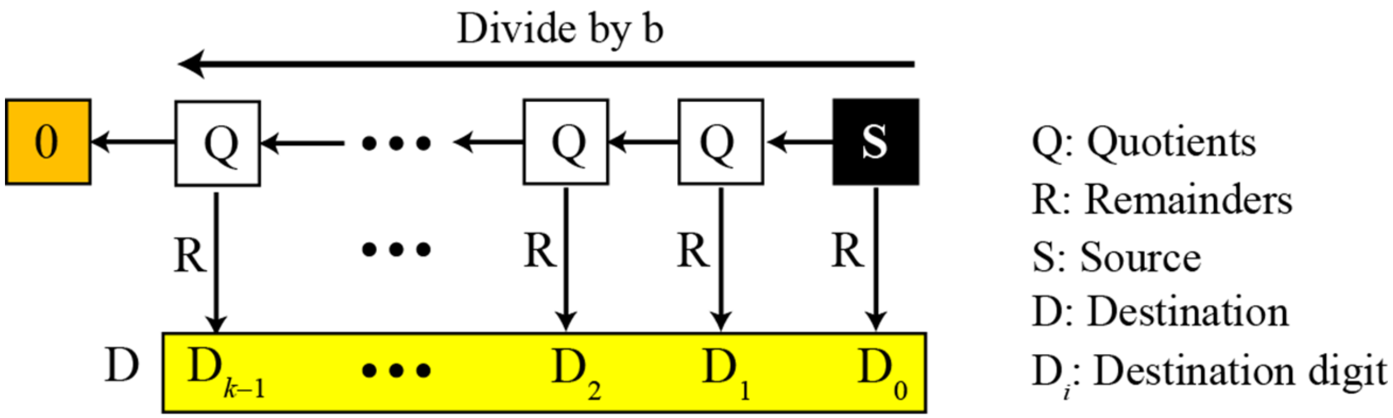
Destination: integral part of converted number

Base: destination base

Condition: quotient is zero

Decimal to any base - integral part

- Try to convert 35 in decimal to binary

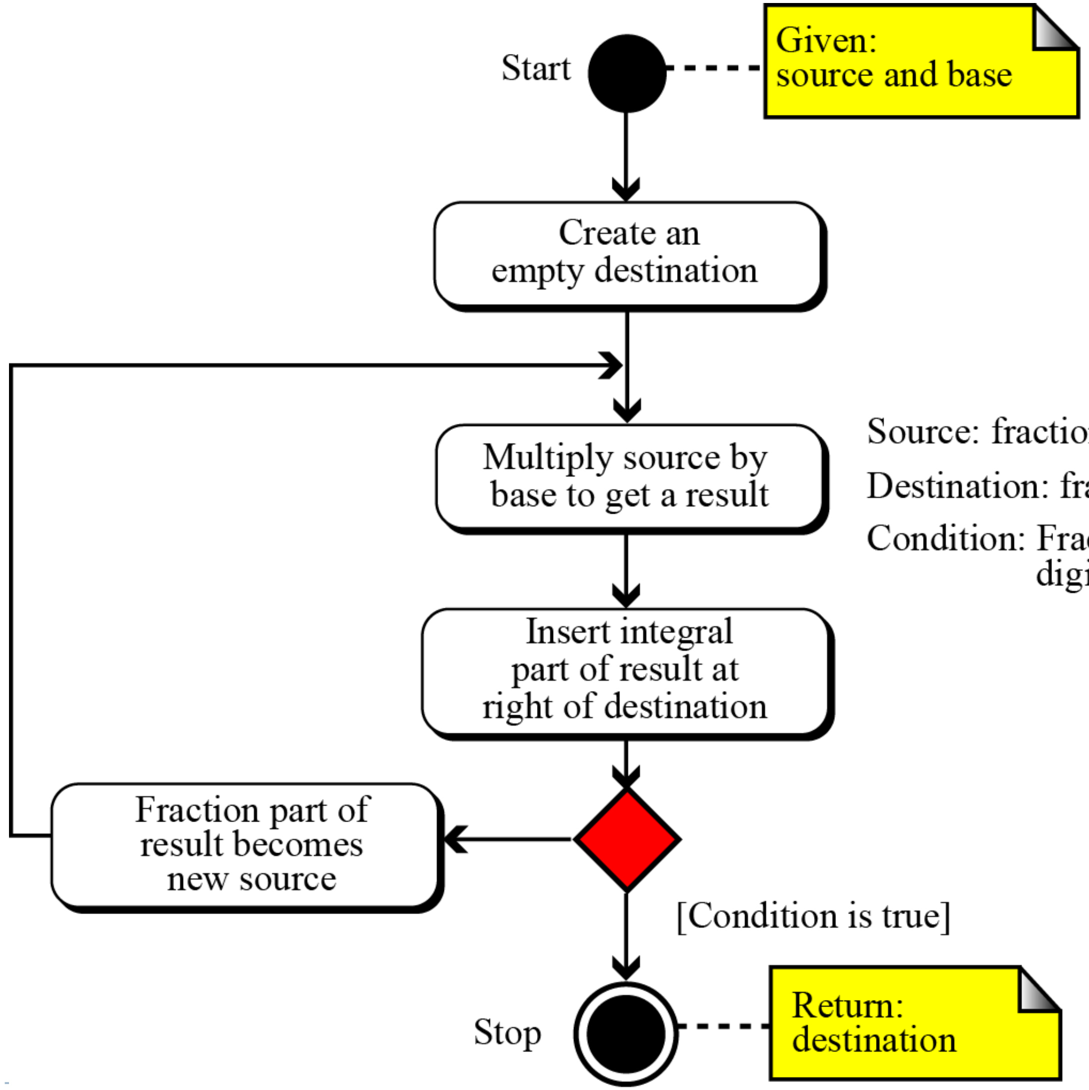


Decimal to any base - integral part

- Try to convert 126 in decimal to octal system

- Try to convert 126 in decimal to hexadecimal system

Decimal to any base - fractional part



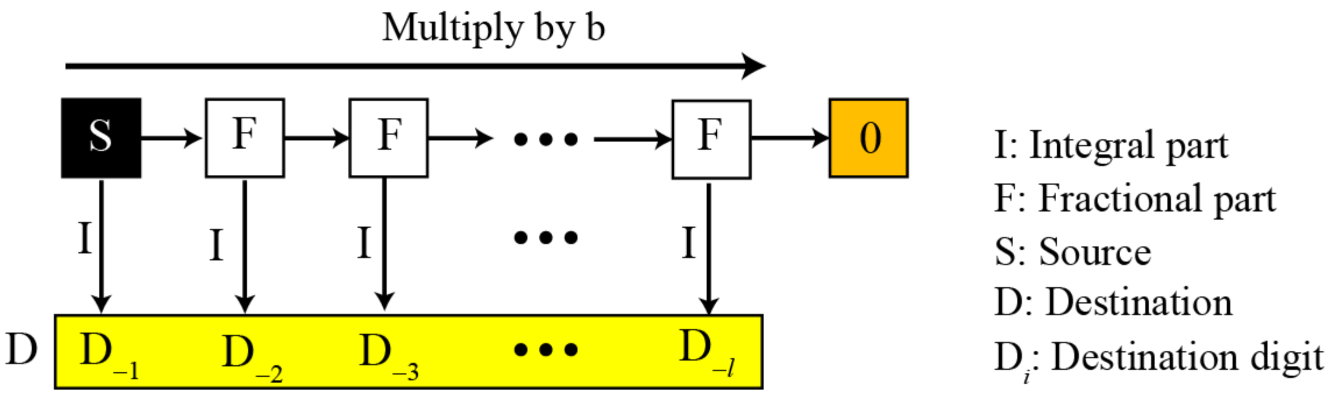
Source: fraction part of decimal number

Destination: fraction part of converted number

Condition: Fraction part is zero or destination digits are enough

Decimal to any base - fractional part

- Try to convert 0.625 in decimal to binary



Note:
The fraction may never become zero.
Stop when enough digits have been created.

Decimal to any base - fractional part

- Try to convert 0.634 in decimal to octal using a maximum of four digits

- Try to convert 178.6 in decimal to hexadecimal using only one digit to the right of the decimal point

Decimal to any base

- An alternative method for converting a small decimal integer (usually less than 256) to binary is to break the number as the sum of numbers that are equivalent to the binary place values shows:

Place values	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Decimal equivalent	128	64	32	16	8	4	2	1

- Using the table, we can convert 165 to $(10100101)_2$

Decimal 165 =	128	+	0	+	32	+	0	+	0	+	4	+	0	+	1
Binary	1		0		1		0		0		1		0		1

Decimal to any base

- A similar method can be used to convert a decimal fraction to binary when the denominator is a power of two:

Place values	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
Decimal equivalent	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$

- Using this table, we convert $\frac{27}{64}$ to $(0.011011)_2$

Decimal $\frac{27}{64}$ =	0	+	$\frac{1}{4}$	+	$\frac{1}{8}$	+	0	+	$\frac{1}{32}$	+	$\frac{1}{64}$
Binary	0		1		1		0		1		1

Number of digits

- In a positional number system with base b, we can always find the number of digits of an integer using the relation

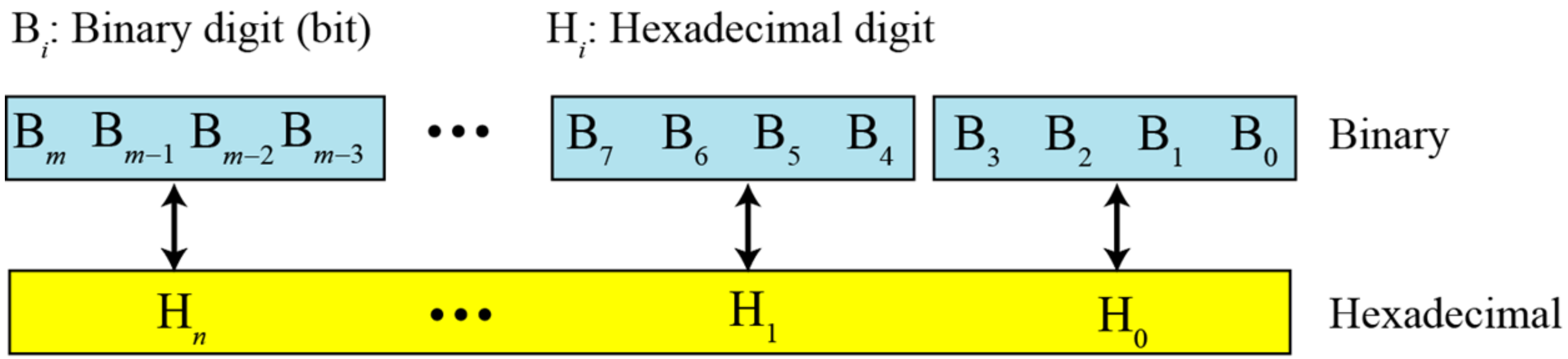
$$K = \lceil \log_b(N + 1) \rceil$$

Where N is the value of the integer in the decimal system

- For example, try to find the required number of digits in the decimal number 234 in all four systems

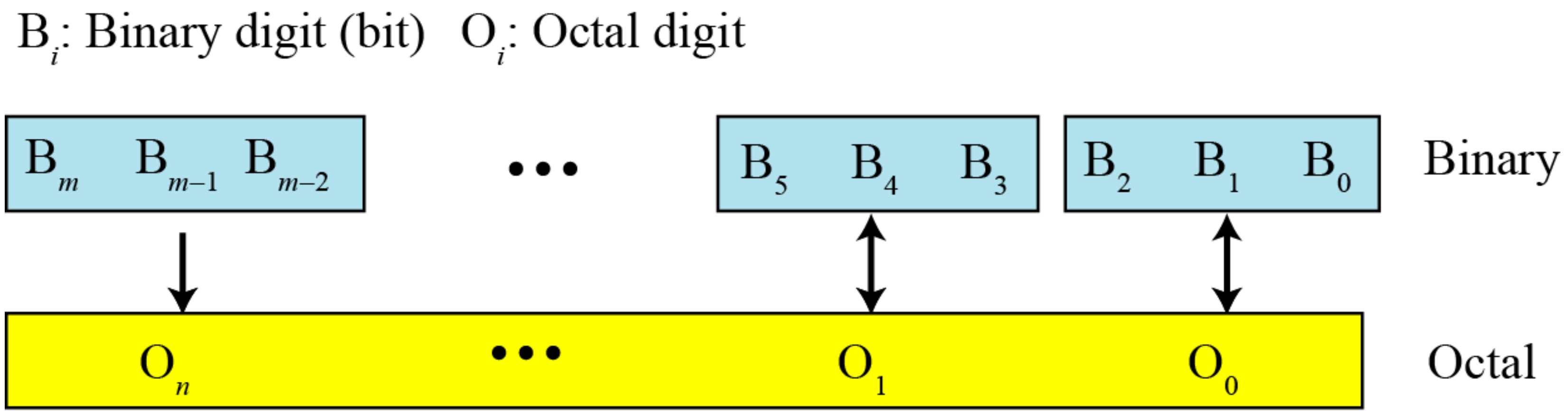
Binary–hexadecimal conversion

- Try to show the hexadecimal equivalent of the binary number $(10011100010)_2$ and the binary equivalent of $(24C)_{16}$



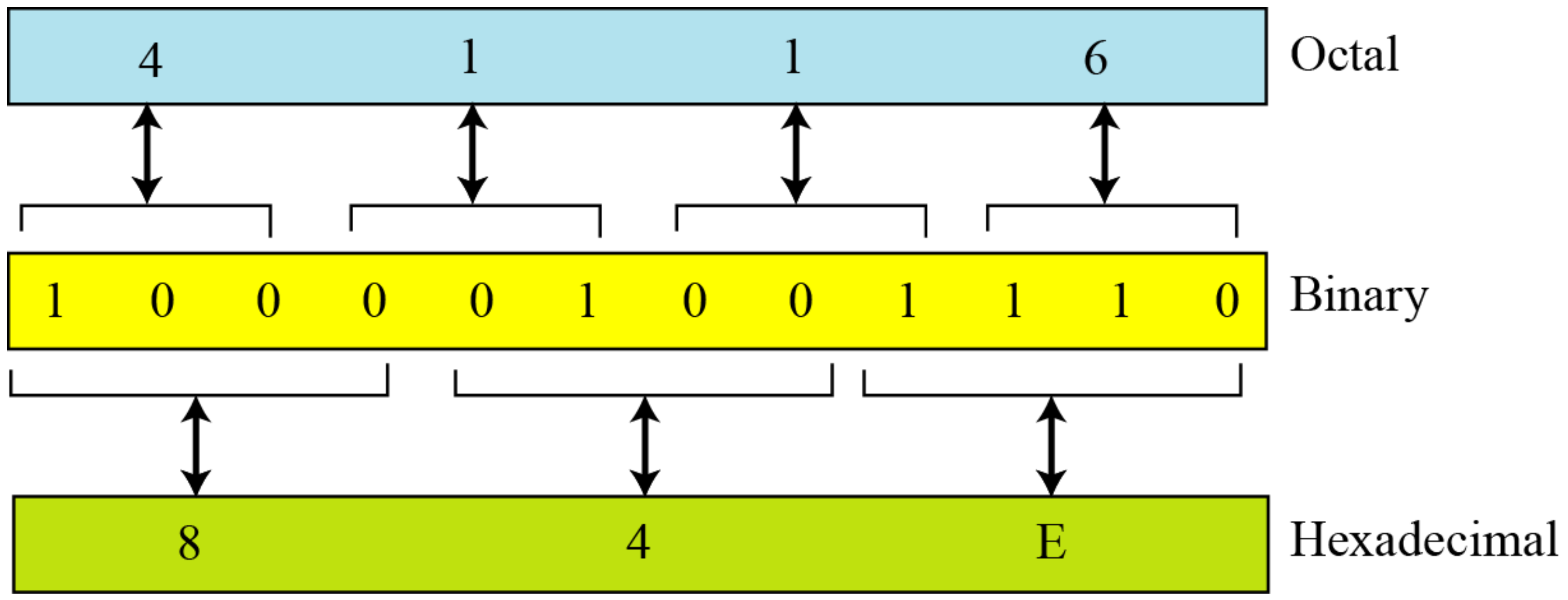
Binary–octal conversion

- Try to show the octal equivalent of the binary number $(101110010)_2$ and the binary equivalent of $(24)_8$



Octal–hexadecimal conversion

- We can use the binary system as the intermediate system



Number of digits from b_1 to b_2 system

- In general, assume that we are using K digits in base b_1 system
  - The maximum number we can represent in this source system is $b_1^K - 1$
  - The maximum number we can represent in the destination system is $b_2^x - 1$
  - Therefore, $b_2^x - 1 \geq b_1^K - 1 \rightarrow x \geq K \times (\frac{\log b_1}{\log b_2})$ or $x = \left\lceil K \times (\frac{\log b_1}{\log b_2}) \right\rceil$
- Try to find the minimum number of binary digits required to store decimal integers with a maximum of six digits

Non-positional Number Systems

- A nonpositional number system still uses a limited number of symbols in which each symbol has a value
  - However, the position a symbol occupies in the number normally bears no relation to its value—*the value of each symbol is fixed*
  - In this system, a number is represented as

$$S_{k-1} \ldots S_2 S_1 S_0 . S_{-1} S_{-2} \ldots S_{-L}$$

and it usually has the value

$$n = \pm \underbrace{S_{K-1} + \ldots + S_1 + S_0}_{\text{Integral part}} + \underbrace{S_{-1} + S_{-2} + \ldots + S_{-L}}_{\text{Fractional part}}$$

Roman number system

- This number system has a set of symbols $S = \{I, V, X, L, C, D, M\}$. The corresponding values are

Table 2.3 Values of symbols in the Roman number system

Symbol	*I*	*V*	*X*	*L*	*C*	*D*	*M*
Value	1	5	10	50	100	500	1000

III	→	1 + 1 + 1	=	3
IV	→	5 – 1	=	4
VIII	→	5 + 1 + 1 + 1	=	8
XVIII	→	10 + 5 + 1 + 1 + 1	=	18
XIX	→	10 + (10 – 1)	=	19
LXXII	→	50 + 10 + 10 + 1 + 1	=	72
CI	→	100 + 1	=	101
MMVII	→	1000 + 1000 + 5 + 1 + 1	=	2007
MDC	→	1000 + 500 + 100	=	1600