1. (24%) Determine whether the series converges absolutely or conditionally, or diverges. In addition, please indicate the test you use.

   (a) $\sum_{n=1}^{\infty}(-1)^n(1-\cos(\frac{1}{n}))$

   (b) $\sum_{n=1}^{\infty}(-1)^n\frac{(n!)^n}{(n^n)^2}$

   (c) $\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n^3+1}+\sqrt{n^3}}$

   (d) $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{n\ln(\ln n)}$

Ans:

(a) $\sum_{n=1}^{\infty}(-1)^n(1-\cos(\frac{1}{n}))=\sum_{n=1}^{\infty}(-1)^n(2sin^2(\frac{1}{2n}))$

Considering $\sum_{n=1}^{\infty}(2sin^2(\frac{1}{2n}))$ since $\lim_{n\to\infty}2\left(\frac{\sin(\frac{1}{2n})}{\frac{1}{2n}}\right)^2=2$ and $\sum_{n=1}^{\infty}\frac{1}{4n^2}$ is converge ($\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a p-series with $p>1$ which is converge) by direct comparison test it is convergent. So the original sequence is absolute convergent.

(b) $\lim_{n\to\infty}\sqrt[n]{|(-1)^n\frac{(n!)^n}{(n^n)^2}|}=\lim_{n\to\infty}\frac{n!}{n^2}=\infty>1$

Therefore, by the root test, the series is diverging.

(c) Since $\lim_{n\to\infty}\frac{\frac{1}{\sqrt{n^3}}}{\frac{1}{\sqrt{n^3+1}+\sqrt{n^3}}}=2$ and $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^3}}$ is a p-series with $p>1$ which is converge. By limit comparison test, $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^3+1}+\sqrt{n^3}}$ is also converge. So $\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n^3+1}+\sqrt{n^3}}$ is absolute converge.

(d) $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{n\ln(\ln n)}$ is converge by the alternating series test, since

$\lim_{n\to\infty}\frac{1}{n\ln(\ln n)}=0$ and $\frac{1}{(n+1)\ln(\ln(n+1))}<\frac{1}{n\ln(\ln n)}$ for $n>2$.

$\sum_{n=2}^{\infty}\frac{1}{n\ln(\ln n)}$ is diverge by direct comparison test since $\frac{1}{n\ln(\ln n)}>\frac{1}{n\ln n}$ and note that $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ is diverge (Let $f(x)=\frac{1}{x\ln(x)}$, $\int_2^{\infty}\frac{1}{x\ln(x)}dx=\ln\ln(x)\,|_2^{\infty}=\infty$, so by integral test the corresponding series is diverge). So the original series is conditionally converging.

2. (16%) Find the interval of convergence of the power series (Be sure to check the for the convergence at the endpoints of the intervals)

(a) $\sum_{n=1}^{\infty} x^n \ln(\frac{n+1}{n})$

(b) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{(-2)^n\sqrt{n}}$

Ans:

(a) $\lim_{n\to\infty} |\frac{u_{n+1}}{u_n}| = \lim_{n\to\infty} |\frac{x^{n+1}\ln(\frac{n+2}{n+1})}{x^n\ln(\frac{n+1}{n})}| = \lim_{n\to\infty} |\frac{x^{n+1}\ln(\frac{1+\frac{2}{n}}{1+\frac{1}{n}})}{x^n\ln(\frac{1+\frac{1}{n}}{1})}| = |x|$ By the ratio test, the series converges for $|x| < 1$

When $x = 1$: $\sum_{n=1}^{\infty} \ln(\frac{n+1}{n})$ is diverge by the n-th term test for divergence since

$\sum_{n=1}^{\infty} \ln(\frac{n+1}{n}) = \sum_{n=1}^{\infty} \ln(n+1) - \ln(n) = \lim_{n\to\infty} \ln(n+1) = \infty$

When $x = -1$: $\sum_{n=1}^{\infty} (-1)^n \ln(\frac{n+1}{n})$ is converge by the alternating series test, since

$\lim_{n\to\infty} \ln(\frac{n+1}{n}) = 0$ and $\ln(\frac{n+2}{n+1}) < \ln(\frac{n+1}{n})$ for $n > 1$.

So the interval of convergence is $[-1,1)$

(b) $\lim_{n\to\infty} |\frac{u_{n+1}}{u_n}| = \lim_{n\to\infty} |\frac{\frac{(x-1)^{n+1}}{(-2)^{n+1}\sqrt{n+1}}}{\frac{(x-1)^n}{(-2)^n\sqrt{n}}}| = \left|\frac{x-1}{-2}\right| \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = \left|\frac{x-1}{-2}\right| < 1 \to |x-1| < 2$

When $x = 3$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is by converge by the alternating series test, since

$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ and $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$ for $n > 1$.

When $x = -1$: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series with $p \leq 1$ which is diverge.

So the interval of convergence is $(-1,3]$

3. (12%) Use a power series to approximate $\int_0^1 cos(x^2)\, dx$ with an error of less than 0.001

Ans: $\int_0^1 \cos(x^2) dx = \int_0^1 \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}\right] dx = \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(4n+1)(2n)!}\right] \begin{matrix} 1 \\ 0 \end{matrix} =$

$\sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+1)(2n)!}$

It is an alternating series, since $\frac{1}{(4*3+1)(2*3)!} < 0.001$ therefore we know that

$\int_0^1 \cos(x^2)dx \approx \sum_{n=0}^{2} \frac{(-1)^n}{(4n+1)(2n)!} = 1 - \frac{1}{10} + \frac{1}{216} \approx \frac{977}{1080}$

4. (18%) Evaluate the following expression (Try to use the Basic series of Taylor series and notice that the power series is a continuous function)

(a) $1 + \frac{3}{1!} + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} + \cdots$

(b) $\sum_{n=1}^{\infty} \frac{n}{(n-1)!}$

(c) $\lim_{x\to 0} \frac{\sin(x)\arctan(x) - x^2 + \frac{x^4}{2}}{x^6}$

Ans:

(a) $1 + \frac{3}{1!} + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{3^n}{n!} = e^3$

(b) Since $e^x = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$

$xe^x = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^n \to xe^x + e^x = \sum_{n=1}^{\infty} \frac{n}{(n-1)!} x^{n-1} \to \sum_{n=1}^{\infty} \frac{n}{(n-1)!} =$

$2e$ (Substitute $x = 1$)

(c) $\lim_{x\to 0} \frac{\sin(x)\arctan(x) - x^2 + \frac{x^4}{2}}{x^6} = \lim_{x\to 0^+} \frac{(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots)(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots) - x^2 + \frac{x^4}{2}}{x^6} =$

$\lim_{x\to 0} \frac{(x^2 - \frac{x^4}{2} + \frac{19x^6}{72} - \cdots) - x^2 + \frac{x^4}{2}}{x^6} = \frac{19}{72}$

5. (12%) Derive the Maclaurin series of $f(x) = arccos(x)$ and $g(x) = arccos(2x^3)$. In addition, calculate $g^{(93)}(0)$

Ans:

$$\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}} = -\sum_{n=0}^{\infty} \binom{\frac{-1}{2}}{n} (-x^2)^n$$

$$\arccos(x) = \sum_{n=0}^{\infty} \binom{\frac{-1}{2}}{n} (-1)^{n+1} \frac{x^{2n+1}}{2n+1} + C$$

Substitute 0 into the equation we have $C = \frac{\pi}{2}$. Therefore,

$$\arccos(\mathrm{x}) = \frac{\pi}{2} + \sum_{n=0}^{\infty} \binom{\frac{-1}{2}}{n} (-1)^{n+1} \frac{x^{2n+1}}{2n+1}$$

$$\arccos(2x^3) = \frac{\pi}{2} + \sum_{n=0}^{\infty} \binom{\frac{-1}{2}}{n} (-1)^{n+1} \frac{2^{2n+1} x^{6n+3}}{2n+1}$$

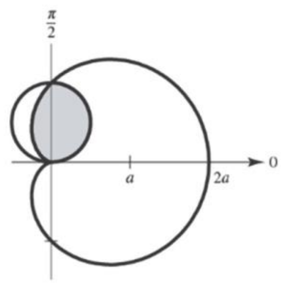
The definition of Maclaurin series of $g(x)$ is $\sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} x^k$

Comparing the coefficient of x^{93} $(n = 15$ since $15 * 6 + 3 = 93)$

We have $\frac{g^{(93)}(0)}{93!} = \binom{\frac{-1}{2}}{15} (-1)^{15+1} \frac{2^{2*15+1}}{2*15+1} = \binom{\frac{-1}{2}}{15} \frac{2^{31}}{31}$

$$g^{(93)}(0) = \binom{\frac{-1}{2}}{15} \frac{2^{31}}{31} 93!$$

6. (10%) Find the area of the shaded region bounded by the curves $r = a(1 + co\,s(\theta))$ and $r = asin(\theta)$



Ans:

$$\mathrm{A} = \frac{\pi a^2}{8} + \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} (a(1 + cos\theta))^2\, d\theta = \frac{\pi a^2}{8} + \frac{a^2}{2} \int_{\frac{\pi}{2}}^{\pi} \frac{3}{2} + 2cos\theta + \frac{\cos(2\theta)}{2} d\theta$$

$$= \frac{\pi a^2}{8} + \frac{a^2}{2} \left[\frac{3}{2}\theta + 2sin\theta + \frac{\sin(2\theta)}{4}\right]_{\frac{\pi}{2}}^{\pi} = \frac{a^2}{2} [\pi - 2]$$

7. (10%) Find the area of the surface formed by revolving the polar graph $r = e^{a\theta}$ about the $\theta = \frac{\pi}{2}$ over the interval $0 \leq \theta \leq \frac{\pi}{2}$

Ans:

$$\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}=\sqrt{(e^{a\theta})^2+(ae^{a\theta})^2}$$

$$\mathrm{S}=2\pi\int_0^{2\pi} e^{a\theta}\cos(\theta)\sqrt{(e^{a\theta})^2+(ae^{a\theta})^2}d\theta$$

$$=\ 2\pi\sqrt{1+a^2}\int_0^{\frac{\pi}{2}} e^{2a\theta}\cos(\theta)\,d\theta\ \ (Integration\ \ by\ \ parts)$$

$$=2\pi\sqrt{1+a^2}\left[\frac{e^{2a\theta}}{4a^2+1}(2acos\theta+sin\theta)\right]_0^{\frac{\pi}{2}}$$

$$=\frac{2\pi\sqrt{1+a^2}}{4a^2+1}(e^{\pi a}-2\mathrm{a})$$

Note that

$$\int e^{2a\theta}\cos(\theta)\,d\theta=\frac{\cos(\theta)}{2a}e^{2a\theta}+\frac{1}{2a}\int e^{2a\theta}\sin(\theta)\,d\theta$$

(Let $\mathrm{u}=\cos(\theta)\,,\mathrm{dv}=e^{2a\theta}d\theta\rightarrow du=-\sin(\theta)\,,v=\frac{1}{2a}e^{2a\theta}$)

$$\int e^{2a\theta}\,sin(\theta)\,d\theta=\frac{sin(\theta)}{2a}e^{2a\theta}-\frac{1}{2a}\int e^{2a\theta}\,cos(\theta)\,d\theta$$

(Let $\mathrm{u}=\sin(\theta)\,,\mathrm{dv}=e^{2a\theta}d\theta\rightarrow du=\cos(\theta)\,,v=\frac{1}{2a}e^{2a\theta}$)

$$\int e^{2a\theta}\cos(\theta)\,d\theta=\frac{e^{2a\theta}}{4a^2+1}(2acos\theta+sin\theta)$$

8. (8%) Determine whether $\int_0^1\frac{\sin(x)}{x}dx$ is converge or diverge.

Ans:

$$\int_0^1\frac{\sin(x)}{x}dx=\int_0^1\frac{x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\cdots}{x}dx=\int_0^1 1-\frac{1}{3!}x^2+\frac{1}{5!}x^4-\cdots dx$$

$$=\left[x-\frac{x^3}{3\times 3!}+\frac{x^5}{5\times 5!}-\cdots\right]_0^1=1-\frac{1}{3\times 3!}+\frac{1}{5\times 5!}\cdot\cdot$$

$=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)\times(2n-1)!}$ is converge by the alternating series test, since

$\lim\limits_{n\to\infty}\frac{1}{(2n-1)\times(2n-1)!}=0$ and $\frac{1}{(2n+1)(2n+1)!}<\frac{1}{(2n-1)(2n-1)!}$ for $\mathrm{n}>1$.