Calculus — Midterm (417) Solutions

1. (30%) Integration Techniques

(a) (5%) $\displaystyle\int \sin^3 x \cos^2 x \, \mathrm{d}x$

Solution. Write $\sin^3 x = (1-\cos^2 x)\sin x$. Let $u = \cos x$, $\mathrm{d}u = -\sin x \, \mathrm{d}x$:

$$\int (1-\cos^2 x)\sin x \cos^2 x \, \mathrm{d}x = -\int (1-u^2)u^2 \, \mathrm{d}u = -\int (u^2 - u^4)\, \mathrm{d}u.$$

$$= -\frac{u^3}{3} + \frac{u^5}{5} + C.$$

Back-substitute $u = \cos x$:

$$\boxed{\int \sin^3 x \cos^2 x \, \mathrm{d}x = -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C.}$$

(b) (5%) $\displaystyle\int e^{2x} \sin x \, \mathrm{d}x$

Solution. Let $I = \int e^{2x}\sin x \, \mathrm{d}x$.

First application of Integration by Parts: $u = \sin x$, $\mathrm{d}v = e^{2x}\,\mathrm{d}x$; $\mathrm{d}u = \cos x \, \mathrm{d}x$, $v = \frac{1}{2}e^{2x}$:

$$I = \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\int e^{2x}\cos x \, \mathrm{d}x.$$

Second application of Integration by Parts: $u = \cos x$, $\mathrm{d}v = e^{2x}\,\mathrm{d}x$; $\mathrm{d}u = -\sin x \, \mathrm{d}x$, $v = \frac{1}{2}e^{2x}$:

$$\int e^{2x}\cos x \, \mathrm{d}x = \frac{1}{2}e^{2x}\cos x + \frac{1}{2}I.$$

Substitute back:

$$I = \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\left(\frac{1}{2}e^{2x}\cos x + \frac{1}{2}I\right) = \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x - \frac{1}{4}I.$$

Collect I: $\frac{5}{4}I = \dfrac{e^{2x}(2\sin x - \cos x)}{4}$.

$$\boxed{\int e^{2x}\sin x \, \mathrm{d}x = \frac{e^{2x}(2\sin x - \cos x)}{5} + C.}$$

(c) (5%) $\displaystyle\int \frac{\sec^2 x}{\tan^2 x + 3\tan x + 2}\, \mathrm{d}x$

Solution. Let $u = \tan x$, $\mathrm{d}u = \sec^2 x \, \mathrm{d}x$:

$$\int \frac{\mathrm{d}u}{u^2 + 3u + 2} = \int \frac{\mathrm{d}u}{(u+1)(u+2)}.$$

Partial fractions: write $\dfrac{1}{(u+1)(u+2)} = \dfrac{A}{u+1} + \dfrac{B}{u+2}$. Multiply through: $1 = A(u+2) + B(u+1)$. Set $u = -1$: $1 = A(1) \Rightarrow A = 1$. Set $u = -2$: $1 = B(-1) \Rightarrow B = -1$.

$$\int \left(\frac{1}{u+1} - \frac{1}{u+2} \right) \mathrm{d}u = \ln|u+1| - \ln|u+2| + C.$$

Back-substitute $u = \tan x$:

$$\boxed{= \ln|\tan x + 1| - \ln|\tan x + 2| + C = \ln\left|\frac{\tan x + 1}{\tan x + 2}\right| + C.}$$

(d) (5%) $\displaystyle\int x^2 e^{-x}\,\mathrm{d}x$

Solution. Apply Integration by Parts with $u = x^2,\ \mathrm{d}v = e^{-x}\,\mathrm{d}x;\ \mathrm{d}u = 2x\,\mathrm{d}x,\ v = -e^{-x}$:

$$\int x^2 e^{-x}\,\mathrm{d}x = -x^2 e^{-x} + 2\int x e^{-x}\,\mathrm{d}x.$$

Apply Integration by Parts again with $u = x,\ \mathrm{d}v = e^{-x}\,\mathrm{d}x;\ \mathrm{d}u = \mathrm{d}x,\ v = -e^{-x}$:

$$\int x e^{-x}\,\mathrm{d}x = -x e^{-x} + \int e^{-x}\,\mathrm{d}x = -x e^{-x} - e^{-x} + C.$$

Substitute back:

$$\int x^2 e^{-x}\,\mathrm{d}x = -x^2 e^{-x} + 2(-x e^{-x} - e^{-x}) + C.$$

$$\boxed{\int x^2 e^{-x}\,\mathrm{d}x = -e^{-x}(x^2 + 2x + 2) + C.}$$

(e) (5%) $\displaystyle\int \sin^4 x\,\mathrm{d}x$

Solution. Use the power-reduction identity $\sin^2 x = \dfrac{1 - \cos 2x}{2}$:

$$\sin^4 x = \left(\frac{1 - \cos 2x}{2}\right)^2 = \frac{1 - 2\cos 2x + \cos^2 2x}{4}.$$

Apply $\cos^2 2x = \dfrac{1 + \cos 4x}{2}$:

$$\sin^4 x = \frac{1 - 2\cos 2x + \frac{1+\cos 4x}{2}}{4} = \frac{3 - 4\cos 2x + \cos 4x}{8}.$$

Integrate term by term:

$$\boxed{\int \sin^4 x\,\mathrm{d}x = \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C.}$$

(f) (5%) $\displaystyle\int_0^2 \frac{\mathrm{d}x}{(4+x^2)^{3/2}}$

Solution. Let $x = 2\tan\theta$, $\mathrm{d}x = 2\sec^2\theta\,\mathrm{d}\theta$, $(4+x^2)^{3/2} = (4\sec^2\theta)^{3/2} = 8\sec^3\theta$.

$$\int \frac{2\sec^2\theta\,\mathrm{d}\theta}{8\sec^3\theta} = \frac{1}{4}\int \cos\theta\,\mathrm{d}\theta = \frac{\sin\theta}{4} + C.$$

Limits: $x = 0 \Rightarrow \theta = 0$; $x = 2 \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$, $\sin\theta = \frac{\sqrt{2}}{2}$.

$$\boxed{= \frac{1}{4}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{8}.}$$

2. (10%) Improper Integrals

(a) (5%) $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,\mathrm{d}x$

Solution. Write the integral as a limit:

$$\int_1^\infty \frac{\ln x}{x^2}\,\mathrm{d}x = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,\mathrm{d}x.$$

Evaluate the definite integral using Integration by Parts with $u = \ln x$, $\mathrm{d}v = x^{-2}\,\mathrm{d}x$; $\mathrm{d}u = \dfrac{1}{x}\,\mathrm{d}x$, $v = -x^{-1}$:

$$\int_1^b \frac{\ln x}{x^2}\,\mathrm{d}x = \left[-\frac{\ln x}{x}\right]_1^b + \int_1^b \frac{\mathrm{d}x}{x^2} = -\frac{\ln b}{b} + \left[-\frac{1}{x}\right]_1^b = -\frac{\ln b}{b} - \frac{1}{b} + 1.$$

Now take the limit as $b\to\infty$: by L'Hôpital's Rule, $\displaystyle\lim_{b\to\infty}\frac{\ln b}{b} = \lim_{b\to\infty}\frac{1/b}{1} = 0$, and $\dfrac{1}{b}\to 0$.

$$\lim_{b\to\infty}\left(-\frac{\ln b}{b} - \frac{1}{b} + 1\right) = 0 - 0 + 1.$$

$$\boxed{\int_1^\infty \frac{\ln x}{x^2}\,\mathrm{d}x = 1.}$$

(b) (5%) $\displaystyle\int_{-1}^1 \frac{\mathrm{d}x}{x^{2/3}}$

Solution. The integrand $x^{-2/3}$ has an interior discontinuity at $x = 0$; split as two improper integrals:

$$\int_{-1}^1 x^{-2/3}\,\mathrm{d}x = \lim_{t\to 0^-}\int_{-1}^t x^{-2/3}\,\mathrm{d}x + \lim_{t\to 0^+}\int_t^1 x^{-2/3}\,\mathrm{d}x.$$

The antiderivative of $x^{-2/3}$ is $3x^{1/3}$. Evaluate each limit separately:

$$\lim_{t\to 0^-}\left[3x^{1/3}\right]_{-1}^t = \lim_{t\to 0^-}\left(3t^{1/3} - 3(-1)^{1/3}\right) = 0 - (-3) = 3.$$

$$\lim_{t\to 0^+}\left[3x^{1/3}\right]_t^1 = \lim_{t\to 0^+}\left(3\cdot 1 - 3t^{1/3}\right) = 3 - 0 = 3.$$

Both limits exist, so the integral converges:

$$\boxed{\int_{-1}^{1} x^{-2/3}\,\mathrm{d}x = 3 + 3 = 6.}$$

3. (21%) Power Series

(a) (4%) First three nonzero terms of the Maclaurin series for $f(x) = e^{-x/2}\ln(1-2x)$.

Solution. Expand each factor separately. For $e^{-x/2}$, substitute $u = -x/2$ into $e^u = 1 + u + u^2/2! + u^3/3! + \cdots$:

$$e^{-x/2} = 1 - \frac{x}{2} + \frac{x^2}{8} - \frac{x^3}{48} + \cdots$$

For $\ln(1-2x)$, substitute $u = -2x$ into $\ln(1+u) = u - u^2/2 + u^3/3 - \cdots$:

$$\ln(1-2x) = -2x - 2x^2 - \frac{8}{3}x^3 - \cdots$$

Multiply, retaining terms up to x^3:

- $1 \cdot \left(-2x - 2x^2 - \frac{8}{3}x^3\right)$
- $\left(-\frac{x}{2}\right)\cdot(-2x) = x^2$
- $\left(-\frac{x}{2}\right)\cdot(-2x^2) = x^3$
- $\frac{x^2}{8}\cdot(-2x) = -\frac{x^3}{4}$

Adding:

$$[x]\text{-terms} : -2x.$$

$$[x^2]\text{-terms} : -2x^2 + x^2 = -x^2.$$

$$[x^3]\text{-terms} : -\frac{8}{3}x^3 + x^3 - \frac{x^3}{4} = \left(-\frac{32}{12} + \frac{12}{12} - \frac{3}{12}\right)x^3 = -\frac{23}{12}x^3.$$

$$\boxed{f(x) = e^{-x/2}\ln(1-2x) = -2x - x^2 - \frac{23}{12}x^3 + \cdots}$$

(b) (4%) Coefficient of x^n in $f(x) = \dfrac{1}{1 - x - 6x^2}$.

Solution. Factor the denominator: $1 - x - 6x^2 = (1-3x)(1+2x)$. Partial fractions: $\dfrac{1}{(1-3x)(1+2x)} = \dfrac{A}{1-3x} + \dfrac{B}{1+2x}$. Multiply through: $1 = A(1+2x) + B(1-3x)$.

Set $x = \frac{1}{3}$: $1 = A\cdot\frac{5}{3} \Rightarrow A = \frac{3}{5}$. Set $x = -\frac{1}{2}$: $1 = B\cdot\frac{5}{2} \Rightarrow B = \frac{2}{5}$.

Expand each term using the geometric series $\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$:

$$\frac{1}{1-3x} = \sum_{n=0}^{\infty} 3^n x^n, \qquad \frac{1}{1+2x} = \sum_{n=0}^{\infty} (-2)^n x^n.$$

Therefore $f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{5} \cdot 3^n + \frac{2}{5} \cdot (-2)^n \right) x^n$.

$$\boxed{[x^n] f(x) = \frac{3^{n+1}}{5} + \frac{2(-2)^n}{5}.}$$

Alternative 1 (Factoring + Direct Multiplication of Two Geometric Series).
Factor and expand each factor as a geometric series without partial fractions:

$$\frac{1}{(1-3x)(1+2x)} = \underbrace{\left(\sum_{j=0}^{\infty} 3^j x^j \right)}_{1/(1-3x)} \cdot \underbrace{\left(\sum_{k=0}^{\infty} (-2)^k x^k \right)}_{1/(1+2x)}.$$

By the Cauchy product formula, the coefficient of x^n in the product is

$$[x^n] f(x) = \sum_{j=0}^{n} 3^j (-2)^{n-j}.$$

This is a finite geometric series with first term $(-2)^n$ and ratio $-3/2$:

$$\sum_{j=0}^{n} 3^j (-2)^{n-j} = (-2)^n \sum_{j=0}^{n} \left(-\frac{3}{2} \right)^j = (-2)^n \cdot \frac{1 - (-3/2)^{n+1}}{1 - (-3/2)} = \frac{2(-2)^n}{5} \left[1 - \left(-\frac{3}{2} \right)^{n+1} \right].$$

Simplify $(-2)^n \cdot (-3/2)^{n+1} = (-1)^{2n+1} \cdot 2^n \cdot 3^{n+1} / 2^{n+1} = -3^{n+1}/2$:

$$= \frac{2(-2)^n}{5} + \frac{3^{n+1}}{5}.$$

$$\boxed{[x^n] f(x) = \sum_{j=0}^{n} 3^j (-2)^{n-j} = \frac{3^{n+1} + 2(-2)^n}{5}.}$$

Alternative 2 (Direct Geometric Series Expansion).
Write $u = x + 6x^2$ so that $1 - x - 6x^2 = 1 - u$. Since $|u| < 1$ near $x = 0$:

$$f(x) = \frac{1}{1-u} = \sum_{k=0}^{\infty} u^k = \sum_{k=0}^{\infty} (x + 6x^2)^k.$$

Factor out x^k:

$$(x + 6x^2)^k = x^k (1 + 6x)^k = x^k \sum_{m=0}^{k} \binom{k}{m} (6x)^m = \sum_{m=0}^{k} \binom{k}{m} 6^m \, x^{k+m}.$$

The power x^n arises when $k+m=n$, i.e. $m=n-k$, requiring $0 \le n-k \le k$, i.e. $\lceil n/2 \rceil \le k \le n$. Therefore:

$$\boxed{[x^n]f(x) = \sum_{k=\lceil n/2 \rceil}^{n} \binom{k}{n-k} 6^{n-k}.}$$

Verification for $n=2$: $k=1$: $\binom{1}{1}6^1 = 6$; $k=2$: $\binom{2}{0}6^0 = 1$; total $= 7 = \dfrac{3^3+2\cdot 4}{5}$. ✓

(c) (9%) Maclaurin series for $f(x) = e^{-x^2}$; then find $f^{(8)}(0)$.

Solution. Use the known series $e^u = \sum_{n=0}^{\infty} \dfrac{u^n}{n!}$ and substitute $u = -x^2$:

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}.$$

$$\boxed{e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots, \quad x \in (-\infty, \infty).}$$

Finding $f^{(8)}(0)$. From the series above, the coefficient of x^8 corresponds to $n=4$: $\dfrac{(-1)^4}{4!} = \dfrac{1}{24}$. Since the Maclaurin coefficient equals $\dfrac{f^{(8)}(0)}{8!}$:

$$\frac{f^{(8)}(0)}{8!} = \frac{1}{24} \;\Rightarrow\; f^{(8)}(0) = \frac{8!}{24} = \frac{40320}{24} = \boxed{1680.}$$

(d) (4%) $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$.

Solution. From the Maclaurin series: $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$. Subtract $1+x$:

$$e^x - 1 - x = \frac{x^2}{2} + \frac{x^3}{6} + \cdots = \frac{x^2}{2}\left(1 + \frac{x}{3} + \cdots\right).$$

Divide by x^2:

$$\frac{e^x - 1 - x}{x^2} = \frac{1}{2} + \frac{x}{6} + \cdots \longrightarrow \boxed{\frac{1}{2}.}$$

4. (23%) Parametric Equations & Polar Curve

(a) (6%) Eliminate the parameter: $x = t+2$, $y = 2\sqrt{t} - 1$.

Solution. From $x = t+2$: $t = x-2$. Substitute into $y = 2\sqrt{t} - 1$:

$$y = 2\sqrt{x-2} - 1.$$

Isolate the square root and square both sides:

$$y + 1 = 2\sqrt{x-2} \;\Rightarrow\; (y+1)^2 = 4(x-2), \quad x \ge 2.$$

$$\boxed{(y+1)^2 = 4(x-2), \quad x \ge 2.}$$

This is a parabola opening to the right with vertex $(2, -1)$.

(b) (7%) Surface area: revolve $x = \dfrac{t^3}{3}$, $y = 2t + 1$ $(0 \le t \le 1)$ about the y-axis.

Solution. The surface area formula for revolution about the y-axis is $S = 2\pi \displaystyle\int_a^b x \, \mathrm{d}s$, where $\mathrm{d}s = \sqrt{(dx/dt)^2 + (dy/dt)^2}\, \mathrm{d}t$.

Compute derivatives: $\dfrac{dx}{dt} = t^2$, $\dfrac{dy}{dt} = 2$.

$$\mathrm{d}s = \sqrt{t^4 + 4}\, \mathrm{d}t.$$

$$S = 2\pi \int_0^1 \frac{t^3}{3} \cdot \sqrt{t^4 + 4}\, \mathrm{d}t = \frac{2\pi}{3} \int_0^1 t^3 \sqrt{t^4 + 4}\, \mathrm{d}t.$$

Let $u = t^4 + 4$, $\mathrm{d}u = 4t^3\, \mathrm{d}t$; limits: $t = 0 \Rightarrow u = 4$, $t = 1 \Rightarrow u = 5$:

$$S = \frac{2\pi}{3} \cdot \frac{1}{4} \int_4^5 \sqrt{u}\, \mathrm{d}u = \frac{\pi}{6} \cdot \frac{2}{3} \big[u^{3/2}\big]_4^5 = \frac{\pi}{9}\big(5^{3/2} - 4^{3/2}\big).$$

Since $5^{3/2} = 5\sqrt{5}$ and $4^{3/2} = 8$:

$$\boxed{S = \frac{\pi(5\sqrt{5} - 8)}{9}.}$$

(c) (10%) $r = 2 - 2\cos\theta$: area and arc length.

Solution. The curve is a cardioid with cusp at the origin.

Area.

$$A = \frac{1}{2} \int_0^{2\pi} r^2\, \mathrm{d}\theta = \frac{1}{2} \int_0^{2\pi} (2 - 2\cos\theta)^2\, \mathrm{d}\theta = \frac{1}{2} \int_0^{2\pi} 4(1 - \cos\theta)^2\, \mathrm{d}\theta$$
$$= 2 \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta)\, \mathrm{d}\theta.$$

Apply $\cos^2\theta = \dfrac{1 + \cos 2\theta}{2}$:

$$A = 2 \int_0^{2\pi} \left(\frac{3}{2} - 2\cos\theta + \frac{\cos 2\theta}{2}\right) \mathrm{d}\theta = 2\left[\frac{3\theta}{2} - 2\sin\theta + \frac{\sin 2\theta}{4}\right]_0^{2\pi} = 2 \cdot 3\pi.$$

$$\boxed{A = 6\pi.}$$

Arc length. $f(\theta) = 2 - 2\cos\theta$, $f'(\theta) = 2\sin\theta$.

$$[f(\theta)]^2 + [f'(\theta)]^2 = (2 - 2\cos\theta)^2 + 4\sin^2\theta = 4(1 - \cos\theta)^2 + 4\sin^2\theta.$$

Expand: $4(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta) = 4(2 - 2\cos\theta) = 8(1 - \cos\theta)$. Use $1 - \cos\theta = 2\sin^2(\theta/2)$:

$$[f(\theta)]^2 + [f'(\theta)]^2 = 16\sin^2\tfrac{\theta}{2}.$$

$$L = \int_0^{2\pi} \sqrt{16\sin^2\tfrac{\theta}{2}}\, \mathrm{d}\theta = 4 \int_0^{2\pi} \left|\sin\tfrac{\theta}{2}\right| \mathrm{d}\theta = 4 \cdot 2 \int_0^{\pi} \sin\tfrac{\theta}{2}\, \mathrm{d}\theta = 8\big[-2\cos\tfrac{\theta}{2}\big]_0^{\pi} = 8(0 + 2).$$

$$\boxed{L = 16.}$$

5. (8%) Polar Coordinates

(a) (4%) $r = 2\cos\theta + 2\sin\theta$ to rectangular.

Solution. Multiply both sides by r: $r^2 = 2r\cos\theta + 2r\sin\theta$. Use $r^2 = x^2+y^2$, $r\cos\theta = x$, $r\sin\theta = y$:

$$x^2 + y^2 = 2x + 2y.$$

Complete the square: $(x-1)^2 + (y-1)^2 = 2$.

$$\boxed{(x-1)^2 + (y-1)^2 = 2.}$$

This is a circle with centre $(1,1)$ and radius $\sqrt{2}$.

(b) (4%) $\dfrac{dy}{dx}$ for $r = 2 + \cos\theta$ at $\theta = \dfrac{\pi}{3}$.

Solution. $f(\theta) = 2 + \cos\theta$, $f'(\theta) = -\sin\theta$.

$$\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta = -\sin^2\theta + (2+\cos\theta)\cos\theta = \cos(2\theta) + 2\cos\theta.$$
$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta = -\sin\theta\cos\theta - (2+\cos\theta)\sin\theta = -2\sin\theta(\cos\theta + 1).$$

At $\theta = \dfrac{\pi}{3}$:

$$\frac{dy}{d\theta} = \cos\frac{2\pi}{3} + 2\cos\frac{\pi}{3} = -\frac{1}{2} + 1 = \frac{1}{2}.$$
$$\frac{dx}{d\theta} = -2\cdot\frac{\sqrt{3}}{2}\left(\frac{1}{2}+1\right) = -\sqrt{3}\cdot\frac{3}{2} = -\frac{3\sqrt{3}}{2}.$$
$$\boxed{\frac{dy}{dx} = \frac{1/2}{-3\sqrt{3}/2} = -\frac{1}{3\sqrt{3}} = -\frac{\sqrt{3}}{9}.}$$

6. (4%) Taylor Polynomial

Solution. Compute derivatives of $f(x) = \ln x$ and evaluate at $c = 2$:

$$f(x) = \ln x, \quad f(2) = \ln 2,$$
$$f'(x) = x^{-1}, \quad f'(2) = \frac{1}{2},$$
$$f''(x) = -x^{-2}, \quad f''(2) = -\frac{1}{4},$$
$$f'''(x) = 2x^{-3}, \quad f'''(2) = \frac{2}{8} = \frac{1}{4}.$$

Apply the Taylor polynomial definition $P_3(x) = \displaystyle\sum_{k=0}^{3} \frac{f^{(k)}(2)}{k!}(x-2)^k$:

$$P_3(x) = \ln 2 + \frac{1/2}{1!}(x-2) + \frac{-1/4}{2!}(x-2)^2 + \frac{1/4}{3!}(x-2)^3.$$

$$\boxed{P_3(x) = \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3.}$$