1. (5%) Determine whether the sequence is convergent or divergent. If convergent, find its limit.

(a)
$$a_n = n^{\frac{1}{n}}$$

(b) $a_n = \frac{\ln(n^3)}{n}$

Ans:

(a) Let $y = \lim_{n \to \infty} a_n = \lim_{n \to \infty} n^{\frac{1}{n}} \to \ln y = \lim_{n \to \infty} \frac{\ln n}{n}$. As *n* becomes large, $\ln n$ grows much more slowly than *n*. In fact, using standard limits or L'Hôpital's rule, we get $\lim_{n \to \infty} \frac{\ln n}{n} = 0$. Therefore, $y = e^0 = 1 \to a_n = n^{\frac{1}{n}}$ converges to 1

(b) Since
$$a_n = \frac{\ln(n^3)}{n} = \frac{3\ln(n)}{n}$$
. Again As *n* becomes large, $\ln n$ grows much

more slowly than *n*, we get $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{3 \ln n}{n} = 0$. Therefore, $a_n = \frac{\ln(n^3)}{n}$ converges to 0.

2. (5%) Determine whether the series is convergent or divergent. In addition, please indicate the test you use.

(a)
$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

(b) $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n-1}\right)^{n+1}$

Ans:

(a) Using the ratio test, $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} \right| = \lim_{n \to \infty} \left| \frac{(n+1)}{(n+1)^{n+1}} n^n \right| =$

$$\lim_{n \to \infty} \left| \frac{1}{(\frac{n+1}{n})^n} \right| = \lim_{n \to \infty} \left(\frac{1}{1 + \frac{1}{n}} \right)^n = \frac{1}{\lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n} = \frac{1}{e} < 1$$

by the Ratio Test the series converges.

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(b) Using the root test, $\lim_{n \to \infty} \sqrt[n]{\left[\left(\frac{2n+3}{3n-1}\right)^{n+1}\right]} = \lim_{n \to \infty} \left(\frac{2n+3}{3n-1}\right)^{\frac{n+1}{n}} = \lim_{n \to \infty} \left(\frac{2n+3}{3n-1}\right)^{\frac{1}{n}} \left(\frac{2n+3}{3n-1}\right)$

Let
$$y = \lim_{n \to \infty} \left(\frac{2+\frac{3}{n}}{3-\frac{1}{n}}\right)^{\frac{1}{n}} \to \ln y = \lim_{n \to \infty} \frac{1}{n} \left(\frac{2+\frac{3}{n}}{3-\frac{1}{n}}\right) = 0 \to y = 1$$

Therefore, $\lim_{n \to \infty} \sqrt[n]{\left|\left(\frac{2n+3}{3n-1}\right)^{n+1}\right|} = \frac{2}{3} \lim_{n \to \infty} \left(\frac{2+\frac{3}{n}}{3-\frac{1}{n}}\right)^{\frac{1}{n}} = \frac{2}{3} < 1$

by the Root Test the series converges.

- (5%) Determine whether the series converges absolutely or conditionally, or diverges. In addition, please indicate the test you use.
 - (a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{5n-3}$ (b) $\sum_{n=1}^{\infty} (-1)^n \frac{\arctan(n)}{n^2+1}$

Ans:

(a) Firstly, we check whether $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{5n-3} \right| = \sum_{n=1}^{\infty} \frac{1}{5n-3}$. By limit comparison test to the divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$, $\lim_{n \to \infty} \frac{\frac{1}{5n-3}}{\frac{1}{n}} = \frac{1}{5}$ which means $\sum_{n=1}^{\infty} \frac{1}{5n-3}$ is also divergent. $\sum_{n=1}^{\infty} \frac{(-1)^n}{5n-3}$ is converge by the alternating series test, since $\lim_{n \to \infty} \frac{1}{5n-3} = 0$ and $\frac{1}{5(n+1)-3} < \frac{1}{5n-3}$ for n > 1. Therefore, the original series is <u>conditionally</u>.

converge.

(b) Firstly, we check whether $\sum_{n=1}^{\infty} \left| (-1)^n \frac{\arctan(n)}{n^2 + 1} \right| = \sum_{n=1}^{\infty} \frac{\arctan(n)}{n^2 + 1}$ is converge or

not. Let
$$f(x) = \frac{\arctan(x)}{x^2 + 1}$$
. Since $f'(x) = \frac{1 - 2x\arctan(x)}{(x+1)^2} < 0$ for $x \ge 1$, f is

positive, continuous and decreasing for $x \ge 1$. $\int_{1}^{\infty} \frac{\arctan(x)}{x^{2}+1} dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} u \, du =$

$$\left[\frac{1}{2}u^2\right]\frac{\pi}{\frac{\pi}{4}} = \frac{3\pi^2}{32}$$
. Therefore, the original series is absolute converge by the integral

test.

4. (8%) Find the interval of convergence of the power series (Be sure to check the for the convergence at the endpoints of the intervals)

(a)
$$\sum_{n=1}^{\infty} \frac{x^n}{2^n (n^2 + 1)}$$

(b) $\sum_{n=1}^{\infty} \frac{n(x+1)^n}{3^n}$

Ans:

(a)
$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} \left| \frac{\frac{x^{n+1}}{2^{n+1}((n+1)^2+1)}}{\frac{x^n}{2^n(n^2+1)}} \right| = \lim_{n \to \infty} \left| \frac{n^2+1}{n^2+2n+2} \frac{1}{2} \right| |x| = \frac{1}{2} |x|.$$
 By the ratio test,

the series converges for |x| < 2

When x = 2: $\sum_{n=1}^{\infty} \frac{x^n}{2^n (n^2 + 1)} = \sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$. It is converging by limit comparison

test to the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

When x = -2: $\sum_{n=1}^{\infty} \frac{x^n}{2^n (n^2+1)} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2+1}$. The series is converge by the alternating series test, since $\lim_{n \to \infty} \frac{1}{n^2+1} = 0$ and $\frac{1}{(n+1)^2+1} < \frac{1}{n^2+1}$ for n > 1. So R = 2 and the interval of convergence is [-2,2]

(b) $\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} \left| \frac{(n+1)(x+1)^{n+1}}{3^{n+1}} \times \frac{3^n}{n(x+1)^n} \right| = \lim_{n \to \infty} \left| \frac{n+1}{n} \right| \left| \frac{x+1}{3} \right| = \left| \frac{x+1}{3} \right|.$ By the ratio test, the series converges for |x+1| < 3When x = 2: $\sum_{n=1}^{\infty} \frac{n(x+1)^n}{3^n} = \sum_{n=1}^{\infty} n$. Which is clearly diverge by the n-th term

Test.

When x = -4: $\sum_{n=1}^{\infty} \frac{n(x+1)^n}{3^n} = \sum_{n=1}^{\infty} (-1)^n n$. Which is clearly diverge by the n-th term Test.

So R = 3 and the interval of convergence is (-4,2)

5. (8%) Use a power series to approximate $\int_0^1 \frac{1-e^{-x^2}}{x^2} dx$ with an error of less than 0.01

Ans:
$$\int_{0}^{1} \frac{1 - e^{-x^{2}}}{x^{2}} dx = \int_{0}^{1} \frac{1 - (1 - x^{2} + \frac{x^{4}}{2!} - \frac{x^{6}}{3!} + \frac{x^{8}}{4!} \dots)}{x^{2}} dx = \int_{0}^{1} \left(1 - \frac{x^{2}}{2!} + \frac{x^{4}}{3!} - \frac{x^{6}}{4!} \dots \right) dx = \left[x - \frac{x^{3}}{3 \times 2!} + \frac{x^{5}}{5 \times 3!} - \frac{x^{7}}{7 \times 4!} \dots \right]_{0}^{1} = 1 - \frac{1}{3 \times 2!} + \frac{1}{5 \times 3!} - \frac{x^{7}}{7 \times 4!} + \dots$$

$$\int_0^1 \frac{1 - e^{-x^2}}{x^2} dx = 1 - \frac{1}{3 \times 2!} + \frac{1}{5 \times 3!} - \frac{1}{7 \times 4!} + \dots + \frac{(-1)^n}{(2n+1)(n+1)!} + \dots$$

It is an alternating series and since $|R_N| \le a_{N+1} = \frac{1}{(2 \times 3 + 1)(3 + 1)!} < 0.01$ when N =

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3. Therefore we know that
$$\int_0^1 \frac{1 - e^{-x^2}}{x} dx \approx 1 - \frac{1}{3 \times 2!} + \frac{1}{5 \times 3!} \approx \frac{13}{15}$$

6. (10%) Evaluate the following expression:

(a) (2.5%)
$$\frac{1}{1\times 2} - \frac{1}{3\times 2^3} + \frac{1}{5\times 2^5} - \frac{1}{7\times 2^7} - \cdots$$

(b) (2.5%) $\sum_{n=1}^{\infty} \left(\frac{1}{e^n} + \frac{1}{n(n+2)}\right)$

(c) (5%) If $\lim_{x \to 0} \frac{\sin(3x) + a + bx + cx^3}{x^3} = 0$, what is the value of *a*, *b*, *c*?

Ans:

(a)
$$\frac{1}{1\times 2} - \frac{1}{3\times 2^3} + \frac{1}{5\times 2^5} - \frac{1}{7\times 2^7} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)2^{2n+1}} = \arctan(\frac{1}{2})$$

(b) $\sum_{n=1}^{\infty} \left(\frac{1}{e^n} + \frac{1}{n(n+2)}\right) = \sum_{n=1}^{\infty} \frac{1}{e^n} + \sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{\frac{1}{e}}{1 - \frac{1}{e}} + \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right) = \frac{1}{e-1} + \frac{1}{2} \left[\frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \dots - \frac{1}{n+1} - \frac{1}{n+2} + \dots\right] = \frac{1}{e-1} + \frac{1}{2} \left[1 + \frac{1}{2}\right] = \frac{1}{e-1} + \frac{3}{4}$

(c)
$$\lim_{x \to 0} \frac{\sin(3x) + a + bx + cx^3}{x^3} = \lim_{x \to 0} \frac{\left(3x - \frac{1}{3!}(3x)^3 + \frac{1}{5!}(3x)^5 \dots\right) + (a + bx + cx^3)}{x^3} = \\\lim_{x \to 0} \frac{a + (3 + b)x + \left(c - \frac{3^3}{3!}\right)x^3 + x^3(\frac{3^5x^2}{5!} - \dots)}{x^3} = \lim_{x \to 0} \frac{a}{x^3} + \frac{3 + b}{x^2} + \left(c - \frac{3^3}{3!}\right) + \left(\frac{3^5x^2}{5!} - \dots\right) = \\\lim_{x \to 0} \frac{a}{x^3} + \frac{3 + b}{x^2} + \left(c - \frac{3^3}{3!}\right) + \lim_{x \to 0} \frac{3^5x^2}{5!} - \dots$$

Since the second term is 0, For the limit to exist (and equal 0) as $x \to 0$, the numerator must have no constant term or x term (since these would lead to terms which blow up) the first term should also be 0, and so $a = 0, b = -3, c = \frac{27}{6} = \frac{9}{2}$.

7. (12%) Find the first three nonzero terms in the Maclaurin Series for each function
(a) f(x) = e^{-x} ln(1 + 2x)
(b) f(x) = √1 + 2xcos(x)
(c) f(x) = 1/(1+x-6x²)

(a)
$$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots$$

$$\ln(1+2x) = 2x - 2x^2 + \frac{8}{3}x^3 + \cdots$$

$$e^{-x}\ln(1+2x) \approx 2x - 4x^2 + \frac{17}{3}x^3$$
(b) $\sqrt{1+2x} = (1+2x)^{\frac{1}{2}} = 1 + \frac{1}{2}2x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}(2x)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}(2x)^3 + \cdots$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots$$

$$\sqrt{1+2x}\cos(x) \approx 1 + x - x^2$$
(c) $\frac{1}{1+x-6x^2} = \frac{1}{1-(-x+6x^2)} = 1 + (-x+6x^2) + (x^2 - 12x^3 + 36x^4) + \cdots \approx$

$$1 - x + 7x^2$$

Or
$$\frac{1}{1+x-6x^2} = \frac{1}{(1+3x)(1-2x)} = \frac{1}{(1+3x)} \frac{1}{(1-2x)} = (1 - 3x + \frac{-1(-2)}{2!}9x^2 + \cdots)(1 - 2x + \frac{-1(-2)}{2!}4x^2 + \cdots) \approx 1 - x + 7x^2$$

8. (6%) Sketch the curve represented by the parametric equations (indicate the orientation of the curve) $x = \sqrt{t} + 1$, y = t - 3, $t \ge 0$ and write the corresponding rectangular equation by eliminating the parameter.

Ans:



9. (9%) Find the area of the surface generated by revolving the curve $x = 4t, y = 3t + 1, 0 \le t \le 2$ about the x-axis

$$\frac{dx}{dt} = 4, \frac{dy}{dt} = 3, \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 5$$

S =
$$2\pi \int_0^2 (3t+1) \, 5dt = 10\pi \left[\frac{3t^2}{2} + t\right]_0^2 = 80\pi$$

10. (10%) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ of the parametric equation $x = 3 + \cos(\theta)$ and $y = 2 + 4\sin(\theta)$ and find the slope and concavity at $\theta = \frac{\pi}{6}$.

Ans:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{4\cos(\theta)}{-\sin(\theta)} = -4\cot(\theta), \\ \frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}[-4\cot(\theta)]}{-\sin(\theta)} = \frac{4\csc^2(\theta)}{-\sin(\theta)} = \frac{4\csc^2(\theta)}{-\sin(\theta)}$$
$$= -4\csc^3(\theta)$$

At $\theta = \frac{\pi}{6}$, the slope is $\frac{dy}{dx} = -4\sqrt{3}$ and $\frac{d^2y}{dx^2} = -32$ which is concave downward.

11. (10%) Sketch the polar graph and find the arc length and area of the polar graph $r = 8(1 + \cos(\theta))$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{(8(1 + \cos(\theta)))^2 + (-8\sin(\theta))^2}$$

$$S = \int_0^{2\pi} \sqrt{(8(1 + \cos(\theta)))^2 + (-8\sin(\theta))^2} = 8\sqrt{2} \int_0^{2\pi} \sqrt{1 + \cos(\theta)} d\theta$$

$$= 8\sqrt{2} \int_0^{2\pi} \sqrt{2\cos^2\left(\frac{\theta}{2}\right)} d\theta = 16 \int_0^{2\pi} \left|\cos\left(\frac{\theta}{2}\right)\right| d\theta$$

$$= 16 \left[\int_0^{\pi} \cos\left(\frac{\theta}{2}\right) d\theta + \int_{\pi}^{2\pi} -\cos\left(\frac{\theta}{2}\right) d\theta\right]$$

$$= 32 \left[\sin\left(\frac{\theta}{2}\right)\right]_0^{\pi} - 32 \left[\sin\left(\frac{\theta}{2}\right)\right]_{\pi}^{2\pi} = 64$$

$$A = \frac{1}{2} \int_0^{2\pi} [8(1 + \cos(\theta))]^2 d\theta = 32 \int_0^{2\pi} (1 + 2\cos(\theta) + \cos^2(\theta)) d\theta$$
$$= 32 \left[\theta + 2\sin(\theta) + \frac{\theta}{2} + \frac{\sin^2(\theta)}{4} \right]_0^{2\pi} = 96\pi$$



12. (12%)

- (a) Find an equation in rectangular coordinates for the surface represented by cylindrical coordinates $z = r^2 sin^2\theta + 3rcos(\theta) + 1$.
- (b) Find an equation in rectangular coordinates for the surface represented by spherical coordinates $\rho = 9 \sec(\theta)$.
- (c) Convert the rectangular equation $x^2 + y^2 + z^2 = 9$ to equation in cylindrical coordinates.
- (d) Convert the rectangular equation $x^2 y^2 = 2z$ to equation in spherical coordinates.

- (a) $z = y^2 + 3x + 1$
- (b) The original meaning Should be $\rho = 9\sec(\Phi)$, therefore, it will be a give away question.
- (c) $r^2 + z^2 = 9$
- (d) $\rho^2 \sin^2 \Phi \cos^2(\theta) \rho^2 \sin^2 \Phi \sin^2 \theta = 2\rho \cos(\Phi) \rightarrow \rho \sin^2 \Phi \cos(2\theta) 2\cos(\Phi) = 0 \rightarrow \rho = 2\sec(2\theta)\cos(\Phi)\csc^2(\Phi)$

Function	Taylor series	Interval of
		convergence
$\frac{1}{x}$	$1 - (x - 1) + (x - 1)^{2} - (x - 1)^{3} + \dots + (-1)^{n}(x - 1)^{n} + \dots$	0 < x < 2
$\frac{1}{1+x}$	$1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots$	-1 < x < 1
ln x	$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \dots + \frac{(-1)^{n-1}(x-1)^n}{n} + \dots$	$0 < x \le 2$
e ^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$	$-\infty < x < \infty$
sin(x)	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots$	$-\infty < x < \infty$
cos(x)	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots$	$-\infty < x < \infty$
arctan(x)	$x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + \frac{(-1)^n x^{2n+1}}{2n+1} + \dots$	$-1 \le x \le 1$
arcsin(x)	$x + \frac{x^3}{2 \times 3} + \frac{1 \times 3x^5}{2 \times 4 \times 5} + \dots + \frac{(2n)! x^{2n+1}}{(2^n n!)^2 (2n+1)} + \dots$	$-1 \le x \le 1$
$(1+x)^k$	$1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!}$	-1 < x < 1
	$+\cdots + \frac{k(k-1)\dots(k-n+1)x^n}{n!} + \cdots$	

Derivative	Integrals
$\frac{d\sin^{-1}u}{dx} = \frac{u'}{\sqrt{1-u^2}}$	$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C$
$\frac{d\cos^{-1}u}{dx} = \frac{-u'}{\sqrt{1-u^2}}$	$\int \frac{du}{a^2 + u^2} = \frac{1}{a}tan^{-1}\frac{u}{a} + C$
$\frac{d\tan^{-1}u}{dx} = \frac{u'}{1+u^2}$	$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}sec^{-1}\frac{ u }{a} + C$
$\frac{d \cot^{-1} u}{dx} = \frac{-u'}{1+u^2}$	
$\frac{d \sec^{-1} u}{dx} = \frac{u'}{ u \sqrt{u^2 - 1}}$	
$\frac{d \csc^{-1} u}{dx} = \frac{-u'}{ u \sqrt{u^2 - 1}}$	