1. (12%) Let $\boldsymbol{r}(\mathrm{t}) = \; e^{-t}\mathbf{i} + \frac{\sin(t)}{t}\mathbf{j} + 4\mathbf{k}$, $\boldsymbol{g}(\mathrm{t}) = e^{t}(t-1)^2\mathbf{i} + (t-1)^3\mathbf{k}$:

(a) Evaluate the limit $\lim_{t\to 0}\boldsymbol{r}(t)$

(b) Find the intervals on which the curve given by $\boldsymbol{g}(t)$ is smooth

(c) Compute $\frac{\mathrm{d}}{\mathrm{dt}}[\boldsymbol{r}(t)\cdot\boldsymbol{g}(t)]$

Ans:

(a) Compute the limit component-wise:

$\lim_{t\to 0} e^{-t} = 1$, $\lim_{t\to 0}\frac{\sin(t)}{t}$ =(L'Hospital rule) $\lim_{t\to 0}\frac{\cos(t)}{1} = 1$, $\lim_{t\to 0} 4 = 4$

The original limit is $\mathbf{i} + \mathbf{j} + 4\mathbf{k}$

(b) $\boldsymbol{g}'(t) = (2e^{t}(t-1) + e^{t}(t-1)^2)\mathbf{i} + 3(t-1)^2\mathbf{k} = e^{t}(t^2-1)\mathbf{i} + 3(t-1)^2\mathbf{k}$

Smoothness requires that every component derivative be continuous and that the derivatives are not all zero simultaneously.

For $\boldsymbol{g}'(t)$, the continuity holds for all t and the components function of $\boldsymbol{g}'(t)$ will be all zeros if $t = 1$. Therefor the function is smooth on $(-\infty, 1)$ and $(1, \infty)$

(c) $\frac{\mathrm{d}}{\mathrm{dt}}[\boldsymbol{r}(t)\cdot\boldsymbol{g}(t)] = \frac{\mathrm{d}}{\mathrm{dt}}\left[e^{-t}e^{t}(t-1)^2 + \frac{\sin(t)}{t}0 + 4(t-1)^3\right] = \frac{\mathrm{d}}{\mathrm{dt}}[(t-1)^2 + 4(t-1)^3] = 2(t-1) + 12(t-1)^2 = 2(t-1)(6t-5)$

2. (12%) Find the following limits:

(a) $\lim_{(x,y)\to(0,0)}\frac{e^{y}sin(x)}{x}$

(b) $\lim_{(x,y)\to(0,0)}\frac{2xycos(x)}{x^2+y^2}$

(c) $\lim_{(x,y)\to(0,0)}\frac{3(x^2+y^2)}{\sqrt{x^2+y^2+4}-2}$ (Using polar coordinates)

Ans:

(a) $\lim_{(x,y)\to(0,0)}\frac{e^{y}sin(x)}{x} = \lim_{(x,y)\to(0,0)} e^{y} \lim_{(x,y)\to(0,0)}\frac{\sin(x)}{x} = 1$

(b) Let $y = mx \to \lim_{(x,y)\to(0,0)}\frac{2xycos(x)}{x^2+y^2} = \lim_{x\to 0}\frac{2mx^2cos(x)}{x^2+m^2x^2} = \frac{2m}{1+m^2}\lim_{x\to 0}\cos(x) = \frac{2m}{1+m^2}$. Which means that if we follow the trajectory of different line $y = mx$

to approach (0,0) we will get different value for different m, therefore, the limit does not exist.

(c) Let $x = rcos(\theta), y = rsin(\theta) \to r^2 = x^2 + y^2$

$$\lim_{(x,y)\to(0,0)} \frac{3(x^2+y^2)}{\sqrt{x^2+y^2+4}-2} = \lim_{r\to0} \frac{3r^2}{\sqrt{r^2+4}-2} = \lim_{r\to0} \frac{3r^2\left(\sqrt{r^2+4}+2\right)}{r^2+4-4}$$

$$= 3\lim_{r\to0} \sqrt{r^2+4}+2 = 12$$

3. (20%)

(a) (6%) Let $f(x,y) = \begin{cases} \frac{4x^2y}{x^3+y^3} & \text{when } (x,y) \neq (0,0) \\ 0 & \text{when } (x,y) = (0,0) \end{cases}$, compute $f_x(0,0)$ and $f_y(0,0)$. In addition, decide whether f differentiable at $(0,0)$

(b) (8%) Let $w = f(x,y,z) = xy + yz + zx, x = rcos(\theta), y = rsin(\theta), z = r\theta$ find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial \theta}$ when $r = 2, \theta = \frac{\pi}{2}$ (Using chain rule)

(c) (6%) Given the surface $x^2 + 2y^2 - 3z^2 = 3$, find the tangent plane and the normal line to the surface at point $(2,-1,1)$

Ans:

(a) For $(x,y) = (0,0)$:

$$f_x(x,y) = \lim_{\Delta x\to0} \frac{f(0+\Delta x,0)-f(0,0)}{\Delta x} = \lim_{\Delta x\to0} \frac{0}{(\Delta x)^3}\frac{1}{\Delta x} = 0$$

$$f_y(x,y) = \lim_{\Delta y\to0} \frac{f(0,0+\Delta y)-f(0,0)}{\Delta y} = \lim_{\Delta y\to0} \frac{0}{(\Delta x)^3}\frac{1}{\Delta y} = 0$$

On the other hand, let $y = mx$, $\lim_{(x,y)\to(0,0)} \frac{4x^2y}{x^3+y^3} = \lim_{x\to0} \frac{4x^2mx}{x^3+m^3x^3} = \lim_{x\to0} \frac{m}{1+m^3}$.

which means that if we follow the trajectory of different line $y = mx$ to approach (0,0) we will get different value for different m, therefore, the limit does not exist. So $f(x,y)$ is not continuous at $(0,0)$. Therefore, it is not differentiable at $(0,0)$.

(b) Using chain rule

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = (y+z)\cdot\cos(\theta) + (x+z)\sin(\theta) + (x+y)\theta$$

At point $r = 2, \theta = \frac{\pi}{2}, \mathrm{x} = 2\cos\left(\frac{\pi}{2}\right) = 0, \mathrm{y} = 2\sin\left(\frac{\pi}{2}\right) = 2, \mathrm{z} = 2\frac{\pi}{2} = \pi, \cos\left(\frac{\pi}{2}\right) = 0, \sin\left(\frac{\pi}{2}\right) = 1$

Hence at point $r=2, \theta=\frac{\pi}{2}$ $\frac{\partial w}{\partial r}=(y+z)\cdot\cos(\theta)+(x+z)\sin(\theta)+$

$(x+y)\theta=(2+\pi)0+(0+\pi)+(0+2)\frac{\pi}{2}=2\pi$

$\frac{\partial w}{\partial \theta}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial \theta}=(y+z)\cdot -rsin(\theta)+(x+z)\,rcos(\theta)+(x+$

$y)r$

At point $r=2, \theta=\frac{\pi}{2}, \mathrm{x}=2\cos\left(\frac{\pi}{2}\right)=0, \mathrm{y}=2\sin\left(\frac{\pi}{2}\right)=2, \mathrm{z}=2\frac{\pi}{2}=$

$\pi, \cos\left(\frac{\pi}{2}\right)=0, \sin\left(\frac{\pi}{2}\right)=1$

Hence at point $r=2, \theta=\frac{\pi}{2}$ $\frac{\partial w}{\partial \theta}=(y+z)\cdot -rsin(\theta)+(x+z)\,rcos(\theta)+$

$(x+y)r=(2+\pi)(-2)+(0+\pi)0+(0+2)2=-2\pi$

(c) Let $F(x,y,z)=x^2+2y^2-3z^2-3=0$

$$\nabla F=2x\boldsymbol{i}+4y\boldsymbol{j}-6z\boldsymbol{k}$$
$$\nabla F(2,-1,1)=4\boldsymbol{i}-4\boldsymbol{j}-\boldsymbol{6k}$$

The equation for the normal line is

$$\frac{x-2}{4}=\frac{y+1}{-4}=\frac{z-1}{-6} \quad \text{or} \quad \frac{x-2}{2}=\frac{y+1}{-2}=\frac{z-1}{-3} \quad \text{or} \quad \begin{cases} x=2+4t \\ y=-1=4t \\ z=1-6t \end{cases}$$

The tangnet plane is

$$4(x-2)-4(y+1)-6(z-1)=0 \rightarrow 2x-2y-3z=3$$

4. (6%) Let $\mathrm{w}=f(x,y,z)=e^{\sqrt{x+y+z^2}}$
   (a) Find the gradient of the function $f(x,y,z)$.
   (b) Find the direction in which f has maximum value of the directional derivative at point $(5,0,2)$ and what is the value?

   Ans:

   (a)

$$\nabla f=\frac{e^{\sqrt{x+y+z^2}}}{2\sqrt{x+y+z^2}}\mathbf{i}+\frac{e^{\sqrt{x+y+z^2}}}{2\sqrt{x+y+z^2}}\mathbf{j}+\frac{ze^{\sqrt{x+y+z^2}}}{\sqrt{x+y+z^2}}\mathbf{k}$$

(b) $\nabla f(5,0,2)=\frac{e^3}{6}\mathbf{i}+\frac{e^3}{6}\mathbf{j}+\frac{2e^3}{3}\mathbf{k} \rightarrow$ The direction is $\mathbf{i}+\mathbf{j}+4\mathbf{k}$

$$\|\nabla f(5,0,2)\| = \sqrt{\frac{e^6}{36}+\frac{e^6}{36}+\frac{4e^6}{9}} = \frac{e^3}{\sqrt{2}}$$

5. (8%) Let $f(x,y) = 9xy - x^3 - y^3$
   (a) Find the critical points of f
   (b) Classify each critical point as a local maximum, local minimum or saddle point

   Ans:

   (a) $f_x = 9y - 3x^2,\ f_y = 9x - 3y^2$.

   Let $f_x = 0$ and $f_y = 0$,

   We get $x^4 = 9y^3 = 9(3x) = 27x \to x(x^3 - 27) = 0$.

   Therefore, the critical points are $(0,0), (3,3)$

   (b)

   Since $f_{xx} = -6x, f_{xy} = f_{yx} = 9, f_{yy} = -6y$.

(x,y)	f_{xx}	f_{xy}	f_{yy}	d	
(0,0)	0	9	0	-81	Saddle point
(3,3)	-18	9	-18	243	local maximum

6. (18%) Evaluate the following expressions
   (a) $\int_0^{2\sqrt{\ln 3}} \int_{\frac{y}{2}}^{\sqrt{\ln 3}} e^{x^2}\, dxdy$ (by reversing the order of integration) .
   (b) Sketch the region R whose region is given by the iterated integral

   $\int_0^3 \int_0^x dydx + \int_3^6 \int_0^{6-x} dydx$ and evaluate it.

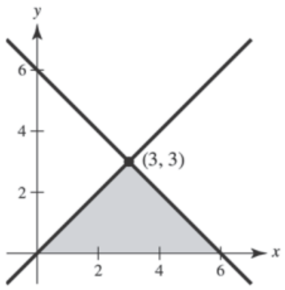
   (c) Evaluate the double integral $\iint_R \sin(x^2+y^2)dA$, where R is the region in the first quadrant between the circles with center the origin and radii 2 and 3 (by changing to polar coordinates) .

Ans:

(a) $\int_0^{2\sqrt{\ln 3}} \int_{\frac{y}{2}}^{\sqrt{\ln 3}} e^{x^2}\, dxdy = \int_0^{\sqrt{\ln 3}} \int_0^{2x} e^{x^2}\, dydx = \int_0^{\sqrt{\ln 3}} ye^{x^2}\Big|_0^{2x} dx =$

$\int_0^{\sqrt{\ln 3}} 2x\, e^{x^2} dx = \int_0^{\ln 3} e^u\, du = e^u\Big|_0^{\ln 3} = 3 - 1 = 2$

(b)



$$\int_0^3\int_0^x dydx + \int_0^6\int_0^{6-x} dydx = \int_0^3 x \ \ dx + \int_3^6 6 - x \ \ dx = \frac{x^2}{2}\Big|_0^3 + \left[6x - \frac{x^2}{2}\right]_3^6$$
$$= 9.$$

(c) $R = \{(x,y)|2 \le x^2 + y^2 \le 3, 0 \le y,\ 0 \le x\} = \left\{(r,\theta)\middle|2 \le r \le 3, 0 \le \theta \le \frac{\pi}{2}\right\}$

$$\iint_R \sin(x^2 + y^2)dA = \int_0^{\frac{\pi}{2}}\int_2^3 sin\,(r^2)\,rdrd\theta = \int_0^{\frac{\pi}{2}}\int_4^9 \frac{1}{2} sin\,(u)\,dud\theta$$
$$= \int_0^{\frac{\pi}{2}}\left[\frac{1}{2}(-\cos(u))\right]_4^9 d\theta = \int_0^{\frac{\pi}{2}}\frac{1}{2}(\cos(4) - \cos(9))\,d\theta$$
$$= \frac{\pi}{4}(\cos(4) - \cos(9))$$

7. (8%) Find the area of the surface given by $z = f(x,y) = x^2 + y + 2$ that lies above the region R where R is a triangular region with vertices $(0,0), (2,0), (2,2)$

Ans:

$$f_x = 2x, f_y = 1$$
$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{2 + 4x^2}$$
$$S = \int_0^2\int_0^x \sqrt{2 + 4x^2}\,dydx = \int_0^2 x\sqrt{2 + 4x^2}\,dx = \frac{1}{12}u^{\frac{3}{2}}\Big|_2^{18} = \frac{1}{12}\left(18\sqrt{18} - 2\sqrt{2}\right)$$
$$= \frac{13\sqrt{2}}{3}$$

Note that we let $u = 2 + 4x^2$

8. (8%) Find the volume of the solid which bounded by the cylinder $x^2 + y^2 = 1$ and the plane $z = y, z = 0, x = 0$ in the first octant.

Ans:

$$V = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^y 1 \ dzdydx = \int_0^1 \int_0^{\sqrt{1-x^2}} y\,dydx = \int_0^1 \frac{y^2}{2}\Big|_0^{\sqrt{1-x^2}} dydx$$

$$= \int_0^1 \frac{1-x^2}{2} dx = \left[\frac{1}{2}x - \frac{x^3}{6}\right]_0^1 = \frac{1}{3}$$

9. (8%) Find the volume of the solid bounded above by $3x^2 + 3y^2 + z^2 = 27$ and below by the xy-plane

Ans: Use cylindrical coordinates

$$z = \sqrt{27 - 3r^2}$$

$$V = \int_0^{2\pi} \int_0^3 \int_0^{\sqrt{27-3r^2}} r \ dzdrd\theta = \int_0^{2\pi} \int_0^3 r(\sqrt{27 - 3r^2})\, drd\theta$$

$$= \frac{1}{6} \int_0^{2\pi} \frac{2}{3} u^{\frac{3}{2}} \Big|_0^{27} \ d\theta = 18\sqrt{3}\pi$$

Note that we let $u = 27 - 3r^2$