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Chapter 13

FUNCTIONS OF SEVERAL VARIABLES

13.1 Summary

Section 13.1 Introduction to functions of several variables ... 3

- A function of two variables** Let D be a set of ordered pairs of real numbers. If to each ordered pair (x, y) in D there corresponds a unique real number $f(x, y)$, then f is called a **function** (函數) of x and y . The set D is the **domain** (定義域) of f , and the corresponding set of values for $f(x, y)$ is the **range** (值域) of f 5

2. A scalar field can be characterized by **level curves** (**等高線**) (or **contour lines** (**等高線**)) along which the value of $f(x, y)$ is constant. 14
3. If f is a function of three variables and c is a constant, the graph of the equation $f(x, y, z) = c$ is a **level surface** (**等位曲面**) of the function f 25
- Section 13.2 Limits and continuity** 33

4. **Limit of a function of two variables** Let f be a function of two variables defined, except possibly at (x_0, y_0) , on an open disk centered at (x_0, y_0) , and let L be a real number. Then

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = L$$

if for each ε there corresponds a $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

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5. **Continuity of a function of two variables** A function f of two variables is **continuous at a point** (x_0, y_0) (在點 (x_0, y_0) 是連續的) at a point (x_0, y_0) in an open region R if $f(x_0, y_0)$ is equal to the limit of $f(x, y)$ as (x, y) approaches (x_0, y_0) . That is,

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = f(x_0, y_0).$$

The function f is **continuous in the open region** R (在開區域 R 是連續的) if it is continuous at every point in R 54

6. **Continuity of a function of two variables** If k is a real number

and f and g are continuous at (x_0, y_0) , then the following function are continuous at (x_0, y_0) .

1. Scalar multiple: kf
2. Product: fg
3. Sum and difference: $f \pm g$
4. Quotient: f/g , if $g(x_0, y_0) \neq 0$.

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7. **Continuity of a composite function** If h is continuous at (x_0, y_0) and g is continuous at $h(x_0, y_0)$, then the **composite function** (合成函數) given by $(g \circ h)(x, y) = g(h(x, y))$ is **continuous** (連續) at (x_0, y_0) .

That is,

$$\lim_{(x,y) \rightarrow (x_0,y_0)} g(h(x, y)) = g(h(x_0, y_0)).$$

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8. **Continuity of a function of three variables** A function f of three variable is **continuous at a point** (x_0, y_0, z_0) (在一點 (x_0, y_0, z_0) 是連

in an open region R if $f(x_0, y_0, z_0)$ is defined and is equal to the limit of $f(x, y, z)$ as (x, y, z) approaches (x_0, y_0, z_0) . That is,

$$\lim_{(x,y,z) \rightarrow (x_0,y_0,z_0)} f(x, y, z) = f(x_0, y_0, z_0).$$

The function f is **continuous in the open region R** (在開區域 R 是連續的) if it is continuous at every point in R 62

Section 13.3 Partial derivatives 63

9. **Partial derivatives of a function of two variables** If $z = f(x, y)$, then the **first partial derivatives** (第一階偏導數) of f with respect to x and y are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \quad \text{and} \quad f_y(x, y) = \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

provided the limits exist. 65

10. **Notation for first partial derivatives** For $z = f(x, y)$, the partial derivatives f_x and f_y are denoted by

$$\frac{\partial}{\partial x} f(x, y) = f_x(x, y) = z_x = \frac{\partial z}{\partial x} \quad \text{and} \quad \frac{\partial}{\partial y} f(x, y) = f_y(x, y) = z_y = \frac{\partial z}{\partial y}.$$

The first partials evaluated at the point (a, b) are denoted by

$$\left. \frac{\partial z}{\partial x} \right|_{(a,b)} = f_x(a, b) \quad \text{and} \quad \left. \frac{\partial z}{\partial y} \right|_{(a,b)} = f_y(a, b).$$

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11. The concept of a partial derivative can be extended naturally to functions of three or more variables. If $w = f(x, y, z)$, there are three partial deriv-

atives, each of which is formed by holding two of the variables constant.

$$\frac{\partial w}{\partial x} = f_x(x, y, z) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$$

$$\frac{\partial w}{\partial y} = f_y(x, y, z) = \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}$$

$$\frac{\partial w}{\partial z} = f_z(x, y, z) = \lim_{\Delta z \rightarrow 0} \frac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}$$

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12. Equality of mixed partial derivatives (混合偏導數的恆等式)

If f is a function of x and y such that f_{xy} and f_{yx} are continuous on an open disk R , then, for every (x, y) in R ,

$$f_{xy}(x, y) = f_{yx}(x, y).$$

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Section 13.4 Differentials 84

13. **Total differential** If $z = f(x, y)$ and Δx and Δy are increments of x and y , then the **differentials** (微分) of the independent variables x and y are

$$dx = \Delta x \quad \text{and} \quad dy = \Delta y$$

and the **total differential** (全微分) of the dependent variable z is

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = f_x(x, y) dx + f_y(x, y) dy.$$

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14. **Differentiability** A function f given by $z = f(x, y)$ is **differentiable** (可微的) at (x_0, y_0) if Δz can be written in the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$$

where both ϵ_1 and $\epsilon_2 \rightarrow 0$ as $(\Delta x, \Delta y) \rightarrow (0, 0)$. The function f is **differentiable in a region R** (在區域 R 上可微的) if it is differentiable at each point in R 89

15. **Sufficient condition for differentiability** If f is a function of x and y , where f_x and f_y are continuous in an open region R , then f is differentiable on R 92

16. A function of three variables $w = f(x, y, z)$ is called **differentiable** (可微的) at (x, y, z) provided that

$$\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$$

can be written in the form

$$\Delta w = f_x \Delta x + f_y \Delta y + f_z \Delta z + \epsilon_1 \Delta x + \epsilon_2 \Delta y + \epsilon_3 \Delta z$$

where ϵ_1 , ϵ_2 , and $\epsilon_3 \rightarrow 0$ as $(\Delta x, \Delta y, \Delta z) \rightarrow (0, 0, 0)$ 98

17. **Sufficient condition for differentiability** If f is a function of x , y , and z , where f , f_x , f_y , and f_z are continuous in an open region R , then f is differentiable on R 100

18. **Differentiability implies continuity** (可微性隱含連續性) If a function of x and y is differentiable at (x_0, y_0) , then it is continuous at (x_0, y_0) 102

Section 13.5 Chain Rules for functions of several variables 106

19. **Chain Rule: one independent variable** (連鎖律：一個獨立變數)
 Let $w = f(x, y)$, where f is a differentiable function of x and y . If $x = g(t)$ and $y = h(t)$, where g and h are differentiable functions of t ,

then w is a differentiable function of t , and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}.$$

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20. If each x_i is a differentiable function of a single variable t , then for

$$w = f(x_1, x_2, \dots, x_n)$$

you have

$$\frac{dw}{dt} = \frac{\partial w}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial w}{\partial x_2} \frac{dx_2}{dt} + \dots + \frac{\partial w}{\partial x_n} \frac{dx_n}{dt}.$$

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21. **Chain Rule: two independent variables** (連鎖律：兩個獨立變數)

Let $w = f(x, y)$, where f is a differentiable function of x and y . If

$x = g(s, t)$ and $y = h(s, t)$ such that the first partial $\frac{\partial x}{\partial s}$, $\frac{\partial x}{\partial t}$, $\frac{\partial y}{\partial s}$, and $\frac{\partial y}{\partial t}$ all exist, then $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ exist and are given by

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t}.$$

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22. If w is a differentiable function of the n variables x_1, x_2, \dots, x_n , where each x_i is a differentiable function of the m variables t_1, t_2, \dots, t_m , then for $w = f(x_1, x_2, \dots, x_n)$ you obtain the following.

$$\begin{aligned} \frac{\partial w}{\partial t_1} &= \frac{\partial w}{\partial x_1} \frac{\partial x_1}{\partial t_1} + \frac{\partial w}{\partial x_2} \frac{\partial x_2}{\partial t_1} + \dots + \frac{\partial w}{\partial x_n} \frac{\partial x_n}{\partial t_1} \\ \frac{\partial w}{\partial t_2} &= \frac{\partial w}{\partial x_1} \frac{\partial x_1}{\partial t_2} + \frac{\partial w}{\partial x_2} \frac{\partial x_2}{\partial t_2} + \dots + \frac{\partial w}{\partial x_n} \frac{\partial x_n}{\partial t_2} \end{aligned}$$

$$\begin{aligned} & \vdots \\ \frac{\partial w}{\partial t_m} &= \frac{\partial w}{\partial x_1} \frac{\partial x_1}{\partial t_m} + \frac{\partial w}{\partial x_2} \frac{\partial x_2}{\partial t_m} + \cdots + \frac{\partial w}{\partial x_n} \frac{\partial x_n}{\partial t_m} \end{aligned}$$

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23. **Chain Rule: implicit differentiation (隱函數微分)** If the equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x , then

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}, \quad F_y(x, y) \neq 0.$$

If the equation $F(x, y, z) = 0$ defines z implicitly as a differentiable function of x and y , then

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}, \quad F_z(x, y, z) \neq 0.$$

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Section 13.6 Directional derivatives and gradients 128

24. **Directional derivative** Let f be a function of two variables x and y and let $\mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$ be a unit vector. Then the **directional derivative** (方向導數) of f in the direction of \mathbf{u} , denoted by $D_{\mathbf{u}}f$, is

$$D_{\mathbf{u}}f(x, y) = \lim_{t \rightarrow 0} \frac{f(x + t \cos \theta, y + t \sin \theta) - f(x, y)}{t}$$

provided this limit exists.....134

25. **Directional derivative** If f is a differentiable function of x and y , then the directional derivative of f in the direction of the unit vector $\mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$ is

$$D_{\mathbf{u}}f(x, y) = f_x(x, y) \cos \theta + f_y(x, y) \sin \theta.$$

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26. **Gradient of a function of two variables** Let $z = f(x, y)$ be a function of x and y such that f_x and f_y exist. Then the **gradient** (**梯度**) of f , denoted by $\nabla f(x, y)$, is the vector

$$\nabla f(x, y) = f_x(x, y) \mathbf{i} + f_y(x, y) \mathbf{j}.$$

∇f is read as "del f " 142

27. **Alternative form of the directional derivative** If f is a differentiable function of x and y , then the directional derivative of f in the direction of the unit vector \mathbf{u} is

$$D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.$$

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28. **Properties of the gradient** Let f be differentiable at the point

(x, y) .

1. If $\nabla f(x, y) = \mathbf{0}$, then $D_{\mathbf{u}}f(x, y) = 0$ for all \mathbf{u} .
2. The direction of maximum increase of f is given by $\nabla f(x, y)$. The maximum value of $D_{\mathbf{u}}f(x, y)$ is $\|\nabla f(x, y)\|$.
3. The direction of minimum increase of f is given by $-\nabla f(x, y)$. The minimum value of $D_{\mathbf{u}}f(x, y)$ is $-\|\nabla f(x, y)\|$.

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29. **Gradient is normal to level curves** If f is differentiable at (x_0, y_0) and $\nabla f(x_0, y_0) \neq \mathbf{0}$, then $\nabla f(x_0, y_0)$ is normal to the level curve through (x_0, y_0)156

30. **Directional derivative and gradient for three variables** Let f be a function of x , y , and z , with continuous first partial derivatives.

The **directional derivative** (方向導數) of f in the direction of a unit vector $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is given by

$$D_{\mathbf{u}}f(x, y, z) = af_x(x, y, z) + bf_y(x, y, z) + cf_z(x, y, z).$$

The **gradient** (梯度) of f is defined as

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}.$$

Properties of the gradient are as follows.

1. $D_{\mathbf{u}}f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$
2. If $\nabla f(x, y, z) = \mathbf{0}$, then $D_{\mathbf{u}}f(x, y, z) = 0$ for all \mathbf{u} .
3. The direction of maximum increase of f is given by $\nabla f(x, y, z)$. The maximum value of $D_{\mathbf{u}}f(x, y, z)$ is

$$\|\nabla f(x, y, z)\|.$$

4. The direction of minimum increase of f is given by $-\nabla f(x, y, z)$.

The minimum value of $D_{\mathbf{u}}f(x, y, z)$ is

$$-\|\nabla f(x, y, z)\|.$$

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Section 13.7 Tangent planes and normal lines 166

31. **Tangent plane and normal line** Let F be differentiable at the point $P(x_0, y_0, z_0)$ on the surface S given by $F(x, y, z) = 0$ such that $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$.

1. The plane through P that is normal to $\nabla F(x_0, y_0, z_0)$ is called the **tangent plane** (切平面) to S at P .

2. The line through P having the direction of $\nabla F(x_0, y_0, z_0)$ is called

the **normal line** (法線) to S at P .

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32. **Equation of tangent plane** If F is differentiable at (x_0, y_0, z_0) , then an equation of the tangent plane to the surface given by $F(x, y, z) = 0$ at (x_0, y_0, z_0) is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

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33. **Gradient is normal to level surfaces** If F is differentiable at (x_0, y_0, z_0) and $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, then $\nabla F(x_0, y_0, z_0)$ is normal to the level surface through (x_0, y_0, z_0) 190

Section 13.8 Extrema of functions of two variables 191

34. **Extreme Value Theorem** (極值定理) Let f be a continuous function of two variables x and y defined on a closed bounded region R in the xy -plane.

1. There is at least one point in R at which f takes on a minimum value.
2. There is at least one point in R at which f takes on a maximum value.

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35. **Relative extrema** Let f be a function defined on a region R containing (x_0, y_0) .

1. The function f has a **relative minimum** (相對極小) at (x_0, y_0) if

$$f(x, y) \geq f(x_0, y_0)$$

for all (x, y) in an open disk containing (x_0, y_0) .

2. The function f has a **relative maximum** (相對極大) at (x_0, y_0) if

$$f(x, y) \leq f(x_0, y_0)$$

for all (x, y) in an open disk containing (x_0, y_0) .

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36. **Critical point** Let f be defined on an open region R containing (x_0, y_0) . The point (x_0, y_0) is a **critical point** (臨界點) of f if one of the following is true.

1. $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$
2. $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

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37. **Relative extrema occur only at critical points** If f has a

relative extremum at (x_0, y_0) on an open region R , then (x_0, y_0) is a critical point of f 200

38. **Second Partial Test** (二階偏導數檢定) Let f have continuous second partial derivatives on an open region containing a point (a, b) for which

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0.$$

To test for relative extrema of f , consider the quantity $d = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$.

1. If $d > 0$ and $f_{xx}(a, b) > 0$, then f has a **relative minimum** (相對極小) at (a, b) .
2. If $d > 0$ and $f_{xx}(a, b) < 0$, then f has a **relative maximum** (相對極大) at (a, b) .

3. If $d < 0$, then $(a, b, f(a, b))$ is a **saddle point** (鞍點).

4. The test is inconclusive if $d = 0$.

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Section 13.9 Applications of extrema of functions of two variables 219

39. **Least squares regression line** The **least squares regression line**

(最小平方迴歸線) for $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots, (x_n, y_n)\}$ is given

by $f(x) = ax + b$, where $S_x = \sum_{i=1}^n x_i$, $S_y = \sum_{i=1}^n y_i$, $S_{xx} = \sum_{i=1}^n x_i^2$, $S_{xy} = \sum_{i=1}^n x_i y_i$, and

$$a = \frac{nS_{xy} - S_x S_y}{nS_{xx} - S_x^2} \quad \text{and} \quad b = \frac{S_y - aS_x}{n}.$$

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Section 13.10 Lagrange multipliers 237

40. **Lagrange's Theorem** (拉格朗日定理) Let f and g have continuous first partial derivatives such that f has an extremum at a point (x_0, y_0) on the smooth constraint curve $g(x, y) = c$. If $\nabla g(x_0, y_0) \neq \mathbf{0}$, then there is a real number λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

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41. **Method of Lagrange Multipliers** (拉格朗日乘數法) Let f and g satisfy the hypothesis of Lagrange's Theorem 13.19, and let f have a minimum or maximum subject to the constraint $g(x, y) = c$. To find the minimum or maximum of f , use the following steps.

1. Simultaneously solve the equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) =$

c by solving the following system of equations.

$$f_x(x, y) = \lambda g_x(x, y) \quad f_y(x, y) = \lambda g_y(x, y) \quad g(x, y) = c$$

2. Evaluate f at each solution point obtained in the first step. The largest value yields the maximum of f subject to the constraint $g(x, y) = c$, and the smallest value yields the minimum of f subject to the constraint $g(x, y) = c$.

Alternative: Let $F(x, y, \lambda) = f(x, y) - \lambda (g(x, y) - c)$. Then solve the free-constrained optimization problem for F 248

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