Contents

List of Tables

List of Figures

Chapter 13

Functions of Several Variables

13.1 Summary

1. **A function of two variables** Let D be a set of ordered pairs of real numbers. If to each ordered pair (x, y) in D there corresponds a unique real number $f(x, y)$, then f is called a **function** (函數) of x and y. The set D is the **domain** (定義域) of f, and the corresponding set of values for $f(x, y)$ is the **range** (值域) of f. 5

4. **Limit of a function of two variables** Let f be a function of two variables defined, except possible at (x_0, y_0), on an open disk centered at (x_0, y_0), and let L be a real number. Then

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

if for each ε there corresponds a $\delta > 0$ such that

$$|f(x,y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta.$$

. . . . 42

5. **Continuity of a function of two variables** A function f of two variables is **continuous at a point (x_0, y_0)** (在點 (x_0, y_0) 是連續的) at a point (x_0, y_0) in an open region R if $f(x_0, y_0)$ is equal to the limit of $f(x,y)$ as (x,y) approaches (x_0, y_0). That is,

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0).$$

The function f is **continuous in the open region R** (在開區域 R 是連續 if it is continuous at every point in R. 54

6. **Continuity of a function of two variables** If k is a real number

and f and g are continuous at (x_0, y_0), then the following function are continuous at (x_0, y_0).

1. Scalar multiple: kf **2.** Product: fg

3. Sum and difference: $f \pm g$ **4.** Quotient: f/g, if $g(x_0, y_0) \neq 0$.

55

7. **Continuity of a composite function** If h is continuous at (x_0, y_0) and g is continuous at $h(x_0, y_0)$, then the **composite function** (**合成函數**) given by $(g \circ h)(x, y) = g(h(x, y))$ is **continuous** (**連續**) at (x_0, y_0). That is,

$$\lim_{(x,y)\to(x_0,y_0)} g(h(x, y)) = g(h(x_0, y_0)).$$

. . . . 56

8. **Continuity of a function of three variables** A function f of three variable is **continuous at a point (x_0, y_0, z_0)** (**在一點 (x_0, y_0, z_0) 是連

in an open region R if $f(x_0, y_0, z_0)$ is defined and is equal to the limit of $f(x, y, z)$ as (x, y, z) approaches (x_0, y_0, z_0). That is,

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z) = f(x_0, y_0, z_0).$$

The function f is **continuous in the open region R** (在開區域 R 是連續 if it is continuous at every point in R. 62

Section 13.3 Partial derivatives 63

9. **Partial derivatives of a function of two variables** If $z = f(x, y)$, then the **first partial derivatives** (第一階偏導數) of f with respect to x and y are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{\Delta x\to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \quad \text{and} \quad f_y(x, y) = \lim_{\Delta y\to 0} \frac{f(x, y + \Delta}{}$$

provided the limits exist. 65

10. **Notation for first partial derivatives** For $z = f(x, y)$, the partial derivatives f_x and f_y are denoted by

$$\frac{\partial}{\partial x} f(x, y) = f_x(x, y) = z_x = \frac{\partial z}{\partial x} \quad \text{and} \quad \frac{\partial}{\partial y} f(x, y) = f_y(x, y) = z_y = \frac{\partial z}{\partial y}.$$

The first partials evaluated at the point (a, b) are denoted by

$$\left.\frac{\partial z}{\partial x}\right|_{(a,b)} = f_x(a, b) \quad \text{and} \quad \left.\frac{\partial z}{\partial y}\right|_{(a,b)} = f_y(a, b).$$

. . . . 67

11. The concept of a partial derivative can be extended naturally to functions of three or more variables. If $w = f(x, y, z)$, there are three partial deriv-

atives, each of which is formed by holding two of the variables constant.

$$\frac{\partial w}{\partial x} = f_x(x, y, z) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$$
$$\frac{\partial w}{\partial y} = f_y(x, y, z) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}$$
$$\frac{\partial w}{\partial z} = f_z(x, y, z) = \lim_{\Delta z \to 0} \frac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}$$

. . . . 79

12. **Equality of mixed partial derivatives** (混合偏導數的恆等式)

If f is a function of x and y such that f_{xy} and f_{yx} are continuous on an open disk R, then, for every (x, y) in R,

$$f_{xy}(x, y) = f_{yx}(x, y).$$

. . . . 83

13. **Total differential** If $z = f(x, y)$ and Δx and Δy are increments of x and y, then the **differentials** (**微分**) of the independent variables x and y are

$$\mathrm{d}x = \Delta x \quad \text{and} \quad \mathrm{d}y = \Delta y$$

and the **total differential** (**全微分**) of the dependent variable z is

$$\mathrm{d}z = \frac{\partial z}{\partial x}\,\mathrm{d}x + \frac{\partial z}{\partial y}\,\mathrm{d}y = f_x(x, y)\,\mathrm{d}x + f_y(x, y)\,\mathrm{d}y.$$

14. **Differentiability** A function f given by $z = f(x, y)$ is **differentiable** (**可微的**) at (x_0, y_0) if Δz can be written in the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where both ϵ_1 and $\epsilon_2 \longrightarrow 0$ as $(\Delta x, \Delta y) \longrightarrow (0,0)$. The function f is **differentiable in a region R** (**在區域 R 上可微的**) if it is differentiable at each point in R. 89

15. **Sufficient condition for differentiability** If f is a function of x and y, where f_x and f_y are continuous in an open region R, then f is differentiable on R. 92

16. A function of three variables $w = f(x, y, z)$ is called **differentiable** (**可微的**) at (x, y, z) provided that

$$\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$$

can be written in the form

$$\Delta w = f_x \, \Delta x + f_y \, \Delta y + f_z \, \Delta z + \epsilon_1 \, \Delta x + \epsilon_2 \, \Delta y + \epsilon_3 \, \Delta z$$

then w is a differentiable function of t, and

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial w}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial w}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}.$$

. . . . 108

20. If each x_i is a differentiable function of a single variable t, then for

$$w = f(x_1, x_2, \ldots, x_n)$$

you have

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial w}{\partial x_1}\frac{\mathrm{d}x_1}{\mathrm{d}t} + \frac{\partial w}{\partial x_2}\frac{\mathrm{d}x_2}{\mathrm{d}t} + \cdots + \frac{\partial w}{\partial x_n}\frac{\mathrm{d}x_n}{\mathrm{d}t}.$$

. . . . 112

21. **Chain Rule: two independent variables** (**連鎖律：兩個獨立變數**)

Let $w = f(x, y)$, where f is a differentiable function of x and y. If

$x = g(s,t)$ and $y = h(s,t)$ such that the first partial $\frac{\partial x}{\partial s}$, $\frac{\partial x}{\partial t}$, $\frac{\partial y}{\partial s}$, and $\frac{\partial y}{\partial t}$ all exist, then $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ exist and are given by

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}.$$

. . . . 117

22. If w is a differentiable function of the n variables x_1, x_2, ..., x_n, where each x_i is a differentiable function of the m variables t_1, t_2, ..., t_m, then for $w = f(x_1, x_2, \ldots, x_n)$ you obtain the following.

$$\frac{\partial w}{\partial t_1} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_1} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_1}$$
$$\frac{\partial w}{\partial t_2} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_2} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_2} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_2}$$

$$\vdots$$

$$\frac{\partial w}{\partial t_m} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_m} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_m} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_m}$$

. . . . 121

23. **Chain Rule: implicit differentiation (隱函數微分)** If the equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x(x, y)}{F_y(x, y)}, \quad F_y(x, y) \neq 0.$$

If the equation $F(x, y, z) = 0$ defines z implicitly as a differentiable function of x and y, then

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}, \quad F_z(x, y, z) \neq 0.$$

. . . . 125

26. **Gradient of a function of two variables** Let $z = f(x, y)$ be a function of x and y such that f_x and f_y exist. Then the **gradient** (梯度) of f, denoted by $\nabla f(x, y)$, is the vector

$$\nabla f(x, y) = f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j}.$$

∇f is read as "del f" 142

27. **Alternative form of the directional derivative** If f is a differentiable function of x and y, then the directional derivative of f in the direction of the unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.$$

.... 144

28. **Properties of the gradient** Let f be differentiable at the point

(x, y).

1. If $\nabla f(x, y) = \mathbf{0}$, then $D_{\mathbf{u}} f(x, y) = 0$ for all $\mathbf{u}$.
2. The direction of maximum increase of f is given by $\nabla f(x, y)$. The maximum value of $D_{\mathbf{u}} f(x, y)$ is $\|\nabla f(x, y)\|$.
3. The direction of minimum increase of f is given by $-\nabla f(x, y)$. The minimum value of $D_{\mathbf{u}} f(x, y)$ is $-\|\nabla f(x, y)\|$.

. . . . 149

29. **Gradient is normal to level curves** If f is differentiable at (x_0, y_0) and $\nabla f(x_0, y_0) \neq \mathbf{0}$, then $\nabla f(x_0, y_0)$ is normal to the level curve through (x_0, y_0). 156

30. **Directional derivative and gradient for three variables** Let f be a function of x, y, and z, with continuous first partial derivatives.

The **directional derivative** (**方向導數**) of f in the direction of a unit vector $\mathbf{u} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ is given by

$$D_{\mathbf{u}} f(x, y, z) = a f_x(x, y, z) + b f_y(x, y, z) + c f_z(x, y, z).$$

The **gradient** (**梯度**) of f is defined as

$$\nabla f(x, y, z) = f_x(x, y, z)\,\mathbf{i} + f_y(x, y, z)\,\mathbf{j} + f_z(x, y, z)\,\mathbf{k}.$$

Properties of the gradient are as follows.

1. $D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$
2. If $\nabla f(x, y, z) = \mathbf{0}$, then $D_{\mathbf{u}} f(x, y, z) = 0$ for all $\mathbf{u}$.
3. The direction of maximum increase of f is given by $\nabla f(x, y, z)$. The maximum value of $D_{\mathbf{u}} f(x, y, z)$ is

$$\|\nabla f(x, y, z)\|.$$

4. The direction of minimum increase of f is given by $-\nabla f(x,y,z)$. The minimum value of $D_{\mathbf{u}}f(x,y,z)$ is

$$-\|\nabla f(x,y,z)\|.$$

31. **Tangent plane and normal line** Let F be differentiable at the point $P(x_0,y_0,z_0)$ on the surface S given by $F(x,y,z)=0$ such that $\nabla F(x_0,y_0,z_0)\neq\mathbf{0}$.

1. The plane through P that is normal to $\nabla F(x_0,y_0,z_0)$ is called the **tangent plane** (切平面) to S at P.
2. The line through P having the direction of $\nabla F(x_0,y_0,z_0)$ is called

34. **Extreme Value Theorem** (**極值定理**) Let f be a continuous function of two variables x and y defined on a closed bounded region R in the xy-plane.

1. There is at least one point in R at which f takes on a minimum value.
2. There is at least one point in R at which f takes on a maximum value.

. . . . 193

35. **Relative extrema** Let f be a function defined on a region R containing (x_0, y_0).

1. The function f has a **relative minimum** (**相對極小**) at (x_0, y_0) if

$$f(x, y) \geq f(x_0, y_0)$$

for all (x, y) in an open disk containing (x_0, y_0).

2. The function f has a **relative maximum** (**相對極大**) at (x_0, y_0) if

$$f(x, y) \leq f(x_0, y_0)$$

for all (x, y) in an open disk containing (x_0, y_0).

....195

36. **Critical point** Let f be defined on an open region R containing (x_0, y_0). The point (x_0, y_0) is a **critical point** (**臨界點**) of f if one of the following is true.

1. $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$
2. $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

....198

37. **Relative extrema occur only at critical points** If f has a

38. **Second Partials Test** (**二階偏導數檢定**) Let f have continuous second partial derivatives on an open region containing a point (a, b) for which

$$f_x(a,b) = 0 \quad \text{and} \quad f_y(a,b) = 0.$$

To test for relative extrema of f, consider the quantity $d = f_{xx}(a,b)f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$.

1. If $d > 0$ and $f_{xx}(a,b) > 0$, then f has a **relative minimum** (**相對極小**) at (a, b).
2. If $d > 0$ and $f_{xx}(a,b) < 0$, then f has a **relative maximum** (**相對極大**) at (a, b).

3. If $d < 0$, then $(a, b, f(a, b))$ is a **saddle point** (鞍點).

4. The test is inconclusive if $d = 0$.

Section 13.9 Applications of extrema of functions of two variables

39. **Least squares regression line** The **least squares regression line** (最小平方迴歸線) for $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)\}$ is given by $f(x) = ax + b$, where $S_x = \sum_{i=1}^{n} x_i$, $S_y = \sum_{i=1}^{n} y_i$, $S_{xx} = \sum_{i=1}^{n} x_i^2$, $S_{xy} = \sum_{i=1}^{n} x_i y_i$, and

$$a = \frac{nS_{xy} - S_x S_y}{nS_{xx} - S_x^2} \quad \text{and} \quad b = \frac{S_y - aS_x}{n}.$$

40. **Lagrange's Theorem** (**拉格朗日定理**) Let f and g have continuous first partial derivatives such that f has an extremum at a point (x_0, y_0) on the smooth constraint curve $g(x, y) = c$. If $\nabla g(x_0, y_0) \neq \mathbf{0}$, then there is a real number λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

. . . . 244

41. **Method of Lagrange Multipliers** (**拉格朗日乘數法**) Let f and g satisfy the hypothesis of Lagrange's Theorem 13.19, and let f have a minimum or maximum subject to the constraint $g(x, y) = c$. To find the minimum or maximum of f, use the following steps.

1. Simultaneously solve the equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) =$

c by solving the following system of equations.

$$f_x(x,y) = \lambda g_x(x,y) \qquad f_y(x,y) = \lambda g_y(x,y) \qquad g(x,y) = c$$

2. Evaluate f at each solution point obtained in the first step. The largest value yields the maximum of f subject to the constraint $g(x,y) = c$, and the smallest value yields the minimum of f subject to the constraint $g(x,y) = c$.

Alternative: Let $F(x,y,\lambda) = f(x,y) - \lambda\,(g(x,y) - c)$. Then solve the free-constrained optimization problem for F. 248

INDEX