Chapter 9 Infinite Series

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Sequences

- A sequence is a function whose domain is the set of positive integers. Although it is a function, it is common to represent sequences by subscript notation.
- For instance, in the sequence

$$\begin{array}{cccccccc} 1, & 2, & 3, & 4, & \ldots, & n, & \ldots & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \text{Sequence} \\ a_1, & a_2, & a_3, & a_4, & \ldots, & a_n, & \ldots & \end{array}$$

  1 is mapped onto a_1, 2 is mapped onto a_2, and so on.
- The numbers $a_1, a_2, a_3, \ldots, a_n, \ldots$ are the terms The number a_n is the *n*th term of the sequence, and the entire sequence is denoted by $\{a_n\}$.

Example 1 (Writing the terms of a sequence)

a. The terms of the sequence $\{a_n\} = \{3 + (-1)^n\}$ are

b. The terms of the sequence $\{b_n\} = \left\{\frac{n}{1-2n}\right\}$ are

c. The terms of the sequence $\{c_n\} = \left\{\frac{n^2}{2^n-1}\right\}$ are

d. The terms of the recursively defined sequence $\{d_n\}$, where $d_1 = 25$ and $d_{n+1} = d_n - 5$, are

- Sequences whose terms approach limiting values are said to converge. For instance, the sequence $\{1/2^n\}$

$$\frac{1}{2},\quad \frac{1}{4},\quad \frac{1}{8},\quad \frac{1}{16},\quad \frac{1}{32},\quad \ldots$$

converges to 0, as indicated in the following definition.

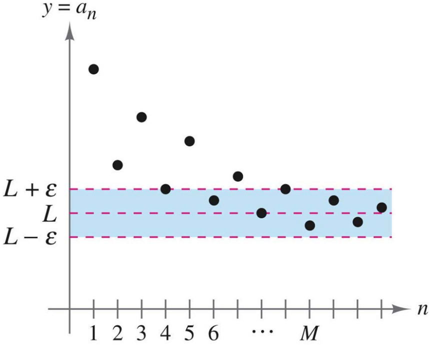
Definition 9.1 (The limit of a sequence)

Let L be a real number. The limit of a sequence $\{a_n\}$ is L, written as

$$\lim_{n\to\infty} a_n = L$$

if for each $\varepsilon > 0$, there exists $M > 0$ such that $|a_n - L| < \varepsilon$ whenever $n > M$. If the limit L of a sequence exists, then the sequence converges to L. If the limit of a sequence does not exist, then the sequence diverges.

- Graphically, this definition says that eventually (for $n > M$ and $\varepsilon > 0$) the terms of a sequence that converges to L will lie within the band between the lines $y = L + \varepsilon$ and $y = L - \varepsilon$ as shown below:



- If a sequence $\{a_n\}$ agrees with a function f at every positive integer, and if $f(x)$ approaches a limit L as $x \to \infty$, the sequence must converge to the same limit L.

Theorem 9.1 (Limit of a sequence)

Let L be a real number. Let f be a function of a real variable such that

$$\lim_{x\to\infty} f(x) = L.$$

If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n, then

$$\lim_{n\to\infty} a_n = L.$$

Example 2 (Finding the limit of a sequence)

Find the limit of the sequence whose nth term is

$$a_n = \left(1 + \frac{1}{n}\right)^n.$$

Theorem 9.2 (Properties of limits of sequences)

Let $\lim_{n\to\infty} a_n = L$ *and* $\lim_{n\to\infty} b_n = K$.

1. *Scalar multiple :* $\lim_{n\to\infty} ca_n = cL$, *c is any real number*

2. *Sum or difference :* $\lim_{n\to\infty} (a_n \pm b_n) = L \pm K$

3. *Product :* $\lim_{n\to\infty}(a_n b_n) = LK$

4. *Quotient :* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{K}$, $b_n \neq 0$ *and* $K \neq 0$

Example 3 (Determining convergence or divergence)

a. $\{a_n\} = \{3 + (-1)^n\}$

b. $\{b_n\} = \{\frac{n}{1-2n}\}$

Example 4 (Using L'Hôpital's Rule to determine convergence)

Show that the sequence whose nth term is $a_n = \frac{n^2}{2^n - 1}$ converges.

- The symbol $n!$ is used to simplify some of the formulas. Let n be a positive integer; then n factorial is defined as $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1) \cdot n$.
- As a special case, zero factorial is defined as $0! = 1$. From this definition, you can see that $1! = 1$, $2! = 1 \cdot 2 = 2$, $3! = 1 \cdot 2 \cdot 3 = 6$, and so on.
- Factorial follows the same conventions for order of operations as exponents. That is, $2n! = 2(n!)$ is different from $(2n)!$

Commonly used ordering If $a > 0$ and $b > 1$, then

$$\ln n \prec n^a \prec b^n \prec n!$$

where $a_n \prec b_n$ denotes that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

Theorem 9.3 (Squeeze Theorem for sequences)

If

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$$

and there exists an integer N such that $a_n \leq c_n \leq b_n$ for all $n > N$, then

$$\lim_{n\to\infty} c_n = L.$$

Example 5 (Using the Squeeze Theorem)

Show that the sequence $\{c_n\} = \left\{(-1)^n \frac{1}{n!}\right\}$ converges, and find its limit.

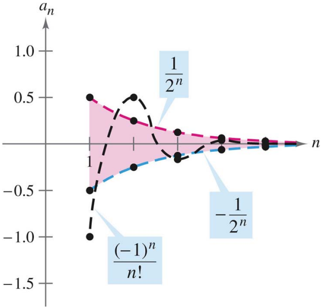


Figure 1: For $n \geq 4$, $(-1)^n/n!$ is squeezed between $-1/2^n$ and $1/2^n$.

Theorem 9.4 (Absolute Value Theorem)

For the sequence $\{a_n\}$, *if*

$$\lim_{n\to\infty} |a_n| = 0 \qquad \textit{then} \qquad \lim_{n\to\infty} a_n = 0.$$

Pattern recognition for sequences

- Sometimes the terms of a sequence are generated by some rule that does not explicitly identify the nth term of the sequence.
- In such cases, you may be required to discover a *pattern* in the sequence and to describe the nth term.
- Once the nth term has been specified, you can investigate the convergence or divergence of the sequence.

Example 6 (Finding the *n*th term of a sequence)

Find a sequence $\{a_n\}$ whose first five terms are

$$\frac{1}{2},\ \frac{3}{4},\ \frac{5}{8},\ \frac{7}{16},\ \frac{9}{32},\ \ldots$$

and then determine whether the sequence converges or diverges.

Example 7 (Finding the *n*th term of a sequence)

Determine an *n*th term for a sequence whose first five terms are

$$-\frac{2}{1},\ \frac{8}{2},\ -\frac{26}{6},\ \frac{80}{24},\ -\frac{242}{120},\ \ldots$$

and then decide whether the sequence converges or diverges.

- The process of determining an nth term from the pattern observed in the first several terms of a sequence is an example of inductive reasoning.

Definition 9.2 (Monotonic sequence)

A sequence $\{a_n\}$ is monotonic if its terms are nondecreasing

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots$$

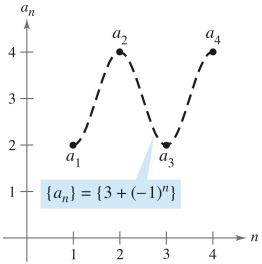
or if its terms are nonincreasing

$$a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots .$$

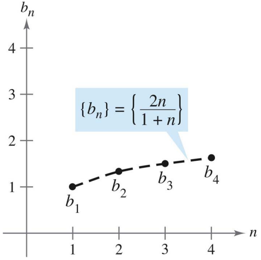
Example 8 (Determining whether a sequence is monotonic)

Determine whether each sequence having the given nth term is monotonic.

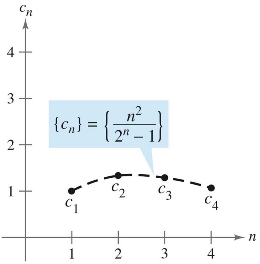
a. $a_n = 3 + (-1)^n$ **b.** $b_n = \frac{2n}{1+n}$ **c.** $\frac{n^2}{2^n-1}$



(a) Not monotonic.



(b) Monotonic.



(c) Not monotonic.

Figure 2: Graphically illustrates three sequences.

Definition 9.3 (Bounded sequence)

1. A sequence $\{a_n\}$ is bounded above if there is a real number M such that $a_n \leq M$ for all n. The number M is called an upper bound of the sequence.
2. A sequence $\{a_n\}$ is bounded below if there is a real number N such that $N \leq a_n$ for all n. The number N is called a lower bound of the sequence.
3. A sequence $\{a_n\}$ is bounded if it is bounded above and bounded below.

- One important property of the real numbers is that they are complete. This means that there are no holes or gaps on the real number line. (The set of rational numbers do not have the completeness property.)
- The completeness axiom for real numbers can be used to conclude that if a sequence has an upper bound, it must have a least upper bound (an upper bound that is smaller than all other upper bounds for the sequence).
- For example, the least upper bound of the sequence $\{a_n\} = \{n/(n+1)\}$,

$$\frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \frac{4}{5},\ \ldots,\ \frac{n}{n+1},\ \ldots$$

is 1.

Theorem 9.5 (Bounded monotonic sequences)

If a sequence $\{a_n\}$ is bounded and monotonic, then it converges.

Example 9 (Bounded and monotonic sequences)

Determine whether or not the following sequences are bounded or convergent.

a. $\{a_n\} = \left\{\frac{1}{n}\right\}$ **b.** $\{b_n\} = \left\{\frac{n^2}{(n+1)}\right\}$ **c.** $\{c_n\} = \{(-1)^n\}$

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Infinite series

- Infinite sequences can be used to represent infinite summations.
- Informally, if $\{a_n\}$ is an infinite sequence, then

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots \qquad \text{Infinite series}$$

  is an infinite series (or simply a series).
- The numbers a_1, a_2, a_3, are the terms of the series.
- For some series, it is convenient to begin the index at $n = 0$ (or some other integer) and it is common to represent an infinite series as simply $\sum a_n$.

- To find the sum of an infinite series, consider the following

$$S_1 = a_1 \qquad S_2 = a_1 + a_2 \qquad S_3 = a_1 + a_2 + a_3$$
$$S_4 = a_1 + a_2 + a_3 + a_4 \qquad S_5 = a_1 + a_2 + a_3 + a_4 + a_5 \qquad \cdots$$
$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

- If this <u>sequence of partial sums</u> converges, the series is said to converge.

Definition 9.4 (Convergent and divergent series)

For the infinite series $\sum_{n=1}^{\infty} a_n$ the nth partial sum is given by

$$S_n = a_1 + a_2 + \cdots + a_n.$$

If the sequence of partial sums $\{S_n\}$ converges to S, then the series $\sum_{n=1}^{\infty} a_n$ converges. The limit S is called the sum of the series.

$$S = a_1 + a_2 + \cdots + a_n + \cdots \qquad S = \sum_{n=1}^{\infty} a_n$$

If $\{S_n\}$ diverges, then the series diverges.

Example 1 (Convergent and divergent series)

a.

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

b.

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots$$

c.

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \cdots$$

Example 2 (Writing a series in telescoping form)

Find the sum of the series $\sum_{n=1}^{\infty} \frac{2}{4n^2-1}$.

Geometric series

- The series $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ is a <u>geometric series</u>.
- In general, the series is given by

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^n + \cdots , \qquad a \neq 0$$

is a geometric series with ratio r.

Theorem 9.6 (Convergence of a geometric series)

A geometric series with ratio r diverges if $|r| \geq 1$. If $0 < |r| < 1$, then the series converges to the sum

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad 0 < |r| < 1.$$

Example 3 (Convergent and divergent geometric series)

a.

$$\sum_{n=0}^{\infty} \frac{3}{2^n} = \sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n = 3(1) + 3\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \cdots$$

b.

$$\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n = 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \cdots$$

Example 4 (A geometric series for a repeating decimal)

Use a geometric series to write $0.\overline{08}$ as the ratio of two integers.

Theorem 9.7 (Properties of infinite series)

Let $\sum a_n$ and $\sum b_n$ be convergent series, and let A, B, and c be real numbers. If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, then the following series converge to the indicated sums.

1. $\sum_{n=1}^{\infty} ca_n = cA$
2. $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$
3. $\sum_{n=1}^{\infty} (a_n - b_n) = A - B$

Theorem 9.8 (Limit of the *n*th term of a convergent series)

If $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n \to \infty} a_n = 0$.

- The contrapositive of Theorem 9.8 provides a useful test for divergence.
- This nth-Term Test for Divergence states that if the limit of the nth term of a series does not converge to 0, the series must diverge.

Theorem 9.9 (nth-term test for divergent)

If $\lim_{n\to\infty} a_n \neq 0$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Example 5 (Using the *n*th-term test for divergent)

a. For the series $\sum_{n=0}^{\infty} 2^n$

b. For the series $\sum_{n=1}^{\infty} \frac{n!}{2n!+1}$

c. For the series $\sum_{n=1}^{\infty} \frac{1}{n}$

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The Integral Test

Theorem 9.10 (The Integral Test)

If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \quad \textit{and} \quad \int_1^{\infty} f(x)\,\mathrm{d}x$$

either both converge or both diverge.

Example 1 (Using the Integral Test)

Apply the Integral Test to the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$.

Example 2 (Using the Integral Test)

Apply the Integral Test to the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

p-series and harmonic series

- A second type of series has a simple arithmetic test for convergence or divergence has the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

which is a p-series, where p is a positive constant.

- For $p = 1$, the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

is the harmonic series.

- A general harmonic series is of the form $\sum \frac{1}{(an+b)}$.

Theorem 9.11 (Convergence of p series)

The p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

1. *converges if* $p > 1$, *and* **2.** *diverges if* $0 < p \leq 1$.

Example 3 (Convergent and divergent p series)

Discuss the convergence or divergence of
a. the harmonic series and **b.** the p-series with $p = 2$.

Example 4 (Testing a series for convergence)

Determine whether the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

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Direct comparison test

- For the convergence tests the terms of the series have to be fairly simple and the series must have special characteristics in order for the convergence tests to be applied.
- A slight deviation from these special characteristics can make a test nonapplicable!
- For example, in the following pairs, the second series cannot be tested by the same convergence test as the first series even though it is similar to the first!
  1. $\sum_{n=0}^{\infty} \frac{1}{2^n}$ is geometric, but $\sum_{n=0}^{\infty} \frac{n}{2^n}$ is not.
  2. $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a p-series, but $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ is not.
  3. $a_n = \frac{n}{(n^2+3)^2}$ is easily integrated, but $b_n = \frac{n^2}{(n^2+3)^2}$.

Theorem 9.12 (Direct Comparison Test)

Let $0 < a_n \leq b_n$ *for all* n*.*

1. *If* $\sum_{n=1}^{\infty} b_n$ *converges, then* $\sum_{n=1}^{\infty} a_n$ *converges.*

2. *If* $\sum_{n=1}^{\infty} a_n$ *diverges, then* $\sum_{n=1}^{\infty} b_n$ *diverges.*

Example 1 (Using the Direct Comparison Test)

Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{1}{2+3^n}$.

Example 2 (Using the Direct Comparison Test)

Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$.

- Remember that both parts of the Direct Comparison Test require that $0 < a_n \leq b_n$. Informally, the test says the following about the two series with nonnegative terms.
    1. If the "larger" series converges, the "smaller" series must also converge.
    2. If the "smaller" series diverges, the "larger" series must also diverge.

Limit comparison test

- Often a given series closely resembles a p-series or a geometric series, yet you cannot establish the term-by-term comparison necessary to apply the Direct Comparison Test. Under these circumstances you may be able to apply a second comparison test, called the Limit Comparison Test.

Theorem 9.13 (Limit Comparison Test)

Suppose that $a_n > 0$, $b_n > 0$, and

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L$$

where L is finite and positive. Then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Example 3 (Using the Limit Comparison Test)

Show that the following general harmonic series diverges.

$$\sum_{n=1}^{\infty} \frac{1}{an+b}, \quad a > 0,\ b > 0$$

- The Limit Comparison Test works well for comparing a "messy" algebraic series with a p-series.

Given Series	Comparison Series	Conclusion
$\sum_{n=1}^{\infty} \frac{1}{3n^2-4n+5}$	$\sum_{n=1}^{\infty} \frac{1}{n^2}$	Both series converge.
$\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n-2}}$	$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$	Both series diverge.
$\sum_{n=1}^{\infty} \frac{n^2-10}{4n^5+n^3}$	$\sum_{n=1}^{\infty} \frac{n^2}{n^5} = \sum_{n=1}^{\infty} \frac{1}{n^3}$	Both series converge.

- When choosing a series for comparison, you can disregard all but the highest powers of n in both the numerator and the denominator.

Example 4 (Using the Limit Comparison Test)

Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$.

Example 5 (Using the Limit Comparison Test)

Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{n2^n}{4n^3+1}$.

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Alternating series

- The simplest series that contains both positive and negative terms is an <u>alternating series</u>, whose terms alternate in sign.
- For example, the geometric series

$$\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n=\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n}=1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\cdots$$

  is an alternating geometric series with $r=-1/2$.
- Alternating series occur in two ways: either the odd terms are negative or the even terms are negative.

Theorem 9.14 (Alternating Series Test)

Let $a_n > 0$. *The alternating series*

$$\sum_{n=1}^{\infty}(-1)^n a_n \quad \textit{and} \quad \sum_{n=1}^{\infty}(-1)^{n+1} a_n$$

converge if the following two conditions are met.

1. $\lim_{n\to\infty} a_n = 0$ **2.** $a_{n+1} \le a_n$, *for all* n

Remark

The second condition in the Alternating Series Test can be modified to require only that $0 < a_{n+1} \le a_n$ for all n greater than some **integer** N.

Example 1 (Using the Alternating Comparison Test)

Determine the convergence or divergence of $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$.

Example 2 (Using the Alternating Series Test)

Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{n}{(-2)^{n-1}}$.

Example 3 (When the Alternating Series Test does not apply)

a.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{n} = \frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \frac{6}{5} - \cdots$$

b.

$$\frac{2}{1} - \frac{1}{1} + \frac{2}{2} - \frac{1}{2} + \frac{2}{3} - \frac{1}{3} + \frac{2}{4} - \frac{1}{4} + \cdots$$

- To conclude that the second series diverges, you can argue that S_{2N} equals the Nth partial sum of the divergent harmonic series.
- This implies that the sequence of partial sums diverges. So, the series diverges.

Alternating series remainder

- For a convergent alternating series, the partial sum S_N can be a useful approximation for the sum S of the series. The error involved in using $S \approx S_N$ is the remainder $R_N = S - S_N$.

Theorem 9.15 (Alternating Series Remainder)

If a convergent alternating series satisfies the condition $a_{n+1} \leq a_n$, then the absolute value of the remainder R_N involved in approximating the sum S by S_N is less than (or equal to) the first neglected term. That is,

$$|S - S_N| = |R_N| \leq a_{N+1}.$$

Example 4 (Approximating the sum of an alternating series)

Approximate the sum of the following series by its first six terms.

$$1 - e^{-1} = \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{n!} \right) = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + \cdots$$

Absolute and conditional convergence

- Occasionally, a series may have both positive and negative terms and not be an alternating series. For instance, the series

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \cdots$$

  is not an alternating series.
- One way to obtain some information about the convergence of this series is to investigate the convergence of the series

$$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right|.$$

- By direct comparison, you have $|\sin n| \leq 1$ for all n, so

$$\left| \frac{\sin n}{n^2} \right| \leq \frac{1}{n^2}, \quad n \geq 1.$$

- Therefore, by the Direct Comparison Test, the series $\sum \left|\frac{\sin n}{n^2}\right|$ converges.

Theorem 9.16 (Absolute convergence)

If the series $\sum |a_n|$ converges, then the series $\sum a_n$ also converges.

- The converse of Theorem 9.16 is not true. For instance, the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges by the Alternating Series Test. Yet the harmonic series diverges. This type of convergence is called <u>conditional</u>.

Definition 9.5 (Absolute and conditional convergence)

1. $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges.
2. $\sum a_n$ is conditionally convergent if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Example 6 (Absolute and conditional convergence)

Determine whether each of the series is convergent or divergent. Classify any convergent series as absolutely or conditionally convergent.

a. $\sum_{n=0}^{\infty} \frac{(-1)^n n!}{2^n} = \frac{0!}{2^0} - \frac{1!}{2^1} + \frac{2!}{2^2} - \frac{3!}{2^3} + \cdots$

b. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} = -\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \cdots$

Example 7 (Absolute and conditional convergence)

Determine whether each of the series is convergent or divergent. Classify any convergent series as absolutely or conditionally convergent.

a. $\sum_{n=1}^{\infty} \frac{(-1)^{n(n+1)/2}}{3^n} = -\frac{1}{3} - \frac{1}{9} + \frac{1}{27} + \frac{1}{81} - \cdots$

b. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)} = -\frac{1}{\ln 2} + \frac{1}{\ln 3} - \frac{1}{\ln 4} + \frac{1}{\ln 5} - \cdots$

Rearrangement of series

- A finite sum such as $(1 + 3 - 2 + 5 - 4)$ can be rearranged without changing the value of the sum.
- This is not necessarily true of an infinite series — it depends on whether the series is absolutely convergent (every rearrangement has the same sum) or conditionally convergent.

Example 8 (Rearrangement of a series)

The alternating harmonic series converges to $\ln 2$. That is,

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}=\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=\ln 2.$$

Rearrange the series to produce a different sum.

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The Ratio Test

- This section begins with a test for absolute convergence — the Ratio Test.

Theorem 9.17 (Ratio Test)

Let $\sum a_n$ be a series with nonzero terms.

1. $\sum a_n$ *converges absolutely if* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$.
2. $\sum a_n$ *diverges if* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ *or* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$.
3. *The Ratio Test is inconclusive if* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$.

Remark

Although the Ratio Test is not a cure for all ills related to testing for convergence, it is particularly useful for series that converge rapidly. Series involving factorials or exponentials are frequently of this type.

Example 1 (Using the Ratio Test)

Determine the convergence or divergence of $\sum_{n=0}^{\infty} \frac{2^n}{n!}$.

Example 2 (Using the Ratio Test)

Determine whether each series converges or diverges.

a. $\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$ **b.** $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Example 3 (A failure of the Ratio Test)

Determine the convergence or divergence of $\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{n+1}$.

- The next test for convergence or divergence of series works especially well for series involving *n*th powers.

Theorem 9.18 (Root Test)

Let $\sum a_n$ *be a series.*

1. $\sum a_n$ *converges absolutely if* $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$.
2. $\sum a_n$ *diverges if* $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ *or* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$.
3. *The Root Test is inconclusive if* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$.

Remark

The Root Test is always inconclusive for any p-series.

Example 4 (Using the Root Test)

Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$.

Strategies for testing series

- Does the nth term approach 0? If not, the series diverges.
- Is the series one of the special types — geometric, p-series, telescoping, or alternating?
- Can the Integral Test, the Root Test, or the Ratio Test be applied?
- Can the series be compared favorably to one of the special types?

Example 5 (Applying the strategies for testing series)

Determine the convergence or divergence of

a. $\sum_{n=1}^{\infty} \frac{n+1}{3n+1}$ **b.** $\sum_{n=1}^{\infty} \left(\frac{\pi}{6}\right)^n$ **c.** $\sum_{n=1}^{\infty} ne^{-n^2}$

d. $\sum_{n=1}^{\infty} \frac{1}{3n+1}$ **e.** $\sum_{n=1}^{\infty} (-1)^n \frac{3}{4n+1}$ **f.** $\sum_{n=1}^{\infty} \frac{n!}{10^n}$

g. $\sum_{n=1}^{\infty} \left(\frac{n+1}{2n+1}\right)^n$

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Polynomial approximations of elementary functions

- To find a polynomial function P that approximates another function f, begin by choosing a number c in the domain of f at which f and P have the same value. That is,

$$P(c) = f(c). \qquad \text{Graphs of } f \text{ and } P \text{ pass through } (c, f(c))$$

- The approximating polynomial is said to be expanded about c or centered at c.
- Geometrically, the requirement that $P(c) = f(c)$ means that the graph of P passes through the point $(c, f(c))$. Of course, there are many polynomials whose graphs pass through the point $(c, f(c))$.

- To find a polynomial whose graph resembles the graph of f near this point. One way to do this is to impose the additional requirement that the slope of the polynomial function be the same as the slope of the graph of f at the point $(c, f(c))$.

$$P'(c) = f'(c). \qquad \text{Graphs of } f \text{ and } P \text{ have the same at } (c, f(c))$$

- With these two requirements, you can obtain a simple linear approximation of f, as shown in Figure 3.

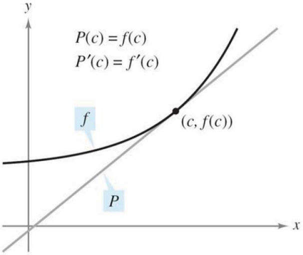


Figure 3: Near $(c, f(c))$, the graph of P can be used to approximate the graph of f.

Example 1 (First-degree polynomial approximation of $f(x) = e^x$)

For the function $f(x) = e^x$, find a first-degree polynomial function

$$P_1(x) = a_0 + a_1 x$$

whose value and slope agree with the value and slope of f at $x = 0$.

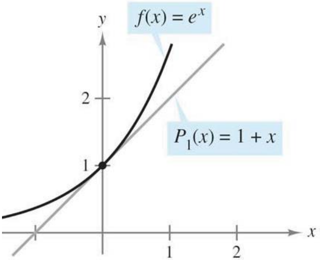


Figure 4: $P_1(x) = 1 + x$ is the first-degree polynomial approximation of $f(x) = e^x$.

Example 2 (Third-degree polynomial approximation of $f(x) = e^x$)

Construct a table comparing the values of the polynomial

$$P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3$$

with $f(x) = e^x$ for several values of x near 0.

x	-1.0	-0.2	-0.1	0	0.1	0.2	1.0
e^x	0.3679	0.81873	0.904837	1	1.105171	1.22140	2.7183
$P_3(x)$	0.3333	0.81867	0.904833	1	1.105167	1.22133	2.6667

Taylor and Maclaurin polynomials

- The polynomial approximation of $f(x) = e^x$ is expanded about $c = 0$. For expansions about an arbitrary value of c, it is convenient to write the polynomial in the form

$$P_n(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots + a_n(x-c)^n.$$

- In this form, repeated differentiation produces

$$\begin{aligned}
P_n'(x) &= a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots + na_n(x-c)^{n-1} \\
P_n''(x) &= 2a_2 + 2(3a_3)(x-c) + \cdots + n(n-1)a_n(x-c)^{n-2} \\
P_n'''(x) &= 2(3a_3) + \cdots + n(n-1)(n-2)a_n(x-c)^{n-3} \\
&\vdots \\
P_n^{(n)}(x) &= n(n-1)(n-2)\cdots(2)(1)a_n.
\end{aligned}$$

- Letting $x = c$, you then obtain

$$P_n(c) = a_0, \quad P_n'(c) = a_1, \quad P_n''(c) = 2a_2, \quad \ldots, \quad P_n^{(n)}(c) = n!a_n.$$

- Because the values of f and its first n derivatives must agree with the values of P_n and its first n derivatives at $x = c$, it follows that

$$f(c) = a_0, \quad f'(c) = a_1, \quad \frac{f''(c)}{2!} = a_2, \quad \ldots, \quad \frac{f^{(n)}(c)}{n!} = a_n.$$

Definition 9.6 (Taylor polynomial and Maclaurin polynomial)

If f has n derivatives at c, then the polynomial

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the <u>nth Taylor polynomial for f at c</u>. If $c = 0$, then

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

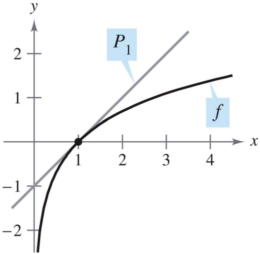
is also called the <u>nth Maclaurin polynomial for f at c</u>.

Example 3 (A Maclaurin polynomial for $f(x) = e^x$)

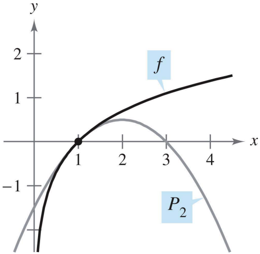
Find the nth Maclaurin polynomial for $f(x) = e^x$.

Example 4 (Finding Taylor polynomials for $\ln x$)

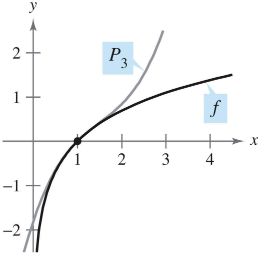
Find the Taylor polynomials P_0, P_1, P_2, P_3, and P_4, for $\ln x$ centered at $c = 1$.



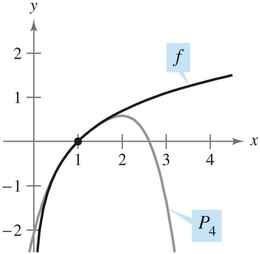
(a) $n = 1$



(b) 2



(c) $n = 3$



(d) $n = 4$

Figure 5: As n increases, the graph of P_n, becomes a better and better approximation of the graph of $f(x) = \ln x$ near $x = 1$.

Example 5 (Finding Maclaurin polynomials for $\cos x$)

Find the Maclaurin polynomials P_0, P_2, P_4, and P_6 for $f(x) = \cos x$. Use $P_6(x)$ to approximate the value of $\cos(0.1)$.

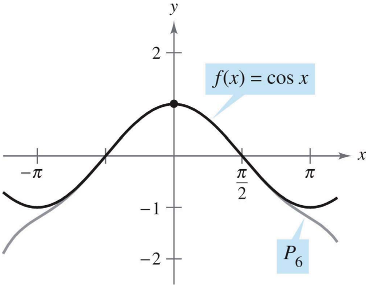


Figure 6: Near $(0, 1)$, the graph of P_6 can be used to approximate the graph of $f(x) = \cos x$.

Example 6 (Finding a Taylor polynomial for $\sin x$)

Find the third Taylor polynomial for $f(x) = \sin x$, expanded about $c = \pi/6$.

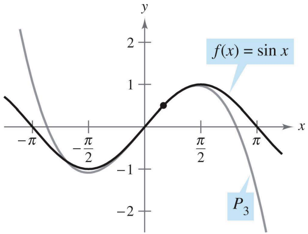


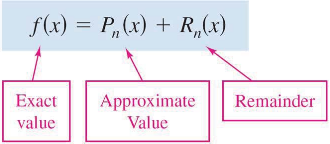
Figure 7: Near $(\pi/6, 1/2)$, the graph of P_3 can be used to approximate the graph of $f(x) = \sin x$.

Example 7 (Approximation using Maclaurin polynomials)

Use a fourth Maclaurin polynomial to approximate the value of $\ln(1.1)$.

Remainder of a Taylor polynomial

- An approximation technique is of little value without some idea of its accuracy. To measure the accuracy of approximating a function value $f(x)$ by the Taylor polynomial $P_n(x)$, you can use the concept of a remainder $R_n(x)$, defined as follows.



- So, $R_n(x) = f(x) - P_n(x)$. The absolute value of $R_n(x)$ is called the error associated with the approximation. That is,

$$\text{Error} = |R_n(x)| = |f(x) - P_n(x)|$$

- The next theorem gives a general procedure for estimating the remainder associated with a Taylor polynomial.

- This important theorem is called Taylor's Theorem, and the remainder given in the theorem is called the Lagrange form of the remainder.

Theorem 9.19 (Taylor's Theorem)

If a function f is differentiable through order $n+1$ in an interval I containing c, then, for each x in I, there exists z between x and c such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}.$$

Example 8 (Determining the accuracy of an approximation)

The third Maclaurin polynomial for $\sin x$ is given by

$$P_3(x) = x - \frac{x^3}{3!}.$$

Use Taylor's Theorem to approximate $\sin(0.1)$ by $P_3(0.1)$ and determine the accuracy of the approximation.

- Using Taylor's Theorem, you have

$$\sin x = x - \frac{x^3}{3!} + R_3(x) = x - \frac{x^3}{3!} + \frac{f^{(4)}(z)}{4!}\, x^4$$

  where $0 < z < 0.1$.

- Therefore,

$$\sin(0.1) \approx 0.1 - \frac{(0.1)^3}{3!} \approx 0.1 - 0.000167 = 0.099833.$$

- Because $f^{(4)}(z) = \sin z$, it follows that the error $|R_3(0.1)|$ can be bounded as follows.

$$0 < R_3(0.1) = \frac{\sin z}{4!}(0.1)^4 < \frac{0.0001}{4!} \approx 0.000004.$$

- This implies that

$$0.099833 < \sin(0.1) = 0.099833 + R_3(x) < 0.099833 + 0.000004$$
$$0.099833 < \sin(0.1) < 0.099837. \quad \blacksquare$$

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Power series

- An important function $f(x) = e^x$ can be represented exactly by an infinite series called a power series. For example, the power series representation for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots .$$

- For each real number x, it can be shown that the infinite series on the right converges to the number e^x.

Definition 9.7 (Power series)

If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

is called a power series. More generally, an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1 (x-c) + a_2 (x-c)^2 + a_3 (x-c)^3 + \cdots + a_n (x-c)^n + \cdots$$

is called a power series centered at c, where c is a constant.

Example 1 (Power series)

a. The following power series is centered at 0.

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

b. The following power series is centered at -1.

$$\sum_{n=0}^{\infty} (-1)^n (x+1)^n = 1 - (x+1) + (x+1)^2 - (x+1)^3 + \cdots$$

c. The following power series is centered at 1.

$$\sum_{n=1}^{\infty} \frac{1}{n}(x-1)^n = (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \cdots$$

■

Radius and interval of convergence

- A power series in x can be viewed as a function of x

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$

where the domain of f is the set of all x for which it converges.

Theorem 9.20 (Convergence of a power series)

For a power series centered at c, precisely one of the following is true.

1. *The series converges only at c.*
2. *There exists a real number $R > 0$ such that the series converges absolutely for $|x - c| < R$, and diverges for $|x - c| > R$.*
3. *The series converges absolutely for all x.*

The number R is the radius of convergence. If the series converges only at c, the radius of convergence is $R = 0$, and if the series converges for all x, the radius of convergence is $R = \infty$. The set of all values of x for which it converges is the interval of convergence of the power series.

Example 2 (Finding the radius of convergence)

Find the radius of convergence of $\sum_{n=0}^{\infty} n!x^n$.

Example 3 (Finding the radius of convergence)

Find the radius of convergence of $\sum_{n=0}^{\infty} 3(x-2)^n$.

Example 4 (Finding the radius of convergence)

Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$.

Differentiation and integration of power series

Theorem 9.21 (Properties of functions defined by power series)

If the function is given by
$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$
has a radius of convergence of $R > 0$*, then, on the interval* $(c-R,\ c+R)$*,* f *is differentiable (and therefore continuous). Moreover, the derivative and antiderivative of* f *are as follows.*

1. $f'(x) = \sum_{n=1}^{\infty} na_n(x-c)^{n-1} = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots$

2.
$\int f(x)\,\mathrm{d}x = C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1} = C + a_0(x-c) + a_1\frac{(x-c)^2}{2} + a_2\frac{(x-c)^3}{3} + \cdots$

Theorem 9.21

- *The radius of convergence of the series obtained by differentiating or integrating a power series is the same as that of the original power series. The interval of convergence, however, may differ as a result of the behavior at the endpoints.*
- *The interval of convergence of the series obtained by differentiating a power series may get worse but cannot get improved. However, the interval of convergence of the series obtained by integrating a power series may get improve but cannot get worse.*

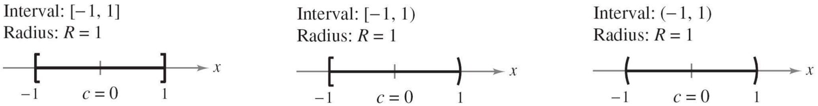
Example 8 (Intervals of convergence for $f(x)$, $f'(x)$, and $\int f(x)\,\mathrm{d}x$)

Consider the function given by

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots .$$

Find the interval of convergence for each of the following.

a. $\int f(x)\,\mathrm{d}x$ **b.** $f(x)$ **c.** $f'(x)$



(a) Interval: $[-1, 1]$ and radius: $R = 1$.

(b) Interval: $[-1, 1)$ and radius: $R = 1$.

(c) Interval: $(-1, 1)$ and radius: $R = 1$.

Figure 8: Intervals of convergence for $f(x)$, $f'(x)$, and $\int f(x)\,\mathrm{d}x$.

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Geometric power series

- Consider the function given by $f(x) = 1/(1-x)$. The form of f closely resembles the sum of a geometric series

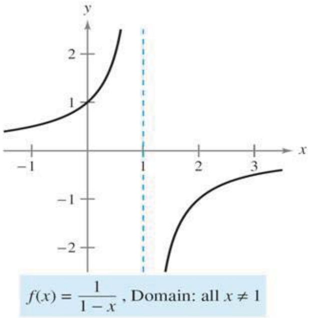
$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad |r| < 1.$$

- In other words, if you let $a = 1$ and $r = x$, a power series representation for $1/(1-x)$, centered at 0, is

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots, \quad |x| < 1.$$

- Of course, this series represents $f(x) = 1/(1-x)$ only on the interval $(-1, 1)$, whereas f is defined for all $x \neq 1$, To represent f in another interval, you must develop a different series.

- For instance, to obtain the power series centered at -1, you could write
$$\frac{1}{1-x} = \frac{1}{2-(x+1)} = \frac{1/2}{1-[(x+1)/2]} = \frac{a}{1-r}$$
which implies that $a = 1/2$ and $r = (x+1)/2$.
- So, for $|x+1| < 2$, you have
$$\begin{aligned}\frac{1}{1-x} &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)\left(\frac{x+1}{2}\right)^n \\ &= \frac{1}{2}\left[1 + \frac{(x+1)}{2} + \frac{(x+1)^2}{4} + \frac{(x+1)^3}{8} + \cdots\right], \quad |x+1| < 2\end{aligned}$$
which converges on the interval $(-3, 1)$.



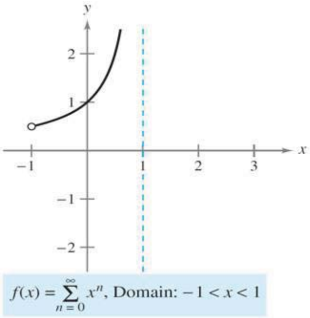


Figure 9: Definition of different ranges with function.

Example 1 (Finding a geometric power series centered at 0)

Find a power series for $f(x) = 4/(x+2)$ centered at 0.

Example 2 (Finding a geometric power series centered at 1)

Find a power series for $f(x) = 1/x$, centered at 1.

Operations with power series

Operations with power series Let $f(x) = \sum a_n x^n$ and $g(x) = \sum b_n x^n$.

1. $f(kx) = \sum_{n=0}^{\infty} a_n k^n x^n$

2. $f(x^N) = \sum_{n=0}^{\infty} a_n x^{nN}$

3. $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$

Example 3 (Adding two power series)

Find a power series, centered at 0, for $f(x) = (3x-1)/(x^2-1)$.

Example 4 (Finding a power series by integration)

Find a power series for $f(x) = \ln x$, centered at 1.

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Taylor series and Maclaurin series

- The development of power series to represent functions is credited to the combined work of many seventeenth and eighteenth-century mathematicians.
- However, the two names that are most commonly associated with power series are Brook Taylor and Colin Maclaurin.

Theorem 9.22 (The form of a convergent power series)

If f is represented by a power series $f(x) = \sum a_n(x-c)^n$ for all x in an open interval I containing c, then $a_n = f^{(n)}(c)/n!$ and

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots .$$

- The coefficients of the power series in Theorem 9.22 are precisely the coefficients of the Taylor polynomials for $f(x)$ at c. For this reason, the series is called the Taylor series for $f(x)$ at c.

Definition 9.8 (Taylor and Maclaurin series)

If a function f has derivatives of all orders at $x = c$, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

is called the <u>Taylor series for $f(x)$ at c</u>. Moreover, if $c = 0$, then the series is the <u>Maclaurin series for f</u>.

Example 1 (Forming a power series)

Use the function $f(x) = \sin x$ to form the Maclaurin series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!} x + \frac{f^{(3)}(0)}{3!} x^3 + \frac{f^{(4)}(0)}{4!} x^4 + \cdots$$

and determine the interval of convergence.

- You cannot conclude that the power series converges to $\sin x$ for all x. You can simply conclude that the power series converges to some function, but you are not sure what function it is. This is a subtle, but important, point in dealing with Taylor or Maclaurin series.
- To persuade yourself that the series

$$f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

might converge to a function other than f, remember that the derivatives are being evaluated at a single point.
- It can easily happen that another function will agree with the values of $f^{(n)}(x)$ when $x = c$ and disagree at other x-values.

- If you formed the power series for the function shown in Figure 10, you would obtain the same series as in Example 1. You know that the series converges for all x, and yet it obviously cannot converge to both $f(x)$ and $\sin x$ for all x.

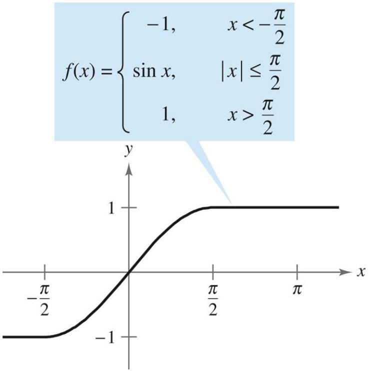


Figure 10: $f(x) \neq \sin x$ for all x but both have the same Taylor series.

- Let f have derivatives of all orders in an open interval I centered at c.
- The Taylor series for f may fail to converge for some x in I. Or, even if it is convergent, it may fail to have $f(x)$ as its sum.
- Nevertheless, Theorem 9.19 tells us that for each n,

$$f(x) = f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n+R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}.$$

Theorem 9.23 (Convergence of Taylor series)

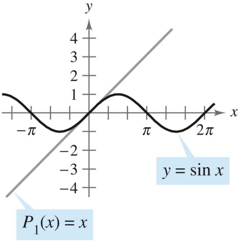
If $\lim_{n\to\infty} R_n = 0$ *for all* x *in the interval* I*, then the Taylor series for* f *converges and equals* $f(x)$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

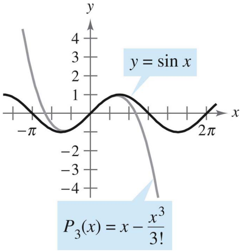
Example 2 (A convergent Maclaurin series)

Show that the Maclaurin series for $f(x) = \sin x$ converges to $\sin x$ for all x.

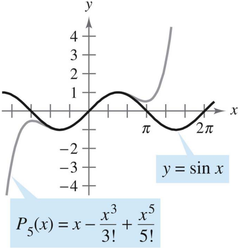
- Figure 11 visually illustrates the convergence of the Maclaurin series for $\sin x$ by comparing the graphs of the Maclaurin polynomials $P_1(x)$, $P_3(x)$, $P_5(x)$, and $P_7(x)$ with the graph of the sine function.



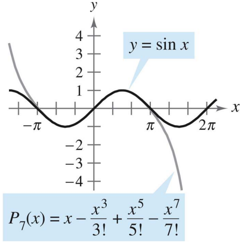
(a) $n = 1$.



(b) $n = 3$.



(c) $n = 5$.



(d) $n = 7$.

Figure 11: As n increases, the graph of P_n more closely resembles the sine function.

Guidelines for finding a Taylor series

1. Differentiate $f(x)$ several times and evaluate each derivative at c.

$$f(c),\quad f'(c),\quad f''(c),\quad f'''(c),\quad \ldots,\quad f^{(n)}(c),\quad \ldots$$

   Try to recognize a pattern in these numbers.

2. Use the sequence developed in the first step to form the Taylor coefficients $a_n = f^{(n)}(c)/n!$, and determine the interval of convergence for the resulting power series

$$f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots.$$

3. Within this interval of convergence, determine whether or not the series converges to $f(x)$.

Example 3 (Maclaurin series for a composite function)

Find the Maclaurin series for $f(x) = \sin x^2$.

Example 4 (Binomial series)

Find the Maclaurin series for $f(x) = (1+x)^k$ and determine its radius of convergence. Assume that k is not a positive integer.

Example 5 (Finding a binomial series)

Find the Maclaurin series for $f(x) = \sqrt[3]{1+x}$.

Deriving Taylor series from a basic list

Function	Interval of convergence
$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \cdots + (-1)^n (x-1)^n + \cdots$	$0 < x < 2$
$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots$	$-1 < x < 1$
$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + \frac{(-1)^{(n-1)}(x-1)^n}{n} + \cdots$	$0 < x \leq 2$
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots$	$-\infty < x < \infty$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$	$-\infty < x < \infty$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$	$-\infty < x < \infty$
$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$	$-1 \leq x \leq 1$
$\arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3 x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5 x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots + \frac{(2n)! x^{2n+1}}{(2^n n!)^2 (2n+1)} + \cdots$	$-1 \leq x \leq 1$
$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \frac{k(k-1)(k-2)(k-3)x^4}{4!} + \cdots$	$-1 < x < 1$

The conevrgence at $x = \pm 1$ depends on the value of k.

Euler's Formula

$$
\begin{aligned}
e^{ix} = \cos x + i \sin x &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}
\end{aligned}
$$

Example 6 (Deriving a power series from a basic list)

Find the power series for $f(x) = \cos\sqrt{x}$.

Example 7 (Multiplication and division of power series)

Find the first three nonzero terms in each Maclaurin series $e^x \arctan x$.

Example 8 (Division of Power Series)

Find the first three nonzero terms in each Maclaurin series $\tan x$.

Example 9 (A power series for $\sin^2 x$)

Find the power series for $f(x) = \sin^2 x$.

Example 10 (Power series approximation of a definite integral)

Use a power series to approximate

$$\int_0^1 e^{-x^2}\,\mathrm{d}x$$

with an error of less than 0.01.