Chapter 15 Vector Analysis

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- We have studied vector-valued functions-functions that assign a vector to a real number where we saw that vector-valued functions of real numbers are useful in representing curves and motion along a curve.
- We will study two other types of vector-valued functions-functions that assign a vector to a **point in the plane** or a **point in space**.
- Such functions are called vector fields, and they are useful in representing various types of orce fields and velocity fields.

Definition 15.1 (Vector field)

A vector field over a plane region R is a function $\mathbf{F}$ that assigns a vector $\mathbf{F}(x, y)$ to each point in R.
A vector field over a solid region Q in space is a function $\mathbf{F}$ that assigns a vector $\mathbf{F}(x, y, z)$ to each point in Q.

- The gradient is one example of a vector field. For example, if

$$f(x, y) = x^2 y + 3xy^2$$

then the gradient of f

$$\begin{aligned} \nabla f(x, y) &= f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j} \\ &= (2xy + 3y^3)\,\mathbf{i} + (x^2 + 9xy^2)\,\mathbf{j} \qquad \text{Vector field in the plane} \end{aligned}$$

is a vector field in the plane.

- The graphical interpretation of this field is a family of vectors, each of which points in the direction of maximum increase along the surface given by $z = f(x, y)$.
- Similarly, if

$$f(x, y, z) = x^2 + y^2 + z^2$$

then the gradient of f

$$\begin{aligned} \nabla f(x, y, z) &= f_x(x, y, z)\,\mathbf{i} + f_y(x, y, z)\,\mathbf{j} + f_z(x, y, z)\,\mathbf{k} \\ &= 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k} \qquad \text{Vector field in the in space} \end{aligned}$$

is a vector field in space.
- Note that the component functions for this particular vector field are $2x$, $2y$, and $2z$.

- A vector field

$$\mathbf{F} = M(x, y, z)\,\mathbf{i} + N(x, y, z)\,\mathbf{j} + P(x, y, z)\,\mathbf{k}$$

is **continuous** at a point if and only if each of its component functions M, N, and P is continuous at that point.
- Some common physical examples of vector fields are velocity fields, gravitational fields, and electric force fields. Figure 1 shows the vector field determined by a wheel rotating on an axle.

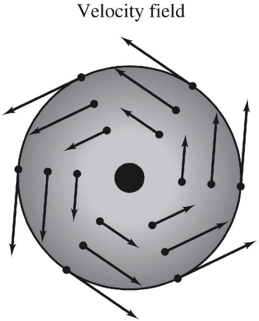

Figure 1: Rotating wheel

- Notice that the velocity vectors are determined by the locations of their initial points-the farther a point is from the axle, the greater its velocity.
- Gravitational fields are defined by Newton's Law of Gravitation, which states that the force of attraction exerted on a particle of mass m_1 located at (x, y, z) by a particle of mass m_2 located at $(0, 0, 0)$ is given by

$$\mathbf{F}(x, y, z) = \frac{-Gm_1 m_2}{x^2 + y^2 + z^2}\mathbf{u}$$

  where G is the gravitational constant and $\mathbf{u}$ is the unit vector in the direction from the origin to (x, y, z).
- In Figure 2, you can see that the gravitational field $\mathbf{F}$ has the properties that $\mathbf{F}(x, y, z)$ always points toward the origin, and that the magnitude of $\mathbf{F}(x, y, z)$ is the same at all points equidistant from the origin.

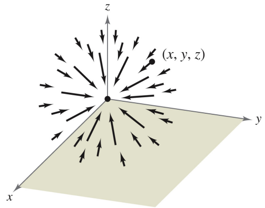

Figure 2: Gravitation force field

- A vector field with these two properties is called a central force field. Using the position vector $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ for the point (x, y, z), you can write the gravitational field $\mathbf{F}$ as

$$\begin{aligned}\mathbf{F}(x, y, z) &= \frac{-Gm_1 m_2}{\|\mathbf{r}\|^2}\left(\frac{\mathbf{r}}{\|\mathbf{r}\|}\right) \\ &= \frac{-Gm_1 m_2}{\|\mathbf{r}\|^2}\mathbf{u}\end{aligned}$$

- Electric force fields are defined by Coulomb's Law, which states that the force exerted on a particle with electric charge q_1 located at (x, y, z) by a particle with electric charge q_2 located at $(0, 0, 0)$ is given by

$$\mathbf{F}(x, y, z) = \frac{cq_1 q_2}{\|\mathbf{r}\|^2}\mathbf{u}$$

where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, $\mathbf{u} = \mathbf{r}/\|\mathbf{r}\|$, and c is a constant that depends on the choice of units for $\|\mathbf{r}\|$, q_1 and q_2.
- Note that an electric force field has the same form as a gravitational field. That is,

$$\mathbf{F}(x, y, z) = \frac{k}{\|\mathbf{r}\|^2}\,\mathbf{u}.$$

- Such a force field is called an <u>inverse square field</u>.

Definition 15.2 (Inverse square field)

Let $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ be a position vector. The vector field $\mathbf{F}$ is an inverse square field if

$$\mathbf{F}(x, y, z) = \frac{k}{\|\mathbf{r}\|^2}\,\mathbf{u}$$

here k is a real number and $\mathbf{u} = \mathbf{r}/\|\mathbf{r}\|$ is a unit vector in the direction of $\mathbf{r}$.

- Because vector fields consist of infinitely many vectors, it is not possible to create a sketch of the entire field.
- When you sketch a vector field, your goal is to sketch representative vectors that help you visualize the field!

Example 2 (Sketching a Vector Field)

Sketch some vectors in the vector field given by

$$\mathbf{F}(x, y) = 2x\,\mathbf{i} + y\,\mathbf{j}$$

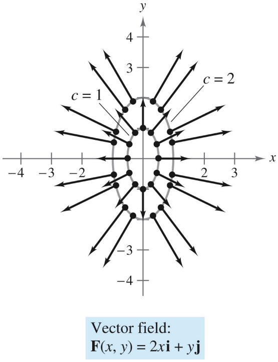

Figure 3: Sketching a vector field

Conservative Vector Fields

- Notice in Figure 3 that all the vectors appear to be normal to the level curve from which they emanate.
- Because this is a property of gradients, it is natural to ask whether the vector field given by $\mathbf{F}(x, y) = 2x\,\mathbf{i} + y\,\mathbf{j}$ is the gradient of some differentiable function f.
- The answer is that some vector fields can be represented as the gradients of differentiable functions and some cannot - those that **can** are called conservative vector field.

Definition 15.3 (Conservative vector field)

A vector field $\mathbf{F}$ is called conservative if there exists a differentiable function f such that $\mathbf{F} = \nabla f$. The function f is called the potential function for $\mathbf{F}$.

Example 4 (Conservative Vector Fields)

a. Show that the vector field given by $\mathbf{F}(x, y) = 2x\,\mathbf{i} + y\,\mathbf{j}$ is conservative.

b. Every inverse square field is conservative.

Theorem 15.1 (Test for conservative vector field in the plane)

Let M and N have continuous first partial derivatives on an open disk R. The vector field given by $\mathbf{F}(x, y) = M\,\mathbf{i} + N\,\mathbf{j}$ is conservative if and only if

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

Example 5 (Test for conservative vector field in the plane)

Decide whether the vector field given by **F** is conservative.

a. $\mathbf{F}(x, y) = x^2y\,\mathbf{i} + xy\,\mathbf{j}$

b. $\mathbf{F}(x, y) = 2x\,\mathbf{i} + y\,\mathbf{j}$

Example 6 (Finding a potential function for $\mathbf{F}(x, y)$)

Find a potential function for

$$\mathbf{F}(x, y) = 2xy\,\mathbf{i} + (x^2 - y)\,\mathbf{j}.$$

Curl of a Vector Field

- Theorem 15.1 has a counterpart for vector fields in space. Before stating that result, the definition of the curl of a vector field in space is given.

Definition 15.4 (Definition of curl of a vector field)

The curl of $\mathbf{F}(x, y, z) = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ is

$$\begin{aligned} \operatorname{curl} \mathbf{F}(x, y, z) &= \nabla \times \mathbf{F}(x, y, z) \\ &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z} \right) \mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k} \end{aligned}$$

If $\operatorname{curl} \mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is said to be irrotational.

- The cross product notation used for curl comes from viewing the gradient ∇f as the result of the differential operator ∇ acting on the function f.
- You can use the following determinant form as an aid in remembering the formula for curl:

$$\begin{aligned} \operatorname{curl} \mathbf{F}(x, y, z) &= \nabla \times \mathbf{F}(x, y, z) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} \\ &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z} \right) \mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k} \end{aligned}$$

Example 7 (Finding the Curl of a Vector Field)

Find curl $\mathbf{F}$ of the vector field given by

$$\mathbf{F}(x, y, z) = 2xy\,\mathbf{i} + (x^2 + z^2)\,\mathbf{j} + 2yz\,\mathbf{k}$$

Is $\mathbf{F}$ irrotational?

- Later in this chapter, you will assign a physical interpretation to the curl of a vector field.
- But for now, the primary use of curl is shown in the following test for conservative vector fields in space.
- The test states that for a vector field in space, the curl is $\mathbf{0}$ at every point in its domain if and only if $\mathbf{F}$ is conservative.
- The proof is similar to that given for Theorem 15.1.

Theorem 15.2 (Test for conservative vector field in space)

Suppose that M, N, and P have continuous first partial derivatives in an open sphere Q in space. The vector field given by $\mathbf{F}(x, y, z) = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ is conservative if and only if

$$\operatorname{curl} \mathbf{F}(x, y, z) = \mathbf{0}.$$

That is, $\mathbf{F}$ *is conservative if and only if*

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z}, \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

- From Theorem 15.2, you can see that the vector field given in Example 7 is conservative because $\text{curl}\,\mathbf{F}(x, y, z) = \mathbf{0}$.
- Try showing that the vector field

$$\mathbf{F}(x, y, z) = x^3y^2z\,\mathbf{i} + x^3z\,\mathbf{j} + x^3y\,\mathbf{k}$$

  is not conservative-you can do this by showing that its curl is

$$\text{curl}\,\mathbf{F}(x, y, z) = (x^3y^2 - 2xy)\,\mathbf{j} + (2xy - 2x^3yz)\,\mathbf{k} \neq \mathbf{0}.$$

- For vector fields in space that pass the test for being conservative, you can find a potential function by following the same pattern used in the plane.

Example 8 (Finding a Potential Function for $\mathbf{F}(x, y, z)$)

Find a potential function for

$$\mathbf{F}(x, y, z) = 2xy\,\mathbf{i} + (x^2 + z^2)\,\mathbf{j} + 2yz\,\mathbf{k}.$$

Divergence of a Vector Field

- You have seen that the curl of a vector field $\mathbf{F}$ is itself a vector field. Another important function defined on a vector field is divergence, which is **a scalar function**.

Definition 15.5 (Divergence of a vector field)

The divergence of $\mathbf{F}(x, y) = M\,\mathbf{i} + N\,\mathbf{j}$ is

$$\operatorname{div}\mathbf{F}(x, y) = \nabla \cdot \mathbf{F}(x, y) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}. \qquad \text{Plane}$$

The divergence of $\mathbf{F}(x, y, z) = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ is

$$\operatorname{div}\mathbf{F}(x, y, z) = \nabla \cdot \mathbf{F}(x, y, z) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}. \qquad \text{Space}$$

If $\operatorname{div}\mathbf{F} = 0$, then $\mathbf{F}$ is said to be divergence-free vector field.

- The dot product notation used for divergence comes from considering ∇ as a differential operator, as follows.

$$\begin{aligned} \nabla \cdot \mathbf{F}(x, y, z) &= \left[\left(\frac{\partial}{\partial x} \right) \mathbf{i} + \left(\frac{\partial}{\partial y} \right) \mathbf{j} + \left(\frac{\partial}{\partial z} \right) \mathbf{k} \right] \cdot (M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}) \\ &= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \end{aligned}$$

- Divergence can be view as a type of derivative of **F**.
- In hydrodynamics, a velocity field that is divergence free is called incompressible and it electricity and magnetism it is called solenoidal.

Example 9 (Finding the Divergence of a Vector Field)

Find the divergence at $(2, 1, -1)$ for the vector field

$$\mathbf{F}(x, y, z) = x^3y^2z\,\mathbf{i} + x^2z\,\mathbf{j} + x^2y\,\mathbf{k}.$$

Theorem 15.3 (Divergence and curl)

If $\mathbf{F}(x, y, z) = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ *is a vector field and* M*,* N*, and* P *have continuous second partial derivatives, then*

$$\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0.$$

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Piecewise Smooth Curves

- A classic property of gravitational fields is that, subject to certain physical constraints, the work done by gravity on an object moving between two points in the field is independent of the path taken by the object.
- One of the constraints is that the path must be a piecewise smooth curve. Recall that a plane curve C given by

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}, \quad a \le t \le b$$

is **smooth** if

$$\frac{\mathrm{d}x}{\mathrm{d}t} \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t}$$

are continuous on $[a, b]$ and not simultaneously 0 on (a, b).

- Similarly, a space curve C given by

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}, \qquad a \le t \le b$$

  is smooth if

$$\frac{\mathrm{d}x}{\mathrm{d}t}, \ \frac{\mathrm{d}y}{\mathrm{d}t}, \quad \text{and} \quad \frac{\mathrm{d}z}{\mathrm{d}t}$$

  are continuous on $[a, b]$ and not simultaneously 0 on (a, b).
- A curve C is piecewise smooth if the interval $[a, b]$ can be partitioned into a finite number of subintervals, on each of which C is smooth.

Example 1 (Finding a Piecewise Smooth Parametrization)

Find a piecewise smooth parametrization of the graph of C shown in Figure 4.

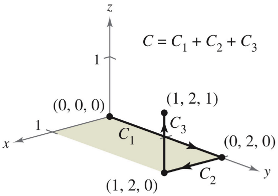


Figure 4: piecewise smooth parametrization

- Recall that parametrization of a curve induces an orientation to the curve.
- In Example 1, the curve is oriented such that the positive direction is from $(0, 0, 0)$, following the curve to $(1, 2, 1)$.
- Try finding a parametrization that induces the opposite orientation.

line integral

- Up to this point in the text, you have studied various types of integrals. For a single integral

$$\int_a^b f(x)\,\mathrm{d}x \qquad \text{Integrate over interval } [a,b]$$

you integrated over the interval $[a,b]$. Similarly, for a double integral

$$\iint_R f(x,y)\,\mathrm{d}A \qquad \text{Integrate over region } R$$

you integrated over the region R in the plane.

- In this section, you will study a new type of integral called a line integral

$$\int_C f(x,y)\,\mathrm{d}s \qquad \text{Integrate over curve } C$$

for which you integrate over a piecewise smooth curve C.

- To introduce the concept of a line integral, consider the mass of a wire of finite length, given by a curve C in space.
- The density (mass per unit length) of the wire at the point (x, y, z) is given by $f(x, y, z)$. Partition the curve C by the points

$$P_0, \ P_1, \ \ldots, \ P_n$$

producing n subarcs, as shown in Figure 5.

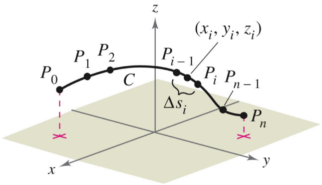


Figure 5: Partitioning of curve C

- The length of the ith subarc is given by Δs_i. Next, choose a point (x_i, y_i, z_i) in each subarc.
- If the length of each subarc is small, the total mass of the wire can be approximated by the sum

$$\text{Mass of wire} \approx \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta s_i$$

- If you let $\|\Delta\|$ denote the length of the longest subarc and let $\|\Delta\|$ approach 0, it seems reasonable that the limit of this sum approaches the mass of the wire.

Definition 15.6 (Line integral)

If f is defined in a region containing a smooth curve C of finite length, then the line integral of f along C is given by

$$\int_C f(x, y)\,\mathrm{d}s = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i, y_i)\Delta s_i \qquad \text{Plane}$$

or

$$\int_C f(x, y, z)\,\mathrm{d}s = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta s_i \qquad \text{Space}$$

provided this limit exists.

- As with the integrals discussed in Chapter 14, evaluation of a line integral is best accomplished by converting it to a definite integral
- It can be shown that if f is continuous, the limit given above exists and is the same for all smooth parametrizations of C.
- To evaluate a line integral over a plane curve C given by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$, use the fact that

$$\mathrm{d}s = \|\mathbf{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}\,\mathrm{d}t.$$

Theorem 15.4 (Evaluation of a line integral as a definite integral)

Let f be continuous in a region containing a smooth curve C. If C is given by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$, where $a \le t \le b$, then

$$\int_C f(x,y)\,\mathrm{d}s = \int_a^b f(x(t),y(t))\sqrt{[x'(t)]^2 + [y'(t)]^2}\,\mathrm{d}t.$$

If C is given by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$, where $a \le t \le b$, then

$$\int_C f(x,y,z)\,\mathrm{d}s = \int_a^b f(x(t),y(t),z(t))\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,\mathrm{d}t.$$

- Note that if $f(x,y,z) = 1$, the line integral gives the arc length of the curve C, as defined in Section 12.5. That is,

$$\int_C 1\,\mathrm{d}s = \int_a^b \|\mathbf{r}'(t)\|\,\mathrm{d}t = \text{length of curve } C.$$

Example 2 (Evaluating a Line Integral)

Evaluate

$$\int_C (x^2 - y + 3z)\,\mathrm{d}s$$

where C is the line segment shown in Figure 6.

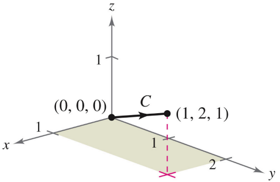


Figure 6: Line segment

- Suppose C is a path composed of smooth curves C_1, C_2, ..., C_n.
- If f is continuous on C, it can be shown that

$$\int_C f(x,y)\,\mathrm{d}s = \int_{C_1} f(x,y)\,\mathrm{d}s + \int_{C_2} f(x,y)\,\mathrm{d}s + \cdots + \int_{C_n} f(x,y)\,\mathrm{d}s.$$

This property is used in Example 3.

Example 3 (Evaluating a Line Integral Over a Path)

Evaluate

$$\int_C x\,\mathrm{d}s$$

where C is the piecewise smooth curve shown in Figure 7.

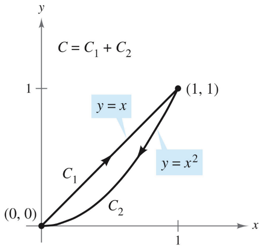


Figure 7: Piecewise smooth curve

- For parametrizations given by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$, it is helpful to remember the form of $\mathrm{d}s$ as

$$\mathrm{d}s = \|\mathbf{r}'(t)\|\,\mathrm{d}t = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,\mathrm{d}t.$$

- This is demonstrated in Example 4.

Example 4 (Evaluating a line Integral)

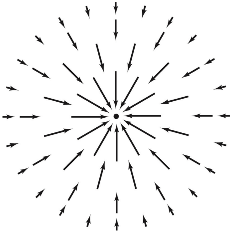
Evaluate

$$\int_C (x+2)\,\mathrm{d}s$$

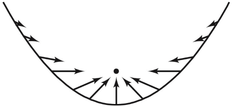
where C is the curve represented by

$$\mathbf{r}(t) = t\,\mathbf{i} + \frac{4}{3}t^{3/2}\,\mathbf{j} + \frac{1}{2}t^2\,\mathbf{k},\ 0 \le t \le 2.$$

- One of the most important physical applications of line integrals is that of finding the work done on an object moving in a force field.
- For example, Figure 8 shows an inverse square force field similar to the gravitational field of the sun.



(a) Inverse square force field **F**



(b) Vectors along a parabolic path in the force field **F**

Figure 8: Force field **F**

- Note that the magnitude of the force along a circular path about the center is constant, whereas the magnitude of the force along a parabolic path varies from point to point.
- To see how a line integral can be used to find work done in a force field **F**, consider an object moving along a path C in the field, as shown in Figure 9.

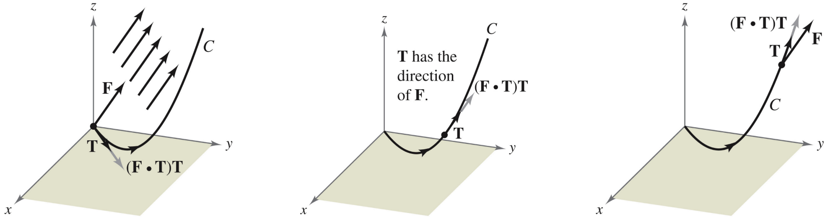


Figure 9: At each point on C, the force in the direction of motion is $(\mathbf{F} \cdot \mathbf{T})\mathbf{T}$

- To determine the work done by the force, you need consider only that part of the force that is acting in the same direction as that in which the object is moving (or the opposite direction).
- This means that at each point on C, you can consider the projection $\mathbf{F} \cdot \mathbf{T}$ of the force vector $\mathbf{F}$ onto the unit tangent vector $\mathbf{T}$.
- On a small sub arc of length Δs_i, the increment of work is

$$\begin{aligned} \Delta W_i &= \text{(force)(distance)} \\ &\approx [\mathbf{F}(x_i, y_i, z_i) \cdot \mathbf{T}(x_i, y_i, z_i)]\Delta s_i \end{aligned}$$

where (x_i, y_i, z_i) is a point in the ith subarc.

- Consequently, the total work done is given by the following integral.

$$W = \int_C \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z)\,\mathrm{d}s$$

- This line integral appears in other contexts and is the basis of the following definition of the line integral of a vector field.
- Note in the definition that

$$\begin{aligned} \mathbf{F} \cdot \mathbf{T}\,\mathrm{d}s &= \mathbf{F} \cdot \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}\|\mathbf{r}'(t)\|\,\mathrm{d}t \\ &= \mathbf{F} \cdot \mathbf{r}'(t)\,\mathrm{d}t \\ &= \mathbf{F} \cdot \,\mathrm{d}\mathbf{r}. \end{aligned}$$

Definition 15.7 (The line integral of a vector field)

Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by $\mathbf{r}(t)$, $a \leq t \leq b$. The line integral of $\mathbf{F}$ on C is given by

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} \, \mathrm{d}s = \int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t) \, \mathrm{d}t$$

Example 6 (Work Done by a Force)

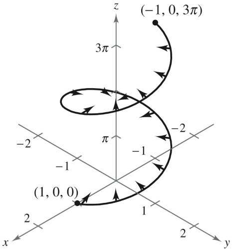
Find the work done by the force field

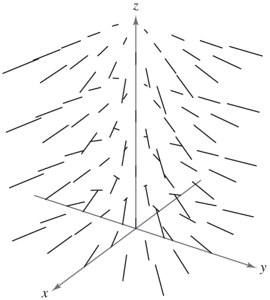
$$\mathbf{F}(x, y, z) = -\frac{1}{2}x\,\mathbf{i} - \frac{1}{2}y\,\mathbf{j} + \frac{1}{4}\,\mathbf{k} \qquad \text{Force field } \mathbf{F}$$

on a particle as it moves along the helix given by

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k} \qquad \text{Space curve } \mathbf{C}$$

from the point $(1, 0, 0)$ to $(-1, 0, 3\pi)$, as shown in Figure below.





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- The computer-generated view of the force field in Example 6 shown in Figure 58 indicates that each vector in the force field points toward the z-axis.
- For line integrals of vector functions, the orientation of the curve C is important.
- If the orientation of the curve is reversed, the unit tangent vector $\mathbf{T}(t)$ is changed to $-\mathbf{T}(t)$, and you obtain

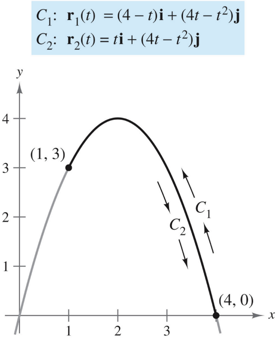
$$\int_{-C} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\int_{C} \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

Example 7 (Orientation and Parametrization of a Curve)

Let $\mathbf{F}(x, y) = y\,\mathbf{i} + x^2\,\mathbf{j}$ and evaluate the line integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ for each parabolic curve shown in Figure below.

a. $C_1 : \mathbf{r}_1(t) = (4 - t)\,\mathbf{i} + (4t - t^2)\,\mathbf{j}, \quad 0 \le t \le 3$

b. $C_2 : \mathbf{r}_2(t) = t\,\mathbf{i} + (4t - t^2)\,\mathbf{j}, \quad 1 \le t \le 4$



Line Integrals in Differential Form

- A second commonly used form of line integrals is derived from the vector field notation used in the preceding section.
- If $\mathbf{F}$ is a vector field of the form $\mathbf{F}(x, y) = M\,\mathbf{i} + N\,\mathbf{j}$, and C is given by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$, then $\mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is often written as $M\,\mathrm{d}x + N\,\mathrm{d}y$.

$$\begin{aligned}
\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_C \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \cdot \mathrm{d}\mathbf{r} \\
&= \int_a^b (M\,\mathbf{i} + N\,\mathbf{j}) \cdot [x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j}]\,\mathrm{d}t \\
&= \int_a^b \left(M\frac{\mathrm{d}x}{\mathrm{d}t} + N\frac{\mathrm{d}y}{\mathrm{d}t} \right) \mathrm{d}t \\
&= \int_C (M\,\mathrm{d}x + N\,\mathrm{d}y)
\end{aligned}$$

- This differential formcan be extended to three variables.
- The parentheses are often omitted, as follows.

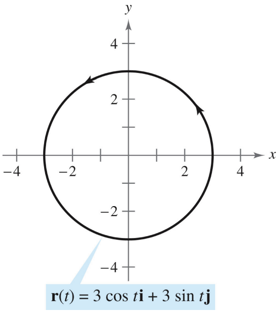
$$\int_C M\,\mathrm{d}x + N\,\mathrm{d}y \text{ and } \int_C M\,\mathrm{d}x + N\,\mathrm{d}y + P\,\mathrm{d}z$$

- Notice how this differential notation is used in Example 8.

Example 8 (Evaluating a line Integral in Differential Form)

Let C be the circle of radius 3 given by $\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$, $0 \le t \le 2\pi$ as shown in Figure below. Evaluate the line integral

$$\int_C y^3\,\mathrm{d}x + (x^3 + 3xy^2)\,\mathrm{d}y.$$



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Fundamental Theorem of line Integrals

- The discussion at the beginning of the preceding section pointed out that in a gravitational field the work done by gravity on an object moving between two points in the field is independent of the path taken by the object.
- You will study an important generalization of this result - it is called the Fundamental Theorem of Line Integrals.
- To begin, an example is presented in which the line integral of a onservative vector field is evaluated over three different paths.

Example 1 (Line integral of a conservative vector field)

Find the work done by the force field

$$\mathbf{F}(x, y) = \frac{1}{2}xy\,\mathbf{i} + \frac{1}{4}x^2\,\mathbf{j}$$

on a particle that moves from $(0, 0)$ to $(1, 1)$ along each path, as shown in Figure below.

a. $C_1 : y = x$ **b.** $C_2 : y = x^2$ **c.** $C_3 : y = x^3$

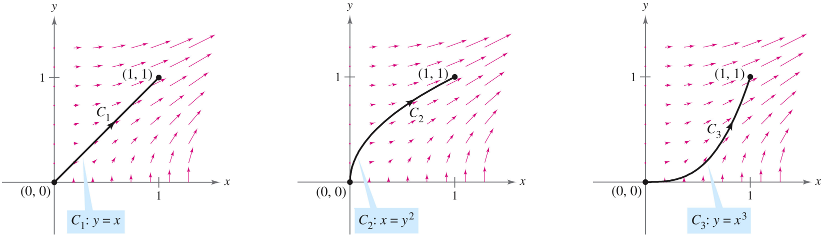

Figure 10: Lower degree polynomial curves

Theorem 15.5 (Fundamental theorem of line integrals)

Let C be a piecewise smooth curve lying in an open region R and given by

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}, \quad a \le t \le b.$$

If $\mathbf{F}(x,y) = M\,\mathbf{i} + N\,\mathbf{j}$ is conservative in R, and M and N are continuous in R, then

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C \nabla f \cdot \mathrm{d}\mathbf{r} = f(x(b), y(b)) - f(x(a), y(a))$$

where f is a potential function of $\mathbf{F}$. That is, $\mathbf{F}(x,y) = \nabla f(x,y)$.

Example 2 (Using the fundamental theorem of line integrals)

Evaluate $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$, where C is a piecewise smooth curve from $(-1, 4)$ to $(1, 2)$ and

$$\mathbf{F}(x, y) = 2xy\,\mathbf{i} + (x^2 - y)\,\mathbf{j}$$

as shown in Figure 11.

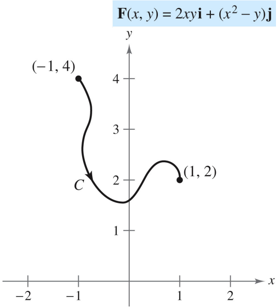

Figure 11: Using the fundamental theorem of line integrals, $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$

Independence of Path

- From the Fundamental Theorem of Line Integrals it is clear that if $\mathbf{F}$ is continuous and conservative in an open region R, the value of $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is the same for every piecewise smooth curve C from one fixed point in R to another fixed point in R.
- This result is described by saying that the line integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is independent of path in the region R.
- A region in the plane (or in space) is connected if any two points in the region can be joined by a piecewise smooth curve lying entirely within the region, as shown in Figure 12.

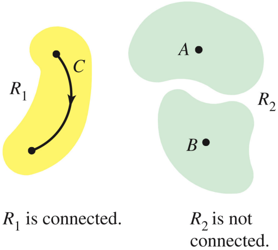

Figure 12: Connected and not connected regions

- In open regions that are connected, the path independence of $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is equivalent to the condition that $\mathbf{F}$ is conservative.

Theorem 15.6 (Independence of path and conservative vector fields)

If $\mathbf{F}$ *is continuous on an open connected region, then the line integral*

$$\int_C \mathbf{F} \cdot \, \mathrm{d}\mathbf{r}$$

is independent of path if and only if $\mathbf{F}$ *is conservative.*

Example 4 (Finding Work in a conservative force field)

For the force field given by

$$\mathbf{F}(x, y, z) = e^x \cos y\,\mathbf{i} - e^x \sin y\,\mathbf{j} + 2\,\mathbf{k}$$

show that $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is independent of path, and calculate the work done by $\mathbf{F}$ on an object moving along a curve C from $(0, \pi/2, 1)$ to $(1, \pi, 3)$.

- How much work would be done if the object in Example 4 moved from the point $(0, \pi/2, 1)$ to $(1, \pi, 3)$ and then back to the starting point $(0, \pi/2, 1)$?
- The Fundamental Theorem of Line Integrals states that there is zero work done. Remember that, by definition, work can be negative. So, by the time the object gets back to its starting point, the amount of work that registers positively is canceled out by the amount of work that registers negatively.
- A curve C given by $\mathbf{r}(t)$ for $a \leq t \leq b$ is closed if $\mathbf{r}(a) = \mathbf{r}(b)$.

- By the Fundamental Theorem of Line Integrals, you can conclude that if $\mathbf{F}$ is continuous and conservative on an open region R, then the line integral over every closed curve C is 0.

Theorem 15.7 (Equivalent conditions)

Let $\mathbf{F}(x, y, z) = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ have continuous first partial derivatives in an open connected region R, and let C be a piecewise smooth curve in R. The following conditions are equivalent.

1. $\mathbf{F}$ *is conservative. That is,* $\mathbf{F} = \nabla f$ *for some function* f.
2. $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ *is independent of path.*
3. $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$ *for every closed curve* C *in* R.

Example 5 (Evaluating a Line Integral)

Evaluate

$$\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

where

$$\mathbf{F}(x, y) = (y^3 + 1)\,\mathbf{i} + (3xy^2 + 1)\,\mathbf{j}$$

and C_1 is the semicircular path from $(0, 0)$ to $(2, 0)$, as shown in Figure 13.

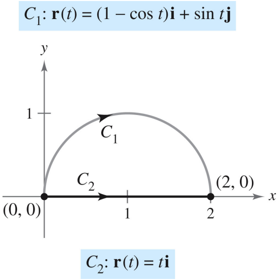

Figure 13: Semicircular path from $(0, 0)$ to $(2, 0)$.

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Green's Theorem

- In this section, you will study Green's Theorem which states that the value of a double integral over a simply connected plane region R is determined by the value of a line integral around the boundary of R.
- A curve C given by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\mathbf{j}$, where $a \leq t \leq b$, is simple if it does not cross itself - that is, $\mathbf{r}(c) \neq \mathbf{r}(d)$ for all c and d in the open interval (a, b).
- A plane region R is simply connected if every simple closed curve in R encloses only points that are in R (see Figure 14).

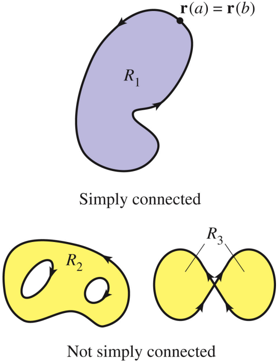


Figure 14: Simple connected and not simple connected regions.

Theorem 15.8 (Green's theorem)

Let R be a simply connected region with a piecewise smooth boundary C, oriented counterclockwise (that is, C is traversed once so that the region R always lies to the left). If M and N have continuous first partial derivatives in an open region containing R, then

$$\int_C M\,\mathrm{d}x + N\,\mathrm{d}y = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\,\mathrm{d}A.$$

Example 1 (Using Green's Theorem)

Use Green's Theorem to evaluate the line integral

$$\int_C y^3\,dx + (x^3 + 3xy^2)\,dy$$

where C is the path from $(0, 0)$ to $(1, 1)$ along the graph of $y = x^3$ and from $(1, 1)$ to $(0, 0)$ along the graph of $y = x$, as shown in Figure 15.

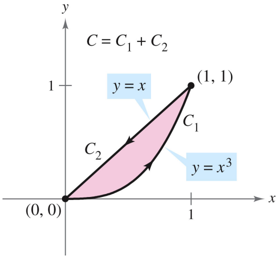


Figure 15: C is simple and closed, and the region R always lies to the left of C.

Example 2 (Using Green's Theorem to Calculate Work)

While subject to the force

$$\mathbf{F}(x, y) = y^3\,\mathbf{i} + (x^3 + 3xy^2)\,\mathbf{j}$$

a particle travels once around the circle of radius 3 shown in Figure 16. Use Green's Theorem to find the work done by $\mathbf{F}$.

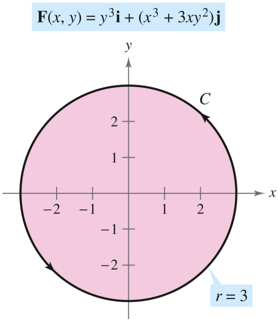


Figure 16: Circle.

Example 3 (Green's Theorem and Conservative Vector Fields)

Evaluate the line integral

$$\int_C y^3\,\mathrm{d}x + 3xy^2\,\mathrm{d}y$$

where C is the path shown in Figure 17.

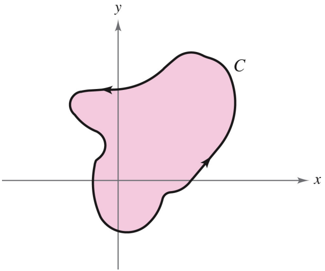


Figure 17: C is closed.

- In Examples 1 and 2 Green's Theorem was used to evaluate line integrals as double integrals.
- You can also use the theorem to evaluate double integrals as line integrals. One useful application occurs when $\partial N/\partial x - \partial M/\partial y = 1$.

$$\begin{aligned} \int_C M\,\mathrm{d}x + N\,\mathrm{d}y &= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathrm{d}A \\ &= \iint_R 1\,\mathrm{d}A \\ &= \text{area of region } R \end{aligned}$$

- Among the many choices for M and N satisfying the stated condition, the choice of $M = -y/2$ and $N = x/2$ produces the following line integral for the area of region R

Theorem 15.9 (Line integral for area)

If R is a plane region bounded by a piecewise smooth simple closed curve C, oriented counterclockwise, then the area of R is given by

$$A = \frac{1}{2}\int_C x\,\mathrm{d}y - y\,\mathrm{d}x.$$

Example 5 (Finding Area by a Line Integral)

Use a line integral to find the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Alternative Forms of Green's Theorem

- This section concludes with the derivation of two vector forms of Green's Theorem for regions in the plane.
- If $\mathbf{F}$ is a vector field in the plane, you can write

$$\mathbf{F}(x, y, z) = M\,\mathbf{i} + N\,\mathbf{j} + 0\,\mathbf{k}$$

so that the curl of $\mathbf{F}$, as described in Section 15.1, is given by

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix}$$

$$= -\frac{\partial N}{\partial z}\,\mathbf{i} + \frac{\partial M}{\partial z}\,\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}.$$

- Consequently,

$$(\operatorname{curl}\mathbf{F}) \cdot \mathbf{k} = \left[-\frac{\partial N}{\partial z}\,\mathbf{i} + \frac{\partial M}{\partial z}\,\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}\right] \cdot \mathbf{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

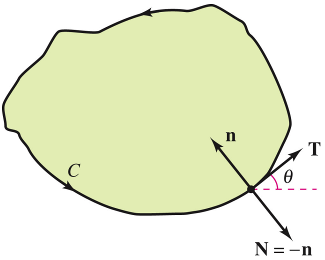
- With appropriate conditions on $\mathbf{F}$, C, and R, you can write Green's Theorem in the vector form

$$\int_c \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathrm{d}A$$
$$= \iint_R (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} \, \mathrm{d}A. \qquad \text{First alternative form}$$

- The extension of this vector form of Green's Theorem to surfaces in space produces Stokes's Theorem, discussed later on.
- For the second vector form of Green's Theorem, assume the same conditions for $\mathbf{F}$, C, and R. Using the arc length parameter s for C, you have $\mathbf{r}(s) = x(s)\,\mathbf{i} + y(s)\,\mathbf{j}$
- So, a unit tangent vector $\mathbf{T}$ to curve C is given by $\mathbf{r}'(s) = \mathbf{T} = x'(s)\,\mathbf{i} + y'(s)\,\mathbf{j}$.

- From Figure below you can see that the outward unit normal vector **N** can then be written as

$$\mathbf{N} = y'(s)\,\mathbf{i} - x'(s)\,\mathbf{j}.$$



$$\begin{aligned}\mathbf{T} &= \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}\\ \mathbf{n} &= \cos\left(\theta + \frac{\pi}{2}\right)\,\mathbf{i} + \sin\left(\theta + \frac{\pi}{2}\right)\,\mathbf{j}\\ &= -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}\\ \mathbf{N} &= \sin\theta\,\mathbf{i} - \cos\theta\,\mathbf{j}\end{aligned}$$

- Consequently, for $\mathbf{F}(x, y) = M\,\mathbf{i} + N\,\mathbf{j}$, you can apply Green's Theorem to obtain

$$\begin{aligned}
\int_C \mathbf{F}\cdot\mathbf{N}\,\mathrm{d}s &= \int_a^b (M\,\mathbf{i} + N\,\mathbf{j})\cdot(y'(s)\,\mathbf{i} - x'(s)\,\mathbf{j})\,\mathrm{d}s \\
&= \int_a^b \left(M\frac{\mathrm{d}y}{\mathrm{d}s} - N\frac{\mathrm{d}x}{\mathrm{d}s}\right)\,\mathrm{d}s \\
&= \int_C M\,\mathrm{d}y - N\,\mathrm{d}x \\
&= \int_C -N\,\mathrm{d}x + M\,\mathrm{d}y \\
&= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\,\mathrm{d}A \qquad \text{Green's Theorem} \\
&= \iint_R \operatorname{div}\mathbf{F}\,\mathrm{d}A.
\end{aligned}$$

- Therefore,

$$\int_C \mathbf{F} \cdot \mathbf{N} \, \mathrm{d}s = \iint_R \operatorname{div} \mathbf{F} \, \mathrm{d}A. \qquad \text{Second alternative form}$$

- The extension of this form to three dimensions is called the Divergence Theorem, discussed in Section 15.7. The physical interpretations of divergence and curl will be discussed later on as well.

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Parametric Surfaces

- You already know how to represent a curve in the plane or in space by a set of parametric equations-or, equivalently, by a vector-valued function.

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} \qquad \text{Plane curve}$$
$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k} \qquad \text{Space curve}$$

- In this section, you will learn how to represent a surface in space by a set of parametric equations - or by a vector-valued function.
- For curves, note that the vector-valued function **r** is a function of a single parameter t. For surfaces, the vector-valued function is a function of two parameters u and v.

Definition 15.8 (Definition of parametric surface)

Let x, y, and z be functions of u and v that are continuous on a domain D in the uv-plane. The set of points (x, y, z) given by

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k} \qquad \text{Parametric surface}$$

is called a parametric surface . The equations

$$x = x(u, v),\ \ y = y(u, v), \quad \text{and} \quad z = z(u, v) \quad \text{Parametric equations}$$

are the parametric equations for the surface.

- If S is a parametric surface given by the vector-valued function $\mathbf{r}$, then S is traced out by the position vector $\mathbf{r}(u, v)$ as the point (u, v) moves throughout the domain D, as shown in Figure 18.

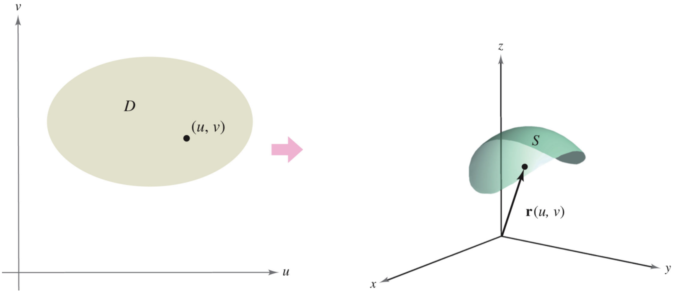

Figure 18: The point (u, v) moves throughout the domain D.

Example 1 (Sketching a Parametric Surface)

Identify and sketch the parametric surface S given by

$$\mathbf{r}(u, v) = 3\cos u\,\mathbf{i} + 3\sin u\,\mathbf{j} + v\,\mathbf{k}$$

where $0 \le u \le 2\pi$ and $0 \le v \le 4$.

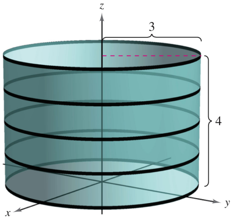

Figure 19: Cylinder.

Finding Parametric Equations for Surfaces

- In Examples 1 you were asked to identify the surface described by a given set of parametric equations.
- The reverse problem - that of writing a set of parametric equations for a given surface - is generally more difficult.
- One type of surface for which this problem is straightforward, however, is a surface that is given by $z = f(x, y)$. You can parametrize such a surface as

$$\mathbf{r}(x, y) = x\,\mathbf{i} + y\,\mathbf{j} + f(x, y)\,\mathbf{k}.$$

Example 3 (Representing a Surface Parametrically)

Write a set of parametric equations for the cone given by

$$z = \sqrt{x^2 + y^2}$$

as shown in Figure 20.

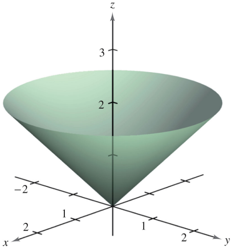

Figure 20: Cone.

Normal Vectors and Tangent Planes

- Let S be a parametric surface given by

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$$

  over an open region D such that x, y, and z have continuous partial derivatives on D.

- The partial derivatives of $\mathbf{r}$ with respect to u and v are defined as

$$\mathbf{r}_u = \frac{\partial x(u, v)}{\partial u}\,\mathbf{i} + \frac{\partial y(u, v)}{\partial u}\,\mathbf{j} + \frac{\partial z(u, v)}{\partial u}\,\mathbf{k}$$

  and

$$\mathbf{r}_v = \frac{\partial x(u, v)}{\partial v}\,\mathbf{i} + \frac{\partial y(u, v)}{\partial v}\,\mathbf{j} + \frac{\partial z(u, v)}{\partial v}\,\mathbf{k}.$$

- Each of these partial derivatives is a vector-valued function that can be interpreted geometrically in terms of tangent vectors.

- For instance, if $v = v_0$ is held constant, then $\mathbf{r}(u, v_0)$ is a vector-valued function of a single parameter and defines a curve C_1 that lies on the surface S.
- The tangent vector to C_1 at the point $(x(u_0, v_0), y(u_0, v_0), z(u_0, v_0))$ is given by

$$\mathbf{r}_u(u_0, v_0) = \left.\frac{\partial x}{\partial u}\right|_{(u_0,v_0)} \mathbf{i} + \left.\frac{\partial y}{\partial u}\right|_{(u_0,v_0)} \mathbf{j} + \left.\frac{\partial z}{\partial u}\right|_{(u_0,v_0)} \mathbf{k}$$

as shown in Figure 21.

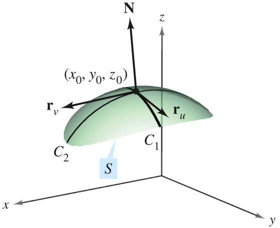

Figure 21: Paraboloid.

- In a similar way, if $u = u_0$ is held constant, then $\mathbf{r}(u_0, v)$ is a vector-valued function of a single parameter and defines a curve C_2 that lies on the surface S.
- The tangent vector to C_2 at the point $(x(u_0, v_0), y(u_0, v_0), z(u_0, v_0))$ is given by

$$\mathbf{r}_v(u_0, v_0) = \left.\frac{\partial x}{\partial v}\right|_{(u_0,v_0)} \mathbf{i} + \left.\frac{\partial y}{\partial v}\right|_{(u_0,v_0)} \mathbf{j} + \left.\frac{\partial z}{\partial v}\right|_{(u_0,v_0)} \mathbf{k}.$$

- If the normal vector $\mathbf{r}_u \times \mathbf{r}_v$ is not $\mathbf{0}$ for any (u, v) in D, the surface S is called **smooth** and will have a tangent plane. Informally, a smooth surface is one that has no sharp points or cusps. For instance, spheres, ellipsoids, and paraboloids are smooth, whereas the cone given in Example 3 is not smooth.

Definition 15.9 (Normal vector to a smooth parametric surface)

Let S be a smooth parametric surface

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$$

defined over an open region D in the uv-plane. Let (u_0, v_0) be a point in D. A normal vector at the point

$$(x_0, y_0, z_0) = (x(u_0, v_0), y(u_0, v_0), z(u_0, v_0))$$

is given by

$$\mathbf{N} = \mathbf{r}_u(u_0, v_0) \times \mathbf{r}_v(u_0, v_0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}.$$

Example 5 (Finding a Tangent Plane to a Parametric Surface)

Find an equation of the tangent plane to the paraboloid given by

$$\mathbf{r}(u,v) = u\,\mathbf{i} + v\,\mathbf{j} + (u^2 + v^2)\,\mathbf{k}$$

at the point $(1, 2, 5)$.

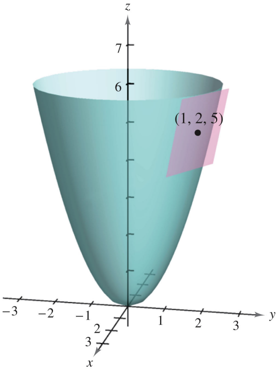

Figure 22: Paraboloids.

- To define the area of a parametric surface, you can use a development that is similar to that given in Section 14.5. Begin by constructing an inner partition of D consisting of n rectangles, where the area of the ith rectangle D_i is $\Delta A_i = \Delta u_i \Delta v_i$, as shown in Figure 23.

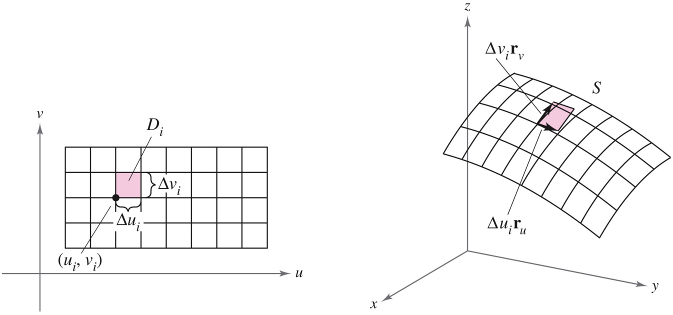

Figure 23: The area of the parallelogram in the tangent plane.

- In each D_i let (u_i, v_i) be the point that is closest to the origin. At the point $(x_i, y_i, z_i) = (x(u_i, v_i), y(u_i, v_i), z(u_i, v_i))$ on the surface S, construct a tangent plane T_i.
- The area of the portion of S that corresponds to $D_i, \Delta T_i$, can be approximated by a parallelogram in the tangent plane. That is, $\Delta T_i \approx \Delta S_i$.
- So, the surface of S is given by $\Sigma \Delta S_i \approx \Sigma \Delta T_i$. The area of the parallelogram in the tangent plane is

$$\|\Delta u_i \mathbf{r}_u \times \Delta v_i \mathbf{r}_v\| = \|\mathbf{r}_u \times \mathbf{r}_v\| \Delta u_i \Delta v_i$$

which leads to the following definition.

Definition 15.10 (Area of a parametric surface)

Let S be a smooth parametric surface

$$\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}$$

defined over an open region D in the uv-plane. If each point on the surface S corresponds to exactly one point in the domain D, then the surface area of S is given by

$$\text{Surface area} = \iint_S \mathrm{d}S = \iint_D \|\mathbf{r}_u \times \mathbf{r}_v\|\,\mathrm{d}A$$

where

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\,\mathbf{i} + \frac{\partial y}{\partial u}\,\mathbf{j} + \frac{\partial z}{\partial u}\,\mathbf{k} \quad \text{and} \quad \mathbf{r}_v = \frac{\partial x}{\partial v}\,\mathbf{i} + \frac{\partial y}{\partial v}\,\mathbf{j} + \frac{\partial z}{\partial v}\,\mathbf{k}.$$

- For a surface S given by $z = f(x, y)$, this formula for surface area corresponds to that given in Section 14.5. To see this, you can parametrize the surface using the vector-valued function

$$\mathbf{r}(x, y) = x\,\mathbf{i} + y\,\mathbf{j} + f(x, y)\,\mathbf{k}$$

defined over the region R in the xy-plane. Using

$$\mathbf{r}_x = \mathbf{i} + f_x(x, y)\,\mathbf{k} \quad \text{and} \quad \mathbf{r}_y = \mathbf{j} + f_y(x, y)\,\mathbf{k}$$

you have

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x(x, y) \\ 0 & 1 & f_y(x, y) \end{vmatrix} = -f_x(x, y)\,\mathbf{i} - f_y(x, y)\,\mathbf{j} + \mathbf{k}$$

and $\|\mathbf{r}_x \times \mathbf{r}_y\| = \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}$.

- This implies that the surface area of S is

$$\begin{aligned} \text{Surface of area} &= \iint_R \|\mathbf{r}_x \times \mathbf{r}_y\|\,\mathrm{d}A \\ &= \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\,\mathrm{d}A. \end{aligned}$$

Example 6 (Finding Surface Area)

Find the surface area of the unit sphere given by

$$\mathbf{r}(u, v) = \sin u \cos v\,\mathbf{i} + \sin u \sin v\,\mathbf{j} + \cos u\,\mathbf{k}$$

where the domain D is given by $0 \le u \le \pi$ and $0 \le v \le 2\pi$.

- If the surface S is a surface of revolution, you can show that the formula for surface area given in Section 7.4 is equivalent to the formula given in this section. For instance, suppose f is a nonnegative function such that f' is continuous over the interval $[a, b]$.
- Let S be the surface of revolution formed by revolving the graph of f, where $a \leq x \leq b$, about the x-axis. From Section 7.4, you know that the surface area is given by

$$\text{Surface area} = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\,\mathrm{d}x.$$

- To represent S parametrically, let $x = u, y = f(u)\cos v$, and $z = f(u)\sin v$, where $a \leq u \leq b$ and $0 \leq v \leq 2\pi$. Then,

$$\mathbf{r}(u, v) = u\,\mathbf{i} + f(u)\cos v\,\mathbf{j} + f(u)\sin v\,\mathbf{k}.$$

Try showing that the formula

$$\text{Surface area} = \iint_D \|\mathbf{r}_u \times \mathbf{r}_v\|\,\mathrm{d}A$$

is equivalent to the formula given above (see Exercise 58).

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Surface Integrals

- The remainder of this chapter deals primarily with surface integrals. You will first consider surfaces given by $z = g(x, y)$. Later in this section you will consider more general surfaces given in parametric form.
- Let S be a surface given by $z = g(x, y)$ and let R be its projection onto the xy-plane, as shown in Figure 24.

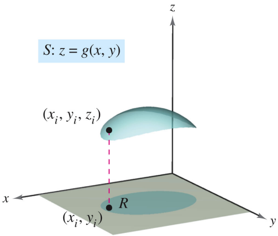


Figure 24: Paraboloids.

- Suppose that g, g_x and g_y are continuous at all points in R and that f is defined on S.
- Employing the procedure used to find surface area in Section 14.5, evaluate f at (x_1, y_1, z_1) and form the sum

$$\sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta S_i$$

where $\Delta S_i \approx \sqrt{1 + [g_x(x_i, y_i)]^2 + [g_y(x_i, y_i)]^2}\Delta A_i$.

- Provided the limit of this sum as $\|\Delta\|$ approaches 0 exists, the surface integral of f over S is defined as

$$\iint_S f(x, y, z)\,\mathrm{d}S = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta S_i.$$

This integral can be evaluated by a double integral.

Theorem 15.10 (Evaluating a surface integral)

Let S be a surface with equation $z = g(x, y)$ and let R be its projection onto the xy-plane. If g, g_x, and g_y are continuous on R and f is continuous on S, then the surface integral of f over S is

$$\iint_S f(x, y, z)\,\mathrm{d}S = \iint_R f(x, y, g(x, y))\sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2}\,\mathrm{d}A.$$

- For surfaces described by functions of x and z (or y and z), you can make the following adjustments to Theorem 15.10. If S is the graph of $y = g(x, z)$ and R is its projection onto the xz-plane, then it is

$$\iint_R f(x, g(x, z), z)\sqrt{1 + [g_x(x, z)]^2 + [g_z(x, z)]^2}\,\mathrm{d}A.$$

- If S is the graph of $x = g(y, z)$ and R is its projection onto the yz-plane, then it is

$$\iint_R f(g(y,z), y, z)\sqrt{1 + [g_y(y,z)]^2 + [g_z(y,z)]^2}\,\mathrm{d}A.$$

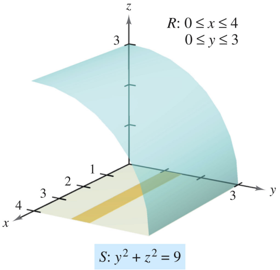
- If $f(x, y, z) = 1$, the surface integral over S yields the surface area of S.
- For instance, suppose the surface S is the plane given by $z = x$, where $0 \leq x \leq 1$ and $0 \leq y \leq 1$. The surface area of S is $\sqrt{2}$ square units. Try verifying that $\iint_S f(x, y, z)\,\mathrm{d}S = \sqrt{2}$.

Example 2 (Evaluating a Surface Integral)

Evaluate the surface integral

$$\iint_S (x + z)\,\mathrm{d}S$$

where S is the first-octant portion of the cylinder $y^2 + z^2 = 9$ between $x = 0$ and $x = 4$, as shown in Figure 130.



- For a surface S given by the vector-valued function

$$\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k} \qquad \text{Parametric surface}$$

  defined over a region D in the uv-plane, you can show that the surface integral of $f(x,y,z)$ over S is given by

$$\iint_D f(x(u,v), y(u,v), z(u,v)) \|\mathbf{r}_u(u,v) \times \mathbf{r}_v(u,v)\| \,\mathrm{d}A.$$

- Note the similarity to a line integral over a space curve C.

$$\int_C f(x,y,z)\,\mathrm{d}s = \int_a^b f(x(t), y(t), z(t)) \|\mathbf{r}'(t)\| \,\mathrm{d}t.$$

- Also, notice that $\mathrm{d}s$ and $\mathrm{d}S$ can be written as

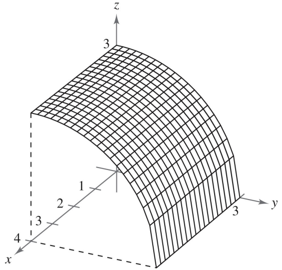
$$\mathrm{d}s = \|\mathbf{r}'(t)\|\,\mathrm{d}t \quad \text{and} \quad \mathrm{d}S = \|\mathbf{r}_u(u,v) \times \mathbf{r}_v(u,v)\|\,\mathrm{d}A.$$

Example 4 (Evaluating a Surface Integral)

Example 2 demonstrated an evaluation of the surface integral

$$\iint_S (x+z)\,\mathrm{d}S$$

where S is the first-octant portion of the cylinder $y^2+z^2=9$ between $x=0$ and $x=4$ (see Figure 133). Reevaluate this integral in parametric form.



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Orientation of a Surface

- Unit normal vectors are used to induce an orientation to a surface S in space. A surface is called orientable if a unit normal vector $\mathbf{N}$ can be defined at every nonboundary point of S in such a way that the normal vectors vary continuously over the surface S. If this is possible, S is called an oriented surface.
- An orientable surface S has two distinct sides. So, when you orient a surface, you are selecting one of the two possible unit normal vectors.
- If S is a closed surface such as a sphere, it is customary to choose the unit normal vector $\mathbf{N}$ to be the one that points outward from the sphere.
- Most common surfaces, such as spheres, paraboloids, ellipses, and planes, are orientable. (See Exercise 43 for an example of a surface that is not orientable.)

- Moreover, for an orientable surface, the gradient vector provides a convenient way to find a unit normal vector. That is, for an orientable surface S given by

$$z = g(x, y) \quad \text{Orientable surface}$$

let

$$G(x, y, z) = z - g(x, y).$$

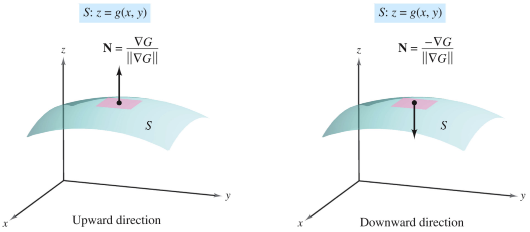
- Then, S can be oriented by either the unit normal vector

$$\begin{aligned} \mathbf{N} &= \frac{\nabla G(x, y, z)}{\|\nabla G(x, y, z)\|} \\ &= \frac{-g_x(x, y)\,\mathbf{i} - g_y(x, y)\,\mathbf{j} + \mathbf{k}}{\sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2}} \quad \text{Upward unit normal vector} \end{aligned}$$

- or the unit normal vector

$$\mathbf{N} = \frac{-\nabla G(x,y,z)}{\|\nabla G(x,y,z)\|} = \frac{g_x(x,y)\,\mathbf{i} + g_y(x,y)\,\mathbf{j} - \mathbf{k}}{\sqrt{1 + [g_x(x,y)]^2 + [g_y(x,y)]^2}} \quad \text{Downward unit normal vector}$$

as shown in Figure 26.



(a) S is oriented in an upward direction.

(b) S is oriented in a downward direction.

Figure 26: Smooth orientable surface.

- If the smooth orientable surface S is given in parametric form by

$$\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k} \quad \text{Parametric surface}$$

the unit normal vectors are given by

$$\mathbf{N} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} \quad \text{Upward unit normal vector}$$

and

$$\mathbf{N} = \frac{\mathbf{r}_v \times \mathbf{r}_u}{\|\mathbf{r}_v \times \mathbf{r}_u\|}. \quad \text{Downward unit normal vector}$$

- For an orientable surface given by

$$y = g(x,z) \quad \text{or} \quad x = g(y,z)$$

you can use the gradient

$$\nabla G(x,y,z) = -g_x(x,z)\,\mathbf{i} + \mathbf{j} - g_z(x,z)\,\mathbf{k} \qquad G(x,y,z) = y - g(x,z)$$

or

$$\nabla G(x,y,z) = \mathbf{i} - g_y(y,z)\,\mathbf{j} - g_z(x,z)\,\mathbf{k} \qquad G(x,y,z) = x - g(y,z)$$

to orient the surface.

Flux Integrals

- One of the principal applications involving the vector form of a surface integral relates to the flow of a fluid through a surface S. Suppose an oriented surface S is submerged in a fluid having a continuous velocity field $\mathbf{F}$.
- Let ΔS be the area of a small patch of the surface S over which $\mathbf{F}$ is nearly constant. Then the amount of fluid crossing this region per unit of time is approximated by the volume of the column of height $\mathbf{F} \cdot \mathbf{N}$, as shown in Figure 140. That is,

$$\Delta V = (\text{height})(\text{area of base}) = (\mathbf{F} \cdot \mathbf{N})\Delta S.$$

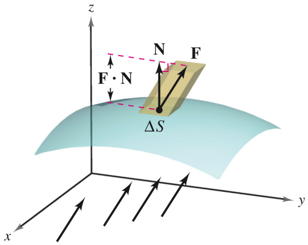


Figure 27: The velocity field **F** indicates the direction of the fluid flow.

- Consequently, the volume of fluid crossing the surface S per unit of time (called the flux of **F** across S) is given by the surface integral in the following definition.

Definition 15.11 (Definition of flux integral)

Let $\mathbf{F}(x, y, z) = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$, where M, N, and P have continuous first partial derivatives on the surface S oriented by a unit normal vector $\mathbf{N}$. The flux integral of $\mathbf{F}$ across S is given by

$$\iint_S \mathbf{F} \cdot \mathbf{N}\,\mathrm{d}S.$$

- Geometrically, a flux integral is the surface integral over S of the normal component of $\mathbf{F}$. If $\rho(x, y, z)$ is the density of the fluid at (x, y, z), the flux integral

$$\iint_S \rho \mathbf{F} \cdot \mathbf{N} \, \mathrm{d}S$$

represents the mass of the fluid flowing across S per unit of time.

- To evaluate a flux integral for a surface given by $z = g(x, y)$, let

$$G(x, y, z) = z - g(x, y).$$

- Then, $\mathbf{N} \, \mathrm{d}S$ can be written as follows.

$$\begin{aligned} \mathbf{N} \, \mathrm{d}S &= \frac{\nabla G(x, y, z)}{\|\nabla G(x, y, z)\|} \, \mathrm{d}S \\ &= \frac{\nabla G(x, y, z)}{\sqrt{(g_x)^2 + (g_y)^2 + 1}} \sqrt{(g_x)^2 + (g_y)^2 + 1} \, \mathrm{d}A \\ &= \nabla G(x, y, z) \, \mathrm{d}A \end{aligned}$$

Theorem 15.11 (Evaluating a flux integral)

Let S be an oriented surface given by $z = g(x, y)$ and let R be its projection onto the xy-plane.

$$\iint_S \mathbf{F} \cdot \mathbf{N} \, \mathrm{d}S = \iint_R \mathbf{F} \cdot [-g_x(x, y)\,\mathbf{i} - g_y(x, y)\,\mathbf{j} + \mathbf{k}] \, \mathrm{d}A \qquad \textit{Upward}$$

$$\iint_S \mathbf{F} \cdot \mathbf{N} \, \mathrm{d}S = \iint_R \mathbf{F} \cdot [g_x(x, y)\,\mathbf{i} + g_y(x, y)\,\mathbf{j} - \mathbf{k}] \, \mathrm{d}A \qquad \textit{Downward}$$

For the first integral, the surface is oriented upward, and for the second integral, the surface is oriented downward.

Example 6 (Finding the Flux of an Inverse Square Field)

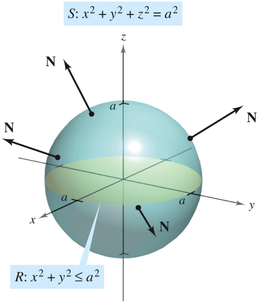
Find the flux over the sphere S given by

$$x^2 + y^2 + z^2 = a^2 \quad \text{Sphere } S$$

where $\mathbf{F}$ is an inverse square field given by

$$\mathbf{F}(x, y, z) = \frac{kq}{\|\mathbf{r}\|^2} \frac{\mathbf{r}}{\|\mathbf{r}\|} = \frac{kq\mathbf{r}}{\|\mathbf{r}\|^3} \quad \text{Inverse square field } \mathbf{F}$$

and $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Assume S is oriented outward, as shown in Figure below


S: x² + y² + z² = a²
z
N
a
N
N
a
a
y
x
N
R: x² + y² ≤ a²

- The result in Example 6 shows that the flux across a sphere S in an inverse square field is independent of the radius of S. In particular, if $\mathbf{E}$ is an electric field, the result in Example 6, along with Coulomb's Law, yields one of the basic laws of electrostatics,known as Gauss's Law:

$$\iint_S \mathbf{E} \cdot \mathbf{N} \, \mathrm{d}S = 4\pi kq \quad \text{Gauss's Law}$$

  where q is a point charge located at the center of the sphere and k is the Coulomb constant.
- Gauss's Law is valid for more general closed surfaces that enclose the origin, and relates the flux out of the surface to the total charge q inside the surface.
- Surface integrals are also used in the study of heat flow.

- Heat flows from areas of higher temperature to areas of lower temperature in the direction of greatest change. As the result, measuring heat flux involves the gradient of the temperature.
- So, assume that the heat flux across a portion of the surface of area ΔS is given by $\Delta H \approx -k \nabla T \cdot \mathbf{N} \, \mathrm{d}S$, where T is the temperature, $\mathbf{N}$ is the unit normal vector to the surface in the direction of the heat flow, and k is the thermal diffusivity of the material.
- The heat flux across the surface is given by

$$H = \iint_S -k \nabla T \cdot \mathbf{N} \, \mathrm{d}S. \qquad \text{Heat flux across } S$$

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Divergence Theorem

- Recall from Section 4 that an alternative form of Green's Theorem is

$$\int_C \mathbf{F} \cdot \mathbf{N} \, \mathrm{d}s = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \mathrm{d}A$$
$$= \iint_R \operatorname{div} \mathbf{F} \, \mathrm{d}A.$$

- In an analogous way, the Divergence Theorem gives the relationship between a triple integral over a solid region Q and a surface integral over the surface of Q.
- In the statement of the theorem, the surface S is closed in the sense that it forms the complete boundary of the solid Q.

- Regions bounded by spheres, ellipsoids, cubes, tetrahedrons, or combinations of these surfaces are typical examples of closed surfaces.
- Assume that Q is a solid region on which a triple integral can be evaluated, and that the closed surface S is oriented by outward unit normal vectors, as shown in Figure 28.

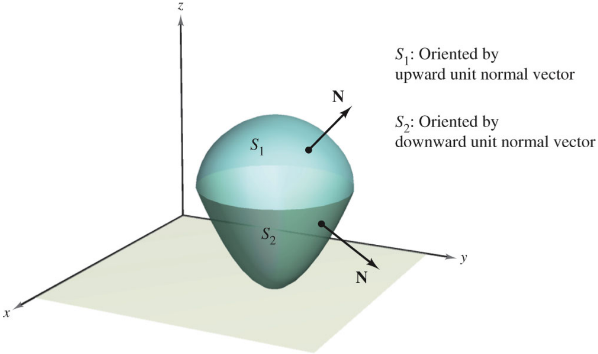

Figure 28: Outward.

Theorem 15.12 (The divergence theorem)

Let Q be a solid region bounded by a closed surface S oriented by a unit normal vector directed outward from Q. If $\mathbf{F}$ is a vector field whose component functions have continuous first partial derivatives in Q, then

$$\iint_S \mathbf{F} \cdot \mathbf{N} \, \mathrm{d}S = \iiint_Q \operatorname{div} \mathbf{F} \, \mathrm{d}V.$$

- Even though the Divergence Theorem was stated for a simple solid region Q bounded by a closed surface, the theorem is also valid for regions that are the finite unions of simple solid regions.
- For example, let Q be the solid bounded by the closed surfaces S_1 and S_2, as shown in Figure 29. To apply the Divergence Theorem to this solid, let $S = S_1 \cup S_2$.

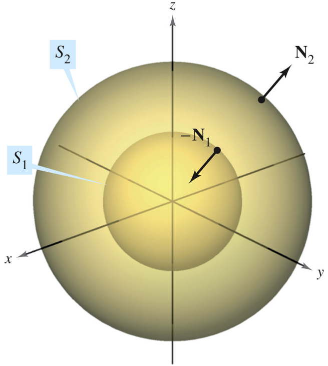

Figure 29: Simple solid regions.

- The normal vector $\mathbf{N}$ to S is given by $-\mathbf{N}_1$ on $S1$ and by $\mathbf{N}_2$ on S_2. So, you can write

$$\begin{aligned}\iiint_Q \operatorname{div}\mathbf{F}\,\mathrm{d}V &= \iint_S \mathbf{F}\cdot\mathbf{N}\,\mathrm{d}S\\ &= \iint_{S_1} \mathbf{F}\cdot(-\mathbf{N}_1)\,\mathrm{d}S + \iint_{S_2} \mathbf{F}\cdot\mathbf{N}_2\,\mathrm{d}S\\ &= -\iint_{S_1} \mathbf{F}\cdot\mathbf{N}_1\,\mathrm{d}S + \iint_{S_2} \mathbf{F}\cdot\mathbf{N}_2\,\mathrm{d}S\end{aligned}$$

Example 1 (Using the Divergence Theorem)

Let Q be the solid region bounded by the coordinate planes and the plane $2x + 2y + z = 6$, and let $\mathbf{F} = x\,\mathbf{i} + y^2\,\mathbf{j} + z\,\mathbf{k}$. Find

$$\iint_S \mathbf{F} \cdot \mathbf{N}\,\mathrm{d}S$$

where S is the surface of Q.

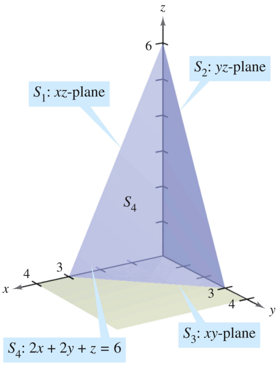

Figure 30: Solid region bounded by the coordinate planes.

Example 2 (Verifying the Divergence Theorem)

Let Q be the solid region between the paraboloid

$$z = 4 - x^2 - y^2$$

and the xy-plane. Verify the Divergence Theorem for

$$\mathbf{F}(x, y, z) = 2z\,\mathbf{i} + x\,\mathbf{j} + y^2\,\mathbf{k}.$$

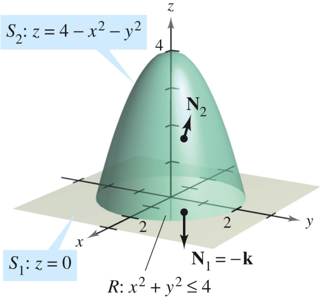

Figure 31: paraboloid.

Flux and the Divergence Theorem

- To help understand the Divergence Theorem, consider the two sides of the equation

$$\iint_S \mathbf{F} \cdot \mathbf{N} \, \mathrm{d}S = \iiint_Q \operatorname{div} \mathbf{F} \, \mathrm{d}V.$$

- You know from Section 6 that the flux integral on the left determines the total fluid flow across the surface S per unit of time. This can be approximated by summing the fluid flow across small patches of the surface.
- The triple integral on the right measures this same fluid flow across S, but from a very different perspective-namely, by calculating the flow of fluid into (or out of) small cubes of volume ΔV_i.

- The flux of the ith cube is approximately

$$\text{Flux of } i\text{th cube} \approx \operatorname{div} \mathbf{F}(x_i, y_i, z_i)\Delta V_i$$

  for some point (x_i, y_i, z_i) in the ith cube.
- Note that for a cube in the interior of Q, the gain (or loss) of fluid through any one of its six sides is offset by a corresponding loss (or gain) through one of the sides of an adjacent cube.
- After summing over all the cubes in Q, the only fluid flow that is not canceled by adjoining cubes is that on the outside edges of the cubes on the boundary. So, the sum

$$\sum_{i=1}^{n} \operatorname{div} \mathbf{F}(x_i, y_i, z_i)\Delta V_i$$

  approximates the total flux into (or out of) Q, and therefore through the surface S.

- To see what is meant by the divergence of **F** at a point, consider ΔV_α to be the volume of a small sphere S_α of radius α and center (x_0, y_0, z_0), contained in region Q, as shown in Figure 32.

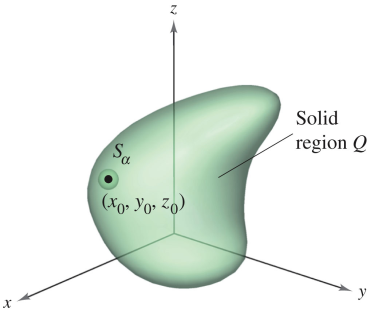

Figure 32: Solid region Q.

- Applying the Divergence Theorem to S_α produces

$$\begin{aligned}\text{Flux of } \mathbf{F} \text{ across } S_\alpha &= \iiint_{Q_\alpha} \operatorname{div} \mathbf{F} \, \mathrm{d}V \\ &\approx \operatorname{div} \mathbf{F}(x_0, y_0, z_0) \Delta V_\alpha\end{aligned}$$

where Q_α is the interior of S_α.

- Consequently, you have

$$\operatorname{div} \mathbf{F}(x_0, y_0, z_0) \approx \frac{\text{flux of } \mathbf{F} \text{ across } S_\alpha}{\Delta V_\alpha}$$

and, by taking the limit as $\alpha \to 0$, you obtain the divergence of $\mathbf{F}$ at the point (x_0, y_0, z_0).

$$\begin{aligned}\operatorname{div} \mathbf{F}(x_0, y_0, z_0) &= \lim_{\alpha \to 0} \frac{\text{flux of } \mathbf{F} \text{ across } S_\alpha}{\Delta V_\alpha} \\ &= \text{flux per unit volume at } (x_0, y_0, z_0)\end{aligned}$$

- The point (x_0, y_0, z_0) in a vector field is classified as a source, a sink, or incompressible, as follows.

1. source, for $\operatorname{div} \mathbf{F} > 0$ See Figure 33(a)
2. sink, for $\operatorname{div} \mathbf{F} < 0$ See Figure 33(b)
3. incompressible, for $Div\mathbf{F} = 0$ See Figure 33(c)

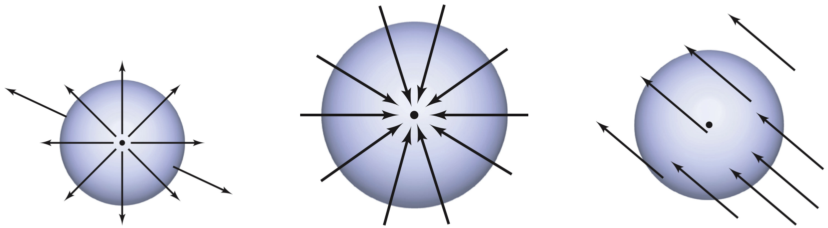

Figure 33: A vector field is classified as a source, a sink, or incompressible.

Example 4 (Calculating Flux by the Divergence Theorem)

Let Q be the region bounded by the sphere $x^2 + y^2 + z^2 = 4$. Find the outward flux of the vector field

$$\mathbf{F}(x, y, z) = 2x^3\,\mathbf{i} + 2y^3\,\mathbf{j} + 2z^3\,\mathbf{k}$$

through the sphere.

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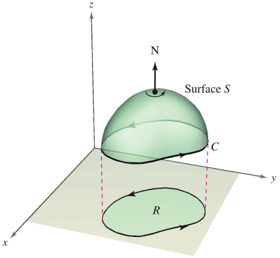
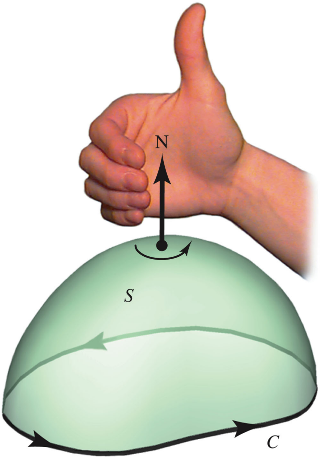
Stokes's Theorem

- A second higher-dimension analog of Green's Theorem is called Stokes's Theorem which gives the relationship between a surface integral over an oriented surface S and a line integral along a closed space curve C forming the boundary of S, as shown in Figure below.
- The positive direction along C is counterclockwise relative to the normal vector **N**.



- That is, if you imagine grasping the normal vector **N** with your right hand, with your thumb pointing in the direction of **N**, your fingers will point in the positive direction C, as shown in Figure below.



Theorem 15.13 (Stokes's Theorem)

Let S be an oriented surface with unit normal vector $\mathbf{N}$, bounded by a piecewise smooth simple closed curve C with a positive orientation. If $\mathbf{F}$ is a vector field whose component functions have continuous first partial derivatives on an open region containing S and C, then

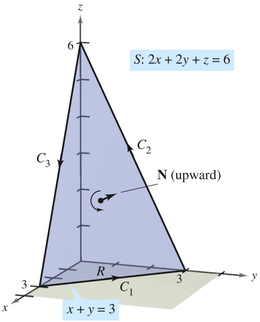
$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{N} \, \mathrm{d}S.$$

Example 1 (Using Stokes's Theorem)

Let C be the oriented triangle lying in the plane $2x + 2y + z = 6$, as shown in Figure below. Evaluate

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

where $\mathbf{F}(x, y, z) = -y^2\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$.

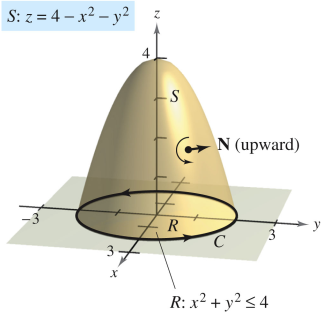


Example 2 (Verifying Stokes's Theorem)

Let S be the portion of the paraboloid $z = 4 - x^2 - y^2$ lying above the xy-plane, oriented upward (see Figure below). Let C be its boundary curve in the xy-plane, oriented counterclockwise. Verify Stokes's Theorem for

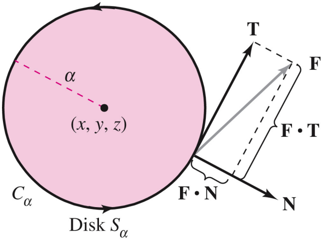
$$\mathbf{F}(x, y, z) = 2z\,\mathbf{i} + x\,\mathbf{j} + y^2\,\mathbf{k}$$

by evaluating the surface integral and the equivalent line integral.



Physical Interpretation of Curl

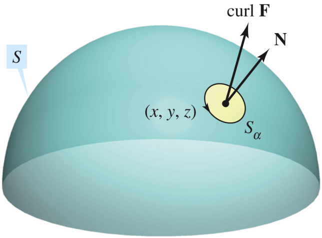
- Stokes's Theorem provides insight into a physical interpretation of curl. In a vector field $\mathbf{F}$, let S, be a small circular disk of radius α, centered at (x, y, z) and with boundary C_α, as shown in Figure below.
- At each point on the circle C_α, $\mathbf{F}$ has a normal component $\mathbf{F} \cdot \mathbf{N}$ and a tangential component $\mathbf{F} \cdot \mathbf{T}$.



- The more closely **F** and **T** are aligned, the greater the value of $\mathbf{F} \cdot \mathbf{T}$. So, a fluid tends to move along the circle rather than across it.
- Consequently, you say that the line integral around C_α, measures the circulation of **F** around C_α. That is,

$$\int_{C_\alpha} \mathbf{F} \cdot \mathbf{T} \, ds = \text{circulation of } \mathbf{F} \text{ around } C_\alpha.$$

- Now consider a small disk S_α, to be centered at some point (x, y, z) on the surface S, as shown in Figure below.



- On such a small disk, curl $\mathbf{F}$ is nearly constant, because it varies little from its value at (x, y, z). Moreover, curl $\mathbf{F} \cdot \mathbf{N}$ is also nearly constant on S_α, because all unit normals to S, are about the same.
- Consequently, Stokes's Theorem yields

$$\begin{aligned}\int_{C_\alpha} \mathbf{F} \cdot \mathbf{T} \, \mathrm{d}s &= \iint_{S_\alpha} (\operatorname{curl} \mathbf{F}) \cdot \mathbf{N} \, \mathrm{d}S \\ &\approx (\operatorname{curl} \mathbf{F}) \cdot \mathbf{N} \iint_{S_\alpha} \mathrm{d}S \\ &\approx (\operatorname{curl} \mathbf{F}) \cdot \mathbf{N} (\pi\alpha^2).\end{aligned}$$

So,

$$\begin{aligned}(\operatorname{curl} \mathbf{F}) \cdot \mathbf{N} &\approx \frac{\int_{C_\alpha} \mathbf{F} \cdot \mathbf{T} \, \mathrm{d}s}{\pi\alpha^2} \\ &= \frac{\text{circulation of } \mathbf{F} \text{ around } C_\alpha}{\text{area of disk } S_\alpha} \\ &= \text{rate of circulation.}\end{aligned}$$

- Assuming conditions are such that the approximation improves for smaller and smaller disks ($\alpha \to 0$), it follows that

$$(\text{curl}\,\mathbf{F}) \cdot \mathbf{N} = \lim_{\alpha \to 0} \frac{1}{\pi \alpha^2} \int_{C_\alpha} \mathbf{F} \cdot \mathbf{T}\, \mathrm{d}s$$

which is referred to as the rotation of **F** about **N**. That is,

$$\text{curl}\,\mathbf{F}(x, y, z) \cdot \mathbf{N} = \text{rotation of } \mathbf{F} \text{ about } \mathbf{N} \text{ at}(x, y, z).$$

- In this case, the rotation of **F** is maximum when curl **F** and **N** have the same direction.

- Normally, this tendency to rotate will vary from point to point on the surface S, and Stokes's Theorem

$$\underbrace{\iint_S (\text{curl}\,\mathbf{F}) \cdot \mathbf{N}\, \mathrm{d}S}_{\text{Surface integral}} = \underbrace{\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}}_{\text{Line integral}}$$

says that the collective measure of this rotational tendency taken over the entire surface S (surface integral) is equal to the tendency of a fluid to circulate around the boundary C (line integral).