1. $f(x) = \frac{1}{1-x}$. Find the interval of convergence for each of the following.

a. $f'(x)$ b. $f''(x)$

a. $f(x) = \frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{k=0}^{\infty} x^k$, $|x|<1$

$\Rightarrow f'(x) = \left(\sum_{k=0}^{\infty} x^k\right)' = \left(\sum_{k=1}^{\infty} x^k\right)' = \sum_{k=1}^{\infty} k \cdot x^{k-1}$, $|x|<1$

For $x=1$, $\sum_{k=1}^{\infty} k$ is div.

For $x=-1$, $\sum_{k=1}^{\infty} (-1)^k \cdot k$ is div.

$\Rightarrow$ interval of conv. $(-1,1)$

b.

$f''(x) = (f'(x))'$, $|x|<1$

$= \left(\sum_{k=1}^{\infty} k \cdot x^{k-1}\right)' = \sum_{k=1}^{\infty} (k \cdot x^{k-1})'$, $|x|<1$

$= \sum_{k=2}^{\infty} k(k-1)x^{k-2}$, $|x|<1$

For $x=1$, $\sum_{k=2}^{\infty} k \cdot (k-1)$ div.

For $x=-1$, $\sum_{k=2}^{\infty} (-1)^k \cdot k \cdot (k-1)$ div.

$\Rightarrow$ interval of conv. $(-1,1)$.

2. Find a power series for $f(x) = \frac{1}{6-x}$, centered at $x=1$

$f(x) = \frac{1}{6-x} = \frac{1}{6-(x-1)-1} = \frac{1}{5-(x-1)}$. Let $u = x-1$

$f(x) = \frac{1}{5-u} = \frac{1}{5(1-\frac{u}{5})} = \frac{1}{5}\left[1 + \frac{u}{5} + \left(\frac{u}{5}\right)^2 + \cdots\right]$, $\left|\frac{1}{5}u\right|<1$

$= \frac{1}{5}\left[1 + \frac{1}{5}(x-1) + \frac{1}{5^2}(x-1)^2 + \cdots\right]$, $\left|\frac{1}{5}(x-1)\right|<1$.

3. Find a power series for $f(x) = e^x \sin x$ centered at $x = 0$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\Rightarrow e^x \sin x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3} + \frac{x^5}{5!} + \cdots\right)$$

$$= x + x^2 + \left(-\frac{1}{3!} + \frac{1}{2!}\right)x^3 + \left(-\frac{1}{3!} + \frac{1}{3!}\right)x^4 + \left(\frac{1}{5!} - \frac{1}{2!3!} + \frac{1}{4!}\right) + \cdots$$

$$= x + x^2 + \frac{x^3}{3!} + 0 - \frac{x^5}{30} + \cdots$$