1° $a_n = \frac{1}{n(\ln n)^p}$, $p > 0$.

1° Let $f(x) = \frac{1}{x(\ln x)^p}$

$$\int_2^\infty f(x)\,dx = \int_2^\infty \frac{1}{x(\ln x)^p}\,dx$$

$$= \int_{\ln 2}^\infty u^{-p}\,du$$

2° $\because \sum_{k=1}^{\infty} \frac{1}{k^p} \begin{cases} \text{Conv}, & \text{where } p > 1 \\ \text{div}, & \text{where } 0 < p \le 1 \end{cases}$ (by p-series)

$\therefore \int_{\ln 2}^\infty u^{-p}\,du \begin{cases} \text{Conv}, & \text{where } p > 0 \\ \text{div}, & \text{where } 0 < p \le 1 \end{cases}$ (by intergal test)

$\therefore \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \begin{cases} \text{Conv}, & \text{where } p > 0 \\ \text{div}, & \text{where } 0 < p \le 1 \end{cases}$ (by intergal test)

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2. $\sum_{n=1}^{\infty} \frac{1}{n!}$

Let $b_n = \frac{1}{n(n-1)}$, $a_n = \frac{1}{n!}$

$$\sum_{n=2}^{k} \frac{1}{n(n-1)} = \sum_{n=2}^{k} \frac{1}{n-1} - \frac{1}{n}$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{k-1} - \frac{1}{k}\right)$$

$$= 1 - \frac{1}{k}$$

$\because \sum_{n=1}^{\infty} \frac{1}{n(n-1)} = \lim_{k\to\infty} \sum_{n=2}^{k} \frac{1}{n(n-1)} = \lim_{k\to\infty} 1 - \frac{1}{k} = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n-1)}$ is conv.

$\therefore \sum_{n=1}^{\infty} \frac{1}{n!}$ is conv.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$

1° $\ln n < n^{3/2}$

$\Rightarrow \dfrac{\ln n}{n^{3/2}} < \dfrac{n^{\alpha}}{n^{3/2}} < \dfrac{1}{n^{3/2-\alpha}}$

If $3/2 - \alpha > 1 \Rightarrow \alpha < \frac{1}{2}$

Take $\alpha = \frac{1}{4}$

$\Rightarrow \dfrac{\ln n}{n^{3/2}} < \dfrac{1}{n^{5/4}}$

$\because \sum \dfrac{1}{n^{5/4}}$ is conv.

$\therefore \dfrac{\ln n}{n^{3/2}}$ is conv.

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4. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

$\because a_{n+1} < a_n$, $\forall n \in \mathbb{N}$.

$\therefore \displaystyle\lim_{n\to\infty} \frac{1}{n} = 0$.

$\therefore$ by alternating series test , $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$ is conv