Example 1 (Sketching a plane curve)
Sketch the plane curve represented by the vector-valued function

$$\mathbf{r}(t) = 3\sin t\mathbf{i} + 2\cos t\mathbf{j}, \quad 0 \le t \le 2\pi.$$

Find the rectangular equation by eliminating the parameter t, and indicate the orientation of the curve.

Solution:
1. Eliminate the parameter t:
The component functions are given by the parametric equations:

$$x = 3\sin t \implies \sin t = \frac{x}{3}$$

$$y = 2\cos t \implies \cos t = \frac{y}{2}$$

Using the trigonometric identity $\sin^2 t + \cos^2 t = 1$:

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1 \implies \frac{x^2}{9} + \frac{y^2}{4} = 1$$

The graph is an **ellipse** centered at the origin with a major axis of length 6 (along the x-axis) and a minor axis of length 4 (along the y-axis).

2. Determine the orientation:
By testing several values of t:

- At $t = 0$: $\mathbf{r}(0) = 3(0)\mathbf{i} + 2(1)\mathbf{j} = (0, 2)$
- At $t = \frac{\pi}{2}$: $\mathbf{r}(\frac{\pi}{2}) = 3(1)\mathbf{i} + 2(0)\mathbf{j} = (3, 0)$
- At $t = \pi$: $\mathbf{r}(\pi) = 3(0)\mathbf{i} + 2(-1)\mathbf{j} = (0, -2)$

As t increases from 0 to 2π, the terminal point of the position vector $\mathbf{r}(t)$ traces the ellipse in a **clockwise** direction.

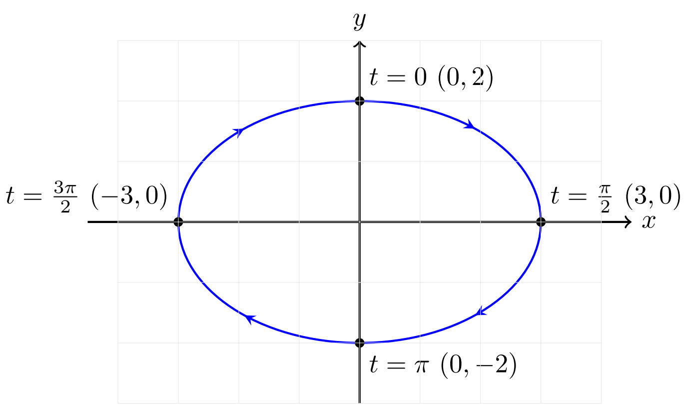

Figure 1: The ellipse is traced clockwise as t increases.

2. Convert the point from cylindrical coordinates to rectangular coordinates

$$(2, -\pi, -4)$$

Solution:
Given the cylindrical coordinates $(r, \theta, z) = (2, -\pi, -4)$.
The formulas to convert from cylindrical to rectangular coordinates (x, y, z) are:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

Substitute $r = 2$, $\theta = -\pi$, and $z = -4$ into the formulas:

$$\begin{aligned} x &= 2\cos(-\pi) = 2(-1) = -2 \\ y &= 2\sin(-\pi) = 2(0) = 0 \\ z &= -4 \end{aligned}$$

Therefore, the point in rectangular coordinates is $(-2, 0, -4)$.

Problem 3: Find an equation in rectangular coordinates for the surface represented by the spherical equation, and explain its graph.

$$\phi = \frac{\pi}{2}$$

Solution:

In spherical coordinates (ρ, θ, ϕ), the relationship between the spherical coordinates and the rectangular coordinate z is given by:

$$z = \rho \cos \phi$$

Given the spherical equation $\phi = \frac{\pi}{2}$, we substitute this value into the conversion formula:

$$z = \rho \cos \left(\frac{\pi}{2}\right)$$

Since $\cos\left(\frac{\pi}{2}\right) = 0$, we have:

$$z = \rho \cdot 0 = 0$$

Therefore, the corresponding equation in rectangular coordinates is:

$$z = 0$$

Explanation of the graph:

The equation $z = 0$ represents the xy-plane. In the spherical coordinate system, the angle ϕ is measured from the positive z-axis. When $\phi = \frac{\pi}{2}$ (or $90°$), it describes the set of all points that are orthogonal to the z-axis, which precisely forms the horizontal plane passing through the origin (the xy-plane).