Calculus — Quiz 2 Solutions

1. (8%) Coordinate Systems

(a) (3%) Cylindrical to rectangular

Sol. For cylindrical coordinates, $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$. Thus

$$x = 4\cos\frac{5\pi}{6} = -2\sqrt{3}, \qquad y = 4\sin\frac{5\pi}{6} = 2, \qquad z = -2.$$

Therefore the rectangular coordinates are

$$\boxed{(-2\sqrt{3},\ 2,\ -2).}$$

(b) (3%) Rectangular to cylindrical and spherical

Sol.

$$r = \sqrt{x^2 + y^2} = \sqrt{3+1} = 2.$$

Since $(x, y) = (\sqrt{3}, -1)$ is in quadrant IV,

$$\theta = \frac{11\pi}{6}, \qquad z = 2.$$

So the cylindrical coordinates are

$$\boxed{\left(2, \frac{11\pi}{6}, 2\right).}$$

For spherical coordinates,

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{3+1+4} = 2\sqrt{2}, \qquad \cos\phi = \frac{z}{\rho} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

Thus $\phi = \pi/4$, and the spherical coordinates are

$$\boxed{\left(2\sqrt{2}, \frac{11\pi}{6}, \frac{\pi}{4}\right).}$$

(c) (2%) Spherical to cylindrical and rectangular

Sol. For spherical coordinates,

$$r = \rho\sin\phi = 6\sin\frac{\pi}{6} = 3, \qquad z = \rho\cos\phi = 6\cos\frac{\pi}{6} = 3\sqrt{3}.$$

Thus the cylindrical coordinates are

$$\boxed{\left(3, \frac{\pi}{3}, 3\sqrt{3}\right).}$$

For rectangular coordinates,

$$x = r\cos\theta = 3\cos\frac{\pi}{3} = \frac{3}{2}, \qquad y = r\sin\theta = 3\sin\frac{\pi}{3} = \frac{3\sqrt{3}}{2}.$$

Therefore

$$\boxed{\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}, 3\sqrt{3}\right).}$$

2. (12%) Vector-valued functions

(a) (4%) $\lim_{t\to 0^+} \mathbf{r}(t)$

Sol.

$$\lim_{t\to 0^+} \mathbf{r}(t) = \left\langle \lim_{t\to 0^+} t^2, \lim_{t\to 0^+} e^t, \lim_{t\to 0^+} \ln(t+1) \right\rangle = \boxed{\langle 0, 1, 0\rangle.}$$

(b) (4%) Domain and smoothness

Sol. The only restriction is $t+1>0$, so the domain is

$$\boxed{(-1,\infty).}$$

Also

$$\mathbf{r}'(t) = \left\langle 2t, e^t, \frac{1}{t+1} \right\rangle,$$

which exists and is continuous for $t > -1$. Since the second component e^t is never 0, $\mathbf{r}'(t) \neq \mathbf{0}$ on the whole domain. Thus

$$\boxed{\mathbf{r} \text{ is smooth on } (-1,\infty).}$$

(c) (4%) $\left.\dfrac{\mathrm{d}}{\mathrm{d}t}\big(\mathbf{r}(t)\cdot\mathbf{g}(t)\big)\right|_{t=1}$, where $\mathbf{g}(t) = \langle e^{-t}, t, t^2\rangle$

Sol.

$$\mathbf{r}'(t) = \left\langle 2t, e^t, \frac{1}{t+1} \right\rangle, \qquad \mathbf{g}'(t) = \langle -e^{-t}, 1, 2t\rangle.$$

Using the product rule for dot products,

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathbf{r}(t)\cdot\mathbf{g}(t)\big) = \mathbf{r}'(t)\cdot\mathbf{g}(t) + \mathbf{r}(t)\cdot\mathbf{g}'(t).$$

At $t=1$,

$$\mathbf{r}(1) = \langle 1, e, \ln 2\rangle, \quad \mathbf{r}'(1) = \left\langle 2, e, \frac{1}{2} \right\rangle,$$

$$\mathbf{g}(1) = \langle e^{-1}, 1, 1\rangle, \quad \mathbf{g}'(1) = \langle -e^{-1}, 1, 2\rangle.$$

Hence

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathbf{r}(t)\cdot\mathbf{g}(t)\big)\right|_{t=1} = \left(2e^{-1} + e + \frac{1}{2}\right) + \left(-e^{-1} + e + 2\ln 2\right).$$

Therefore

$$\boxed{e^{-1} + 2e + 2\ln 2 + \frac{1}{2}.}$$

3. (12%) Limits

(a) (6%) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sqrt{4+x^2+y^2}-2}{x^2+y^2}$

Sol. Rationalize the numerator:

$$\frac{\sqrt{4+x^2+y^2}-2}{x^2+y^2}\cdot\frac{\sqrt{4+x^2+y^2}+2}{\sqrt{4+x^2+y^2}+2}=\frac{1}{\sqrt{4+x^2+y^2}+2}.$$

Therefore

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{\sqrt{4+x^2+y^2}-2}{x^2+y^2}=\frac{1}{4}.}$$

(b) (6%) $\displaystyle\lim_{(x,y)\to(0,0)} \left(x^2+y^2\right)\sin\left(\frac{1}{x^2+y^2}\right)$

Sol. Since $|\sin u|\le 1$ for all u,

$$\left|\left(x^2+y^2\right)\sin\left(\frac{1}{x^2+y^2}\right)\right|\le x^2+y^2\to 0.$$

Also

$$\lim_{(x,y)\to(0,0)} 0=0,\qquad \lim_{(x,y)\to(0,0)}\left(x^2+y^2\right)=0.$$

Therefore, by the Squeeze Theorem,

$$\boxed{\lim_{(x,y)\to(0,0)} \left(x^2+y^2\right)\sin\left(\frac{1}{x^2+y^2}\right)=0.}$$

4. (18%) Ch13 differential block

(a) (6%) Partials and differentiability

Sol. By the definition of partial derivatives at the origin,

$$f_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{0}{h}=0,$$

and

$$f_y(0,0)=\lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k}=\lim_{k\to 0}\frac{0}{k}=0.$$

Now check continuity at the origin. Along $y=0$,

$$\lim_{x\to 0} f(x,0)=\lim_{x\to 0} 0=0,$$

so the limit along $y=0$ is 0. Along $y=x$,

$$\lim_{x\to 0} f(x,x)=\lim_{x\to 0}\frac{x^2}{x^2+x^2}=\frac{1}{2},$$

so the limit along $y=x$ is $\frac{1}{2}$. Since these two path limits are different,

$$\lim_{(x,y)\to(0,0)} f(x,y) \text{ does not exist.}$$

Therefore f is not continuous at $(0,0)$. Since differentiability implies continuity, f is not differentiable at $(0,0)$.

$$\boxed{f_x(0,0)=0,\quad f_y(0,0)=0,\quad f \text{ is not differentiable at } (0,0).}$$

(b) (6%) Chain rule

Sol.

$$z_x = e^y, \qquad z_y = xe^y.$$

At $(u, v) = (1, 1)$,

$$x = 1^2 + 1 = 2, \qquad y = 1 - 1^2 = 0,$$

so $z_x = 1$ and $z_y = 2$. Also

$$x_u = 2u = 2, \quad y_u = 1, \quad x_v = 1, \quad y_v = -2v = -2.$$

Thus

$$z_u = z_x x_u + z_y y_u = (1)(2) + (2)(1) = \boxed{4},$$

and

$$z_v = z_x x_v + z_y y_v = (1)(1) + (2)(-2) = \boxed{-3}.$$

(c) (6%) Tangent plane

Sol. Let

$$F(x, y, z) = x^2 - y^2 + z^2.$$

Then

$$\nabla F(x, y, z) = \langle 2x,\ -2y,\ 2z \rangle, \qquad \nabla F(1, 1, 1) = \langle 2, -2, 2 \rangle.$$

The tangent plane is

$$2(x - 1) - 2(y - 1) + 2(z - 1) = 0.$$

Therefore

$$\boxed{x - y + z = 1.}$$

5. (6%) Gradient and directional derivative

Sol.

$$\nabla f(x, y) = \left\langle \frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2} \right\rangle.$$

At $P = (1, 2)$,

$$\nabla f(1, 2) = \left\langle \frac{1}{3}, \frac{2}{3} \right\rangle.$$

The direction from P to Q is

$$\vec{PQ} = \langle 3, 4 \rangle, \qquad \mathbf{u} = \frac{\langle 3, 4 \rangle}{5}.$$

Thus

$$D_{\mathbf{u}} f(1, 2) = \nabla f(1, 2) \cdot \mathbf{u} = \left\langle \frac{1}{3}, \frac{2}{3} \right\rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle = \boxed{\frac{11}{15}}.$$

The steepest increase direction is the unit vector in the gradient direction:

$$\boxed{\frac{\nabla f(1, 2)}{\|\nabla f(1, 2)\|} = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle.}$$

The maximum rate of increase is

$$\boxed{\|\nabla f(1,2)\| = \frac{\sqrt{5}}{3}.}$$

6. (8%) Critical-point classification

Sol.

$$f_x = 3x^2 - 3, \qquad f_y = 2y - 4.$$

Critical points satisfy

$$x^2 = 1, \qquad y = 2.$$

Thus the critical points are

$$(-1,2), \qquad (1,2).$$

Now

$$f_{xx} = 6x, \qquad f_{yy} = 2, \qquad f_{xy} = 0,$$

so

$$D = f_{xx}f_{yy} - (f_{xy})^2 = 12x.$$

At $(-1,2)$, $D = -12 < 0$, so $(-1,2)$ is a saddle point.

At $(1,2)$, $D = 12 > 0$ and $f_{xx} = 6 > 0$, so $(1,2)$ is a local minimum.

$$\boxed{(-1,2) \text{ is a saddle point; } (1,2) \text{ is a local minimum.}}$$

7. (16%) Double integrals

(a) (5%) Reverse order and evaluate

Sol. The original region is

$$0 \le x \le 1, \qquad x \le y \le 1.$$

Equivalently, $0 \le y \le 1$ and $0 \le x \le y$. Therefore

$$\int_0^1 \int_x^1 2y \, \mathrm{d}y \, \mathrm{d}x = \int_0^1 \int_0^y 2y \, \mathrm{d}x \, \mathrm{d}y.$$

Evaluate:

$$\int_0^1 \int_0^y 2y \, \mathrm{d}x \, \mathrm{d}y = \int_0^1 2y[y] \, \mathrm{d}y = \int_0^1 2y^2 \, \mathrm{d}y = \boxed{\frac{2}{3}}.$$

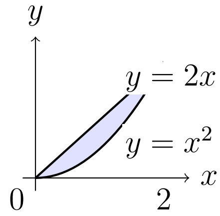
(b) (6%) Sketch or describe D, and evaluate $\iint_D x \, \mathrm{d}A$

Sol. The curves $y = x^2$ and $y = 2x$ intersect when

$$x^2 = 2x \quad \Longrightarrow \quad x = 0, 2.$$

On $0 \le x \le 2$, the line $y = 2x$ lies above the parabola $y = x^2$. Hence

$$D = \{(x,y) : 0 \le x \le 2,\ x^2 \le y \le 2x\}.$$



Thus
$$\iint_D x\,\mathrm{d}A = \int_0^2 \int_{x^2}^{2x} x\,\mathrm{d}y\,\mathrm{d}x = \int_0^2 x(2x - x^2)\,\mathrm{d}x.$$
Therefore
$$\int_0^2 (2x^2 - x^3)\,\mathrm{d}x = \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2 = \frac{16}{3} - 4 = \boxed{\frac{4}{3}}.$$

(c) (5%) Polar integral

Sol. For the first-quadrant annular region E, use
$$1 \le r \le 2, \qquad 0 \le \theta \le \frac{\pi}{2}.$$
Since $x^2 + y^2 = r^2$ and $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$,
$$\iint_E (x^2 + y^2)\,\mathrm{d}A = \int_0^{\pi/2} \int_1^2 r^2 \cdot r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{\pi/2} \int_1^2 r^3\,\mathrm{d}r\,\mathrm{d}\theta.$$
Thus
$$\int_0^{\pi/2} \left[\frac{r^4}{4}\right]_1^2 \mathrm{d}\theta = \int_0^{\pi/2} \frac{15}{4}\,\mathrm{d}\theta = \boxed{\frac{15\pi}{8}}.$$

8. (6%) Surface area

Sol. For $z = f(x, y) = 2x + 3y$,
$$f_x = 2, \qquad f_y = 3.$$
Thus
$$S = \int_0^1 \int_0^2 \sqrt{1 + f_x^2 + f_y^2}\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \int_0^2 \sqrt{14}\,\mathrm{d}y\,\mathrm{d}x.$$
The rectangle has area 2, so
$$\boxed{S = 2\sqrt{14}.}$$

9. (6%) Volume by double integral

Sol. The required volume is
$$V = \int_0^1 \int_0^2 (6 - x - 2y)\,\mathrm{d}y\,\mathrm{d}x.$$
Evaluate the inner integral:
$$\int_0^2 (6 - x - 2y)\,\mathrm{d}y = \left[(6 - x)y - y^2\right]_0^2 = 12 - 2x - 4 = 8 - 2x.$$

Therefore

$$V = \int_0^1 (8-2x)\,\mathrm{d}x = \left[8x - x^2\right]_0^1 = \boxed{7}.$$

10. (8%) Triple integral in cylindrical coordinates

Sol. The solid lies above the xy-plane and below $z = 9 - r^2$. The intersection with the xy-plane is

$$0 = 9 - r^2 \quad \Longrightarrow \quad r = 3.$$

Thus

$$V = \int_0^{2\pi} \int_0^3 \int_0^{9-r^2} r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta.$$

Evaluate:

$$V = 2\pi \int_0^3 r(9-r^2)\,\mathrm{d}r = 2\pi \left[\frac{9r^2}{2} - \frac{r^4}{4}\right]_0^3 = 2\pi \left(\frac{81}{2} - \frac{81}{4}\right).$$

$$\boxed{V = \frac{81\pi}{2}.}$$