1. (12%) Let $\boldsymbol{r}(\mathrm{t}) = \frac{1}{\sqrt{t+1}}\mathbf{i} + \frac{\sin(2t)}{t}\mathbf{j} + \frac{1}{t}\mathbf{k}$, $\boldsymbol{g}(\mathrm{t}) = \sqrt{1+t}\mathbf{i} + t\mathbf{j} + \frac{t}{t-2}\mathbf{k}$:

(a) Evaluate the limit $\lim_{t\to 0} \boldsymbol{r}(t)$

(b) Find the intervals on which the curve given by $\boldsymbol{g}(t)$ is smooth

(c) Compute $\frac{\mathrm{d}}{\mathrm{dt}}[\boldsymbol{r}(t)\cdot\boldsymbol{g}(t)]$

Ans:

(a) Compute the limit component-wise:

$\lim_{t\to 0}\frac{1}{\sqrt{t+1}} = 1$, $\lim_{t\to 0}\frac{\sin(2t)}{t}$ =(L'Hospital rule) $\lim_{t\to 0}\frac{2\cos(2t)}{1} = 2$, $\lim_{t\to 0}\frac{1}{t}$ diverges.

The original limit does not exist, because $\lim_{t\to 0}\frac{1}{t} = \pm\infty$ does not exist.

(Its norm $\|\boldsymbol{r}(t)\|$ tends to $\pm\infty$ but we don't write $\lim_{t\to 0}\mathbf{r}(\mathrm{t}) = \pm\infty$. Since $\pm\infty$ is not a vector.)

(b) $\boldsymbol{g}'(t) = \frac{1}{2\sqrt{1+t}}\mathbf{i} + \mathbf{j} + \frac{-2}{(t-2)^2}\mathbf{k}$

Smoothness requires that every component derivative be continuous and that the derivatives **<u>are not all zero simultaneously</u>**.

For $\boldsymbol{g}'(t)$, the continuity holds in its domain when $\mathrm{t}+1>0$ and $\mathrm{t}\neq 2$.

Since the components function of $\boldsymbol{g}'(t)$ **<u>can not be all zeros</u>**.

The function is smooth on (-1,2) and $(2,\infty)$

(c) $\frac{\mathrm{d}}{\mathrm{dt}}[\boldsymbol{r}(t)\cdot\boldsymbol{g}(t)] = \frac{\mathrm{d}}{\mathrm{dt}}\left[\frac{1}{\sqrt{t+1}}\sqrt{1+t} + \frac{\sin(2t)}{t}t + \frac{1}{t}\frac{t}{t-2}\right] = \frac{\mathrm{d}}{\mathrm{dt}}\left[1+\sin(2t) + \frac{1}{t-2}\right] = 2\cos(2t) - \frac{1}{(t-2)^2}$

[Optional: where t is in (-1,0), (0,2) and$(2,\infty)$ (Since when $\mathrm{t}=0, \boldsymbol{r}(t)$ is undefined and when $t=2$, $\boldsymbol{g}(t)$ is undefined)]

2. (8%) Find the following limits:

(a) $\lim_{(x,y)\to(0,0)}\frac{2x}{x^2-y^2}$

(b) $\lim_{(x,y)\to(0,0)}\frac{(x^2+y^2)}{4}\cdot\ln(x^2+y^2)$

Ans:

(a) Take two different approaches to (0,0):

Along the x-axis $(y=0)$

$$\lim_{(x,y)\to(0,0)} \frac{2x}{x^2 - y^2} = \lim_{(x,0)\to(0,0)} \frac{2x}{x^2} = \lim_{x\to 0} \frac{2}{x} = \pm\infty$$

Along the y-axis ($x = 0$)

$$\lim_{(x,y)\to(0,0)} \frac{2x}{x^2 - y^2} = \lim_{(0,y)\to(0,0)} \frac{0}{-y^2} = 0$$

Because the function approaches different (or unbounded) values along different approaches to (0,0), the limit does not exist.

(b) Let $x = rcos(\theta), y = rsin(\theta) \to r^2 = x^2 + y^2$

$$\lim_{(x,y)\to(0,0)} \frac{(x^2 + y^2)}{4} \cdot \ln(x^2 + y^2) = \lim_{r\to 0} \frac{r^2}{4} \ln r^2 = \frac{1}{2} \lim_{r\to 0} \frac{\ln r}{r^{-2}}$$

$$= (\text{L'Hospital rule}) \frac{1}{2} \lim_{r\to 0} \frac{r^{-1}}{-2r^{-3}} = \frac{1}{2} \lim_{r\to 0} \frac{r^2}{-2} = 0$$

3. (15%)

(a) Let $f(x,y) = \begin{cases} \frac{x^2 y}{x^4 + y^2} & \text{when } (x,y) \neq (0,0) \\ 0 & \text{when } (x,y) = (0,0) \end{cases}$, compute $f_x(0,0)$ and $f_y(0,0)$.

In addition, decide whether f differentiable at $(0,0)$

(b) Given the equation $xz^2 - ysin(z) = 0$, find the first–order partial derivatives $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ using implicit differentiation.

(c) Considering the level surface defined by $z^5 + (sin(x))z^3 + yz = 4$. Find an equation of the tangent plane at the point $(0,3,1)$

Ans:

(a) For $(x,y) = (0,0)$:

$$f_x(x,y) = \lim_{\Delta x\to 0} \frac{f(0 + \Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{0 - 0}{(\Delta x)^4} \frac{1}{\Delta x} = \lim_{\Delta x\to 0} \frac{0}{(\Delta x)^5} = 0$$

$$f_y(x,y) = \lim_{\Delta y\to 0} \frac{f(0,0 + \Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y\to 0} \frac{0 - 0}{(\Delta y)^2} \frac{1}{\Delta y} = 0$$

On the other hand, let $y = mx^2$, $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2} = \lim_{x\to 0} \frac{x^2 m x^2}{x^4 + m^2 x^4} = \lim_{x\to 0} \frac{m}{1 + m^2}$.

which means that if we follow the trajectory of different line $y = mx^2$ to approach (0,0) we will get different value for different m, therefore, the limit does not exist. So $f(x,y)$ is not continuous at $(0,0)$. Therefore, it is not differentiable at $(0,0)$.

(b) Let $F(x,y,z) = xz^2 - ysin(z) = 0$

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-z^2}{2xz - y\cos(z)}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{\sin(z)}{2xz - y\cos(z)}$$

(c) Let $F(x, y, z) = z^5 + (sin(x))z^3 + yz - 4$

$$\nabla F = cos(x)\, z^3\boldsymbol{i} + z\boldsymbol{j} + (5z^4 + 3(sin(x))z^2 + y)\boldsymbol{k}$$

$$\nabla F(0,3,1) = 1\boldsymbol{i} + 1\boldsymbol{j} + 8\boldsymbol{k}$$

$$(x - 0) + (y - 3) + 8(z - 1) = 0$$

$$x + y + 8z = 11$$

4. (10%) Let $f(x, y) = e^{-x}\cos(y)$
   (a) Compute the directional derivative of f at (1,0) in the direction from $P(1,0)$ to $Q(6,12)$
   (b) Find the direction in which f has minimum increase at (1,0). What is the minimum rate of increase?

Ans:

(a)

$$\nabla f = -e^{-x}cos(y)\mathbf{i} - e^{-x}sin(y)\mathbf{j}$$

$\overrightarrow{PQ} = (6,12) - (1,0) = 5\mathbf{i} + 12\mathbf{j}$. u $= \frac{5}{13}\mathbf{i} + \frac{12}{13}\mathbf{j}$. $D_u f(1,0) = \nabla f(1,0) \cdot u =$

$(-e^{-1}, 0) \cdot \left(\frac{5}{13}, \frac{12}{13}\right) = \frac{-5}{13e}$

(b) $\nabla f(1,0) = -e^{-1}\mathbf{i} + 0\mathbf{j} = -\frac{1}{e}\mathbf{i}$

$$||\nabla f(1,0)|| = \sqrt{(-\frac{1}{e})^2 + 0^2} = \frac{1}{e}$$

The direction that has minimum increase $-\nabla f(1,0) = \frac{1}{e}\mathbf{i}$ (or **i**) and the

minimum rate of increase is $-||\nabla f(1,0)|| = \frac{-1}{e}$

5. (10%) Let $f(x, y) = x^3 + y^3 - 3xy$
   (a) Find the critical points of f
   (b) Classify each critical point as a local maximum, local minimum or saddle point

Ans:

(a) $f_x = 3x^2 - 3y,\ f_y = 3y^2 - 3x$.

Let $f_x = 0$ and $f_y = 0$,

From $f_y = 0$ we know $x = y^2$. Subsittue into $f_x = 0$, we get $3y^2 - 3y = 0 \rightarrow 3y(y-1) = 0$.

Therefore, the critical points are $(0,0), (1,1)$

(b)

Since $f_{xx} = 6x, f_{xy} = f_{yx} = -3, f_{yy} = 6y$.

(x,y)	f_{xx}	f_{xy}	f_{yy}	d	
$(0,0)$	0	-3	0	-9	Saddle point
$(1,1)$	6	-3	6	27	local minimum

6. (15%) Evaluate the following expressions

(a) $\int_0^4 \int_{\sqrt{x}}^2 \sin(\frac{x}{y})\, dydx$

(b) $\int_0^2 \int_0^{\sqrt{4-x^2}} \cos(\sqrt{x^2+y^2})\, dydx$

(c) $\int_0^{\frac{\pi}{4}} \int_0^6 \int_0^{6-r} rz\ \ dzdrd\theta$

Ans:

(a) $\int_0^4 \int_{\sqrt{x}}^2 \sin(\frac{x}{y})\, dydx = \int_0^2 \int_0^{y^2} \sin(\frac{x}{y})\, dxdy = \int_0^2 -ycos(\frac{x}{y})|_0^{y^2}\, dy =$

$\int_0^2 -ycos(y) + y\, dy = -ysin(y)\Big|_0^2 + \int_0^2 \sin(y)\, dy + \frac{y^2}{2}\Big|_0^2 = 3 - 2\sin(2) - \cos(2)$

Note that we use integration by parts and let $u = y$ and $dv = cos(y)dy \rightarrow du = dy, v = \sin(y)$.

(b) $\mathrm{R} = \{(x,y)|0 \le x \le 2, 0 \le y \le \sqrt{4-x^2}\} = \{(r,\theta)\Big|0 \le r \le 2, 0 \le \theta \le \frac{\pi}{2}\}$

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \sin(\sqrt{x^2+y^2})\, dydx = \int_0^{\frac{\pi}{2}} \int_0^2 \cos(r)\, r\, drd\theta$$

$$= \int_0^{\frac{\pi}{2}} [rsin(r) + cos(r)]_0^2\, d\theta = \int_0^{\frac{\pi}{2}} 2\sin(2) + \cos(2) - 1\, d\theta$$

$$= \frac{\pi}{2}(2\sin(2) + \cos(2) - 1)$$

Note that we use integration by parts and let $u = r$ and $dv = cos(r)dr \rightarrow du = dr, v = \sin(r)$.

(c) $\int_0^{\frac{\pi}{4}}\int_0^6\int_0^{6-r} rz\ dzdrd\theta = \int_0^{\frac{\pi}{4}}\int_0^6 \frac{rz^2}{2}\Big|_0^{6-r} drd\theta = \int_0^{\frac{\pi}{4}}\int_0^6 \frac{1}{2}(r^3 - 12r^2 + 36r)\, drd\theta = \int_0^{\frac{\pi}{4}} \frac{1}{2}[\frac{r^4}{4} - 4r^3 + 18r^2]_0^6\, d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{2}(108)d\theta = \frac{27\pi}{2}$

7. (10%) Find the area of the surface given by $z = f(x,y) = \frac{1}{2}y^2$ that lies above the region R where R is a square with vertices $(0,0), (3,0), (0,3), (3,3)$

Ans:

$$f_x = 0, f_y = y$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + y^2}$$

$$\mathrm{S} = \int_0^3\int_0^3 \sqrt{1+y^2}\, dxdy = \int_0^3 \left[\sqrt{1+y^2}x\right]_0^3 dy = \int_0^3 3\sqrt{1+y^2}dy$$

Let $y = tan(\theta), dy = sec^2(\theta)d\theta$

$$\int \sqrt{1+y^2}dy = \int \sqrt{1+tan^2(\theta)}sec^2(\theta)d\theta = \int sec^3(\theta)d\theta$$

Let $u = \sec(\theta), \mathrm{dv} = sec^2(\theta) \rightarrow du = \sec(\theta)\tan(\theta)\, d\theta, v = \tan(\theta)$

$$\int sec^3(\theta)d\theta = \sec(\theta)\tan(\theta) - \int \sec(\theta)\, tan^2(\theta)\, d\theta$$

$$= \sec(\theta)\tan(\theta) - \int \sec(\theta)\,(sec^2(\theta) - 1)\, d\theta$$

$$= \sec(\theta)\tan(\theta) - \int sec^3(\theta)\, d\theta + \int sec(\theta)\, d\theta$$

$$\rightarrow 2\int sec^3(\theta)d\theta = \sec(\theta)\tan(\theta) + \int sec(\theta)\, d\theta$$

$$= \sec(\theta)\tan(\theta) + \ln|\sec(\theta) + \tan(\theta)| + C$$

$$\int \sqrt{1+y^2}dy = \frac{1}{2}[\sec(\theta)\tan(\theta) + \ln|\sec(\theta)+\tan(\theta)|] + \frac{C}{2}$$

$$= \frac{1}{2}\left[\sqrt{1+y^2}\mathrm{y} + \ln|\sqrt{1+y^2} + \mathrm{y}|\right] + \frac{C}{2}$$

$$\mathrm{S} = 3\int_0^3 \sqrt{1+y^2}dy = \frac{3}{2}\left[\sqrt{1+y^2}\mathrm{y} + \ln|\sqrt{1+y^2} + \mathrm{y}|\right]_0^3$$

$$= \frac{3}{2}[3\sqrt{10} + \ln|\sqrt{10} + 3|]$$

8. (10%) Find the volume of the solid bounded above by $x^2+y^2+z^2=36$ and below by $z=\sqrt{x^2+y^2}$

Ans: Use cylindrical coordinates

Note that $r^2=x^2+y^2 \to z=r, z=\sqrt{36-r^2}$

Find the intersection of $z=r$ and $z=\sqrt{36-r^2} \to r=3\sqrt{2}$

$$V=\int_0^{2\pi}\int_0^{3\sqrt{2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{36-(x^2+y^2)}} rdzd\,r\,d\theta=\int_0^{2\pi}\int_0^{3\sqrt{2}}\int_r^{\sqrt{36-r^2}} rdzd\,r\,d\theta$$

$$=\int_0^{2\pi}\int_0^{3\sqrt{2}} r(\sqrt{36-r^2}-r)\,drd\theta$$

$$=\int_0^{2\pi}(72-18\sqrt{2})-18\sqrt{2}\;\;d\theta=72\pi(2-\sqrt{2})$$

Or use spherical coordinates

Intersection of sphere and cone: $z^2=x^2+y^2 \to x^2+y^2+z^2=2z^2=36 \to z=3\sqrt{2}$. Since $x^2+y^2+z^2=36 \to \rho=6$. (Since by definition $\rho \geq 0$)

And $z=\rho\cos(\Phi) \to 3\sqrt{2}=6\cos(\Phi) \to \Phi=\frac{\pi}{4}$ (Since by definition $\pi \geq \Phi \geq 0$)

$$V=\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^6 \rho^2\sin(\Phi)d\rho d\,\Phi d\theta=\int_0^{2\pi}\int_0^{\frac{\pi}{4}} 72\sin(\Phi)\,d\Phi d\theta$$

$$=\int_0^{2\pi} 72-72\frac{\sqrt{2}}{2}d\theta=72\pi(2-\sqrt{2})$$

9. (10%) Use the change of variables to find the volume of the solid region lying below the surface $z=f(x,y)=\ln(x^2y+x)$ and above the plane region R wher R is a region bounded by $xy=1, xy=3, x=1, x=e$.

Ans:

Let $u=xy, v=x$

$$\frac{\partial(x,y)}{\partial(u,v)}=\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}=\frac{-1}{v}$$

$$\int\int_R \ln(x^2y + x) \quad \mathrm{dA}$$

$$= \int_1^e \int_1^3 \ln(uv + v)\frac{1}{v}dudv = \int_1^e \int_1^3 \frac{\ln(u+1) + \ln v}{v}dudv =$$

$$= \int_1^e [(u+1)\ln(u+1) - u]_1^3 \frac{1}{v} + (3-1)\frac{\ln v}{v}dv =$$

$$= \ln e\,[(3+1)\ln(3+1) - 3 - 2\ln 2 + 1] + \frac{(3-1)}{2}(\ln e)^2$$

$$= 8\ln 2 - 4 - 2\ln 2 + 2 + 1 = 6\ln 2 - 1$$

Note that we let $u' = \ln(u+1)$ $\;$ *and* $\; dv' = \mathrm{du} \rightarrow \mathrm{du}' = \frac{1}{u+1}du,\; v' = u$

$$\int ln(u+1)\,du = uln(u+1) - \int \frac{u}{u+1}du = uln(u+1) - \int 1 - \frac{1}{u+1}du$$

$$= uln(u+1) - u + ln(u+1) = (u+1)\,ln(u+1) - u$$