1. (24%) Determine whether the series converges absolutely or conditionally, or diverges. In addition, please indicate the test you use.
   (a) $\sum_{n=1}^{\infty}(-1)^{n+1}ne^{-n^2}$
   (b) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(\sqrt{n})}{n^{\frac{3}{2}}}$
   (c) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2n^2}{n^3+3}$
   (d) $\sum_{n=1}^{\infty}\left(\frac{n+1}{n}\right)^{n^2}$

Ans:

(a) Since $(ne^{-n^2})' = (1-2n^2)e^{-n^2} < 0$ which is decreasing and $\lim_{n\to\infty} ne^{-n^2} = 0$

(exponential is much faster). Therefore, by the alternating series test $\sum_{n=1}^{\infty}(-1)^{n+1}ne^{-n^2}$ converges.

$$\sum_{n=1}^{\infty}(-1)^{n+1}ne^{-n^2} = \sum_{n=1}^{\infty}|ne^{-n^2}| = \sum_{n=1}^{\infty}ne^{-n^2}$$

Let $f(\mathrm{x}) = xe^{-x^2}$, $f'(x) = (1-2x^2)e^{-x^2} < 0$ for $\mathrm{x} \geq 1$. f is positive, continuous and decreasing for $\mathrm{x} \geq 1$

$$\int_1^{\infty} xe^{-x^2}dx = \lim_{b\to\infty}\int_1^{b} xe^{-x^2}dx = \lim_{b\to\infty}\frac{1}{2}\int_{-b^2}^{-1}e^{u}du$$

$(Let\ u = -x^2, du = -2xdx) = \frac{1}{2}e^{-1}$ which is converge

Therefore, $\sum_{n=1}^{\infty}(-1)^{n+1}ne^{-n^2}$ is absolute converges

(b) $\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{\sin(\sqrt{n})}{n^{\frac{3}{2}}}\right| = \sum_{n=1}^{\infty}\frac{\sin(\sqrt{n})}{n^{\frac{3}{2}}}$

Since $\frac{\sin(\sqrt{n})}{n^{\frac{3}{2}}} < \frac{1}{n^{\frac{3}{2}}}$ and $\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}$ is a p-series with $\mathrm{p} > 1$ which is converge.

By comparison test, $\sum_{n=1}^{\infty}\frac{\sin(\sqrt{n})}{n^{\frac{3}{2}}}$ is converge.

Therefore, $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(\sqrt{n})}{n^{\frac{3}{2}}}$ is absolute converge.

(c) $(\frac{2n^2}{n^3+3})' = \frac{-2n^4+12n}{(n^3+3)^2} < 0$ for $\mathrm{n} \geq 2$, and $\lim_{n\to\infty}\frac{2n^2}{n^3+3} = 0$. Therefore, by the alternating series test $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{2n^2}{n^3+3}$ converges. So is $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2n^2}{n^3+3}$

(Since finite term does not affect the convergence or divergence)

$$\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{2n^2}{n^3+3}\right| = \sum_{n=1}^{\infty}\frac{2n^2}{n^3+3}$$

Since $\lim_{n\to\infty}\frac{\frac{2n^2}{n^3+3}}{\frac{1}{n}} = 2$ and since $\sum_{n=1}^{\infty}\frac{1}{n}$ is a p-series with $p \le 1$ which is diverge. By the limit comparison test, $\sum_{n=1}^{\infty}\frac{2n^2}{n^3+3}$ is diverge.

Therefore, $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2n^2}{n^3+3}$ is conditionally converge.

(d) $\sum_{n=1}^{\infty}\sqrt[n]{\left(\frac{n+1}{n}\right)^{n^2}} = \sum_{n=1}^{\infty}\left(\frac{n+1}{n}\right)^{n} = e > 1$ which is diverge.

2. (16%) Find the interval of convergence of the power series (Be sure to check the for the convergence at the endpoints of the intervals)

(a) $\sum_{n=1}^{\infty}\frac{n}{n+1}\frac{(x)^n}{(2x+1)^n}$

(b) $\sum_{n=1}^{\infty}\frac{n!(x+1)^n}{3^n}$

Ans:

(a) $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)}{(n+2)}(\frac{x}{2x+1})^{n+1}}{\frac{n}{(n+1)}(\frac{x}{2x+1})^{n}}\right| = \left|\frac{x}{2x+1}\right|$ By the ratio test, the series converges for $\left|\frac{x}{2x+1}\right| < 1$

When $\frac{x}{2x+1} < 1 \rightarrow x > \frac{-1}{2}$ or $x < -1$

When $\frac{x}{2x+1} > -1 \rightarrow x > \frac{-1}{3}$ or $x < \frac{-2}{3}$

The intersection is $x > \frac{-1}{3}$ and $x < -1$

When $x = \frac{-1}{3}$: $\sum_{n=1}^{\infty}\frac{n}{n+1}(-1)^n$ is diverge by the n-th term test for divergence since $\lim_{n\to\infty}\frac{n}{n+1}(-1)^n \neq 0$

When $x = -1$: $\sum_{n=1}^{\infty}\frac{n}{n+1}$ is diverge by the n-th term test for divergence since $\lim_{n\to\infty}\frac{n}{n+1} \neq 0$

So the interval of convergence is $x > \frac{-1}{3}$ and $x < -1$

(b) $\lim_{n\to\infty} |\frac{u_{n+1}}{u_n}| = \lim_{n\to\infty} |\frac{\frac{(n+1)!(x+1)^{n+1}}{3^{n+1}}}{\frac{n!(x+1)^n}{3^n}}| = |x+1| \lim_{n\to\infty} \frac{n+1}{3} = \infty$. which implies that the series converges only at the center -1.

3. (10%) Let $f(x) = \sqrt{1+x} + \sqrt{1-x}$, what is $f^{(10)}(0) =$?

Ans:

$$f(x) = 1 + \frac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}x^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{2!}x^3 + \cdots..+1 - \frac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}x^2$$

$$-\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}x^3 + \cdots = f(0) + f'(0)x + \frac{1}{2!}f''(x)x^2 + \cdots.$$

$$f^{(10)}(0) = 2\left(\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\left(-\frac{17}{2}\right)\right)$$

4. (18%) Evaluate the following expression (Try to use the Basic series of Taylor series and notice that the power series is a continuous function)

(a) $\sum_{n=0}^{\infty} \frac{3^{n+1}}{n!}$

(b) $\frac{1}{1\times 2} - \frac{1}{2\times 2^2} + \frac{1}{3\times 2^3} - \frac{1}{4\times 2^4} + \ldots$

(c) $\lim_{x\to 0^+} \frac{\arctan(2x)-\sin(2x)}{sinx-x}$

Ans:

(a) Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\mathrm{x}e^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$

$$\sum_{n=0}^{\infty} \frac{3^{n+1}}{n!} = 3e^3$$

(b) $\frac{1}{1\times 2} - \frac{1}{2\times 2^2} + \frac{1}{3\times 2^3} - \frac{1}{4\times 2^4} + \ldots\ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}\left(\frac{1}{2}\right)^{n+1} = \ln(1+\frac{1}{2}) = \ln\frac{3}{2}$

(c) $\lim_{x\to 0^+} \frac{\arctan(2x)-sin(2x)}{sinx-x} = \lim_{x\to 0^+} \frac{(2x-\frac{1}{3}(2x)^3+\cdots)-(2x-\frac{1}{3!}(2x)^3+\cdots)}{\left(x-\frac{1}{3!}x^3+\cdots\right)-x} = \lim_{x\to 0^+} \frac{-\frac{8}{6}}{-\frac{1}{6}} = 8$

5. (12%) Derive the Maclaurin series of $f(x) = arccot(2x)$

Ans:

$(arccot(x))' = \frac{-1}{1+x^2} = \frac{-1}{1-(-x^2)} = -\sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^{n+1}x^{2n} \quad |x| < 1$

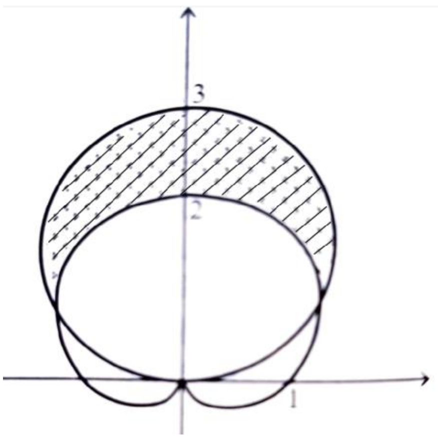
$$arccot(x) = C + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}x^{2n+1}}{2n+1}$$

Substitute $x = 0$, we get, $C = \frac{\pi}{2}$

$$arccot(x) = \frac{\pi}{2} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}x^{2n+1}}{2n+1}$$

$$arccot(2x) = \frac{\pi}{2} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2x)^{2n+1}}{2n+1} = \frac{\pi}{2} - 2x + \frac{8x^3}{3} - \frac{32x^5}{5} + \cdots.$$

6. (10%) Find the area of the shaded region bounded by the curves $r = 1 + sin(\theta)$ and $r = 3sin(\theta)$



Ans:

Solve $r = 3\sin(\theta)$ and $r = 1 + \sin(\theta)$ we get $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$$A = 2\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}[(3sin(\theta))^2 - (1 + sin(\theta))^2]\,d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}[8sin^2\theta - 1 - 2sin\theta]\,d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left[8\frac{1-\cos(2\theta)}{2} - 1 - 2sin\theta\right]d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}[3 - 4\cos(2\theta) - 2\sin(\theta)]\,d\theta = 3\theta - 2\sin(2\theta) + 2\cos(\theta)]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \pi$$

7. (10%) Find the area of the surface formed by revolving the polar graph $r = 2(1 + sin(\theta))$ about the $\theta = \frac{\pi}{2}$ over the interval $\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$

Ans:

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{4(1 + sin\theta)^2 + 4cos^2\theta} = 2\sqrt{2}\sqrt{1 + sin\theta}$$

$$S = 2\pi\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 2(1 + \sin(\theta))\cos(\theta)\, 2\sqrt{2}\sqrt{1 + sin\theta}\,d\theta$$

$$= 8\sqrt{2}\pi\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}(1 + \sin(\theta))^{\frac{3}{2}}\cos(\theta)\ (Let\ u = 1 + \sin(\theta), du$$

$$= \cos(\theta)d\theta) = 8\sqrt{2}\pi\int_0^2 (u)^{\frac{3}{2}}\,du = \frac{16\sqrt{2}\pi}{5}\left[u^{\frac{5}{2}}\right]_0^2 = \frac{128\pi}{5}$$