Calculus — Final Exam Solutions

1. (8%) Coordinate Systems

(a) (4%) Cylindrical to rectangular

Sol. We use the cylindrical-to-rectangular formulas

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z.$$

Here $r = 2$, $\theta = 7\pi/6$, and $z = 5$, so

$$x = r\cos\theta = 2\cos\frac{7\pi}{6} = -\sqrt{3}, \qquad y = r\sin\theta = 2\sin\frac{7\pi}{6} = -1, \qquad z = 5.$$

Thus the rectangular coordinates are

$$\boxed{(-\sqrt{3}, -1, 5).}$$ 錯一個扣一分

(b) (4%) Rectangular to cylindrical and spherical

Sol. For cylindrical coordinates, use

$$r = \sqrt{x^2+y^2}, \qquad \tan\theta = \frac{y}{x}.$$

For spherical coordinates, use

$$\rho = \sqrt{x^2+y^2+z^2}, \qquad \cos\phi = \frac{z}{\rho}.$$

Now

$$r = \sqrt{x^2+y^2} = \sqrt{4+4} = 2\sqrt{2}.$$

Since $(x, y) = (-2, 2)$ is in quadrant II,

$$\theta = \frac{3\pi}{4}, \qquad z = 2\sqrt{2}.$$

Thus the cylindrical coordinates are

$$\boxed{\left(2\sqrt{2}, \frac{3\pi}{4}, 2\sqrt{2}\right).}$$ 錯一個扣1分(最多扣2)

For spherical coordinates,

$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{4+4+8} = 4, \qquad \cos\phi = \frac{z}{\rho} = \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2}.$$

Hence $\phi = \pi/4$, so

$$\boxed{\left(4, \frac{3\pi}{4}, \frac{\pi}{4}\right).}$$ 錯一個扣1分(最多扣2)

2. (12%) Vector-Valued Functions

(a) (4%) Evaluate $\lim_{t\to 0} \mathbf{r}(t)$.

Sol. A vector-valued limit is evaluated component by component:

$$\lim_{t\to 0} \mathbf{r}(t) = \left\langle \lim_{t\to 0} e^{2t}, \lim_{t\to 0} \frac{\sin(2t)}{t}, \lim_{t\to 0} 3 \right\rangle.$$

Using continuity of the exponential function and the standard limit $\lim_{u\to 0} \sin u/u = 1$,

$$\lim_{t\to 0} e^{2t} = 1, \qquad \lim_{t\to 0} \frac{\sin(2t)}{t} = 2 \lim_{t\to 0} \frac{\sin(2t)}{2t} = 2.$$

Therefore

$$\boxed{\lim_{t\to 0} \mathbf{r}(t) = \langle 1, 2, 3\rangle.}$$ 錯一個扣1分

(b) (4%) Smoothness intervals for $\mathbf{g}(t)$

Sol. A vector-valued curve is smooth on an interval if $\mathbf{g}'(t)$ exists, has continuous components, and $\mathbf{g}'(t) \neq \mathbf{0}$ on that interval. Here

$$\mathbf{g}(t) = \langle e^{-t}(t+2)^2, 0, (t+2)^3\rangle.$$

$$\underline{\mathbf{g}'(t) = \left\langle -t(t+2)e^{-t}, 0, 3(t+2)^2 \right\rangle}.$$ 2分

1分

The derivative is continuous for all real t, and $\mathbf{g}'(t) = \mathbf{0}$ only when $\underline{t = -2}$. Hence the curve is smooth on

$$\boxed{(-\infty, -2) \cup (-2, \infty).}$$ 1分

(c) (4%) Dot-product derivative

Sol. We use the dot product first and then differentiate. Equivalently, this is the product rule for dot products:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}\cdot\mathbf{g}) = \mathbf{r}'\cdot\mathbf{g} + \mathbf{r}\cdot\mathbf{g}'.$$

In this problem it is simpler to expand the dot product:

$$\mathbf{r}(t)\cdot\mathbf{g}(t) = e^{2t}e^{-t}(t+2)^2 + 3(t+2)^3 = e^t(t+2)^2 + 3(t+2)^3.$$

Since $\mathbf{r}(t)$ contains $\sin(2t)/t$, this derivative is taken on the domain $t \neq 0$. Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathbf{r}(t)\cdot\mathbf{g}(t)\big) = e^t(t+2)^2 + 2e^t(t+2) + 9(t+2)^2,$$

or

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathbf{r}(t)\cdot\mathbf{g}(t)\big) = e^t(t+2)(t+4) + 9(t+2)^2, \qquad t \neq 0.}$$ 4分

3. (12%) Multivariable Limits

(a) (4%) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$

Sol. We use the path test: if two paths approaching the same point give different limiting values, then the two-variable limit does not exist. Along $y = 0$,

$$\frac{x^2}{x^2+y^2} = \frac{x^2}{x^2} = 1.$$ 2分

Along $x = 0$,

$$\frac{x^2}{x^2+y^2} = \frac{0}{y^2} = 0.$$ 2分

The two path limits are different. Therefore

$$\boxed{\text{the limit does not exist.}}$$ 結論錯扣1分

(b) (4%) $\displaystyle \lim_{(x,y)\to(0,0)} x^2 \cos\left(\frac{1}{x^2+y^2}\right)$

Sol. We use the Squeeze Theorem. Since $|\cos u| \le 1$ for all real u,

$$\left| x^2 \cos\left(\frac{1}{x^2+y^2}\right) \right| \le x^2 \le x^2+y^2 \to 0.$$ 過程2分

By the Squeeze Theorem,

$$\boxed{\lim_{(x,y)\to(0,0)} x^2 \cos\left(\frac{1}{x^2+y^2}\right) = 0.}$$ 答案2分

(c) (4%) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$

Sol. Let

$$u = x^2+y^2.$$

As $(x,y) \to (0,0)$, we have $u \to 0^+$. Then

$$\frac{\sin(x^2+y^2)}{x^2+y^2} = \frac{\sin u}{u}.$$ 1分

Using the standard one-variable limit $\underline{\lim_{u\to 0} \sin u/u = 1}$, we get

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{u\to 0^+} \frac{\sin u}{u} = 1.}$$ 3分

4. (22%) Ch13 Differential Block

(a) (6%) Partials, differential, and linear approximation

Sol. We use

$$dz = f_x(a,b)\,dx + f_y(a,b)\,dy$$

and the linear approximation formula

$$f(a+\Delta x, b+\Delta y) \approx f(a,b) + f_x(a,b)\Delta x + f_y(a,b)\Delta y.$$

First,

$$f_x = 2xe^{xy} + x^2ye^{xy}, \qquad f_y = x^3e^{xy}.$$

At $(1,0)$,

1分 1分

$$\underline{f_x(1,0) = 2}, \qquad \underline{f_y(1,0) = 1}, \qquad f(1,0) = 1.$$

Thus

$$\boxed{dz = 2\,\mathrm{d}x + \,\mathrm{d}y.}$$ 2分

For $(1.02, -0.01)$, $\Delta x = 0.02$ and $\Delta y = -0.01$, so

$$f(1.02, -0.01) \approx 1 + 2(0.02) - 0.01 = \boxed{1.03}.$$ 2分

(b) (6%) Chain Rule

Sol. For $w = w(x,y,z)$ with $x = x(r,\theta)$, $y = y(r,\theta)$, and $z = z(r,\theta)$, the Chain Rule gives

$$w_r = w_x x_r + w_y y_r + w_z z_r,$$
$$w_\theta = w_x x_\theta + w_y y_\theta + w_z z_\theta.$$

First compute

$$w_x = y+z, \qquad w_y = x+z, \qquad w_z = y+x.$$

At $r=1$ and $\theta = \pi$,

$$x=-1, \qquad y=0, \qquad z=\pi,$$

so

$$w_x = \pi, \qquad w_y = \pi - 1, \qquad w_z = -1.$$

Also,

$$x_r = \cos\theta = -1, \quad y_r = \sin\theta = 0, \quad z_r = 2r\theta = 2\pi,$$
$$x_\theta = -r\sin\theta = 0, \quad y_\theta = r\cos\theta = -1, \quad z_\theta = r^2 = 1.$$

Therefore

$$\frac{\partial w}{\partial r} = w_x x_r + w_y y_r + w_z z_r = \pi(-1) + (\pi-1)(0) + (-1)(2\pi) = \boxed{-3\pi},$$ 3分

and

$$\frac{\partial w}{\partial \theta} = w_x x_\theta + w_y y_\theta + w_z z_\theta = \pi(0) + (\pi-1)(-1) + (-1)(1) = \boxed{-\pi}.$$ 3分

(c) (6%) Tangent plane and normal line

Sol. For a level surface $F(x,y,z) = c$, the gradient $\nabla F(P)$ is normal to the surface at P. The tangent plane formula is

$$\nabla F(P) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0,$$

and a normal line is

$$(x,y,z) = P + \lambda \nabla F(P).$$

Let $F = x^2 + 2y^2 + 3z^2$. Then

2 分

$$\nabla F = \langle 2x, 4y, 6z \rangle, \qquad \underline{\nabla F(1,2,1) = \langle 2,8,6 \rangle}.$$

The tangent plane is

$$2(x-1) + 8(y-2) + 6(z-1) = 0,$$

or

$$\boxed{x + 4y + 3z = 12.}$$ 2 分

A normal line is

$$\boxed{(x,y,z) = (1,2,1) + \lambda\langle 2,8,6 \rangle, \qquad \lambda \in \mathbb{R}.}$$ 2 分

(d) (4%) Partial derivative by definition

Sol. By the definition of partial derivative with respect to y,

$$\underline{f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}}.$$ 定義 2 分

For $f(x,y) = x^2y^2 - xy^3$,

$$\frac{f(x,y+h) - f(x,y)}{h} = \frac{x^2\big((y+h)^2 - y^2\big) - x\big((y+h)^3 - y^3\big)}{h}.$$

Thus

$$\frac{f(x,y+h) - f(x,y)}{h} = x^2(2y+h) - x(3y^2 + 3yh + h^2).$$

Taking $h \to 0$ gives

$$\boxed{f_y(x,y) = 2x^2y - 3xy^2.}$$ 答案 2 分

5. (10%) Gradient and Directional Derivative

(a) (7%) Gradient and directional derivative

Sol. The gradient is

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle.$$

The directional derivative in a unit direction $\mathbf{u}$ is

$$D_{\mathbf{u}} f(P) = \nabla f(P) \cdot \mathbf{u}.$$

Here

2 分

$$\underline{\nabla f(x,y) = \langle e^y, xe^y + 2y \rangle}, \qquad \nabla f(2,0) = \langle 1,2 \rangle.$$

Thus

$$\boxed{\nabla f(2,0) = \langle 1,2 \rangle.}$$ 2 分

Therefore

$$D_{\mathbf{u}} f(2,0) = \langle 1,2 \rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle = \boxed{\frac{11}{5}}.$$ 3 分

(b) (3%) Steepest decrease

Sol. The theorem for steepest change says that the direction of steepest increase is $\nabla f(P)/\|\nabla f(P)\|$, while the direction of steepest decrease is

$$\boxed{-\frac{\nabla f(2,0)}{\|\nabla f(2,0)\|} = \frac{\langle -1,-2\rangle}{\sqrt{5}}.}$$ 2分

The rate of steepest decrease is

$$\boxed{-\|\nabla f(2,0)\| = -\sqrt{5}.}$$ 1分

6. (8%) Critical Point Classification

Sol. Critical points occur where $f_x = f_y = 0$. Then we use the second derivative test:

$$D = f_{xx}f_{yy} - (f_{xy})^2.$$ 1 3 2分 計算錯扣1分

If $D > 0$ and $f_{xx} > 0$, the point is a relative minimum. If $D > 0$ and $f_{xx} < 0$, the point is a relative maximum. If $D < 0$, the point is a saddle point. If $D = 0$, the test is inconclusive.

Now compute the first partial derivatives:

$$f_x = 2x + y - 3, \qquad f_y = x + 2y - 3.$$

Set $f_x = f_y = 0$:

$$2x + y = 3, \qquad x + 2y = 3,$$

so $(x,y) = (1,1)$. Also

$$f_{xx} = 2, \qquad f_{yy} = 2, \qquad f_{xy} = 1.$$

Thus

$$D = f_{xx}f_{yy} - (f_{xy})^2 = 2\cdot 2 - 1 = 3 > 0, \qquad f_{xx} > 0.$$

Therefore

$$\boxed{(1,1) \text{ is a relative minimum.}}$$ 2分

There are no relative maximum points and no saddle points. 各2分

7. (20%) Double Integrals

(a) (5%) Reverse the order and evaluate

Sol. The original integral is 一定要先交換順序不然一律0分

$$\int_0^1 \int_x^1 (x+y)\,dy\,dx.$$

From the order $dy\,dx$, we read the region as

$$0 \le x \le 1, \qquad x \le y \le 1.$$

This is the triangular region above the line $y = x$ and below $y = 1$ in the first quadrant. To reverse the order, let y be the outer variable. The smallest and largest values of y

are 0 and 1. For a fixed y with $0 \le y \le 1$, the variable x runs from the y-axis to the line $x = y$:

$$0 \le y \le 1, \qquad 0 \le x \le y.$$

Hence

$$\int_0^1 \int_x^1 (x+y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \int_0^y (x+y)\,\mathrm{d}x\,\mathrm{d}y.$$

Now evaluate the inner integral first:

$$\int_0^y (x+y)\,\mathrm{d}x = \left[\frac{x^2}{2} + yx\right]_{x=0}^{x=y} = \frac{y^2}{2} + y^2.$$

Therefore

1

$$\int_0^1 \int_0^y (x+y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \left(\frac{y^2}{2} + y^2\right)\mathrm{d}y = \underline{\frac{3}{2}\int_0^1 y^2\,\mathrm{d}y} = \boxed{\frac{1}{2}}.$$

1

(b) (5%) Volume enclosed by a surface and coordinate planes

Sol. We use the volume formula

$$V = \iint_R (\text{top} - \text{bottom})\,\mathrm{d}A.$$

Here the top surface is $z = 2 + x^2 y e^y$ and the bottom surface is $z = 0$. The planes $x = \pm 2$, $y = 0$, and $y = 1$ give the rectangular projection

$$R = \{(x, y) : -2 \le x \le 2,\ 0 \le y \le 1\}.$$

上下界 3分

Therefore

$$\underline{V = \int_{-2}^2 \int_0^1 (2 + x^2 y e^y)\,\mathrm{d}y\,\mathrm{d}x.}$$

Evaluate the y-integral first. Since x is constant with respect to y,

$$\int_0^1 (2 + x^2 y e^y)\,\mathrm{d}y = 2 + x^2 \int_0^1 y e^y\,\mathrm{d}y.$$

Using integration by parts, or the antiderivative

$$\int y e^y\,\mathrm{d}y = (y-1)e^y,$$

we get

$$\int_0^1 y e^y\,\mathrm{d}y = [(y-1)e^y]_0^1 = 1,$$

so the volume becomes

2

$$V = \int_{-2}^2 (2 + x^2)\,\mathrm{d}x = \left[2x + \frac{x^3}{3}\right]_{-2}^2 = 8 + \frac{16}{3} = \boxed{\frac{40}{3}}.$$

(c) (5%) Polar integral over the quarter disk

Sol. Use polar coordinates:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad x^2+y^2=r^2, \qquad \mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta.$$

The region R is the first-quadrant quarter disk $x^2+y^2 \le 4$, so

$$0 \le r \le 2, \qquad 0 \le \theta \le \frac{\pi}{2}.$$

The integrand x^2+y^2 becomes r^2, and the area element contributes another factor of r. Therefore

3

$$\iint_R (x^2+y^2)\,\mathrm{d}A = \underline{\int_0^{\pi/2}\int_0^2 r^3\,\mathrm{d}r\,\mathrm{d}\theta} = \int_0^{\pi/2} 4\,\mathrm{d}\theta = \boxed{2\pi}.$$

2

(d) (5%) Volume bounded by a cylinder and planes

Sol. Use cylindrical coordinates:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad \mathrm{d}V = r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta.$$

The first octant gives

$$0 \le \theta \le \frac{\pi}{2}, \qquad r \ge 0, \qquad z \ge 0.$$

The cylinder $x^2+y^2=4$ becomes $r=2$, so

$$0 \le \theta \le \frac{\pi}{2}, \qquad 0 \le r \le 2.$$

The planes $z=0$ and $y=2z$ give

$$0 \le z \le \frac{y}{2} = \frac{r\sin\theta}{2}.$$

Thus

3

$$V = \underline{\int_0^{\pi/2}\int_0^2\int_0^{r\sin\theta/2} r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta}.$$

Evaluate first with respect to z:

$$\int_0^{r\sin\theta/2} r\,\mathrm{d}z = \frac{r^2\sin\theta}{2}.$$

Therefore

$$V = \int_0^{\pi/2}\int_0^2 \frac{r^2\sin\theta}{2}\,\mathrm{d}r\,\mathrm{d}\theta = \frac{1}{2}\left(\int_0^{\pi/2}\sin\theta\,\mathrm{d}\theta\right)\left(\int_0^2 r^2\,\mathrm{d}r\right) = \boxed{\frac{4}{3}}.$$

2

8. (5%) Surface Area

Sol. For a graph $z = g(x,y)$ over a region R, the surface area formula is

$$S = \iint_R \underline{\sqrt{1+g_x^2+g_y^2}\,\mathrm{d}A}.$$

公式 2分

Write the plane as

$$z = 6 - 2x - 3y.$$

Then

$$z_x = -2, \qquad z_y = -3,$$

so the surface area factor is

$$\sqrt{1 + z_x^2 + z_y^2} = \sqrt{14}.$$

The first-octant projection is the triangle

$$R = \{(x, y) : x \geq 0,\ y \geq 0,\ 2x + 3y \leq 6\},$$

with vertices $(0, 0)$, $(3, 0)$, and $(0, 2)$. Its area is

$$\frac{1}{2}(3)(2) = 3.$$

2

Therefore

$$\boxed{S = 3\sqrt{14}.}$$

1

9. (5%) Volume by Triple Integral

Sol. We use the volume formula

$$V = \iiint_E 1 \, \mathrm{d}V.$$

The planes $z = 0$ and $y + z = 2$ give

$$0 \leq z \leq 2 - y.$$

Since $z \geq 0$, we have $y \leq 2$. Together with the cylinder $y = 2x^2$, the top boundary in the projection is $y = 2$. The curves $y = 2x^2$ and $y = 2$ intersect when

$$2x^2 = 2, \qquad x = \pm 1.$$

Therefore the projection onto the xy-plane is

$$-1 \leq x \leq 1, \qquad 2x^2 \leq y \leq 2.$$

Thus

$$V = \int_{-1}^{1} \int_{2x^2}^{2} \int_{0}^{2-y} \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x.$$

3

Evaluating,

$$V = \int_{-1}^{1} \int_{2x^2}^{2} (2 - y) \, \mathrm{d}y \, \mathrm{d}x = \int_{-1}^{1} \left[2y - \frac{y^2}{2} \right]_{2x^2}^{2} \mathrm{d}x = \int_{-1}^{1} (2 - 4x^2 + 2x^4) \, \mathrm{d}x = \boxed{\frac{32}{15}}.$$

2

10. (8%) Volume in cylindrical or spherical coordinates

Sol. In spherical coordinates,

$$x = \rho \sin\phi \cos\theta, \qquad y = \rho \sin\phi \sin\theta, \qquad z = \rho \cos\phi,$$

and

$$\mathrm{d}V = \rho^2 \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta.$$

The sphere $x^2 + y^2 + z^2 = 16$ becomes $\rho = 4$. The cone $3z^2 = x^2 + y^2$ is the upper nappe, so

$$z = \frac{1}{\sqrt{3}} \sqrt{x^2 + y^2}.$$

In spherical coordinates this is

$$\rho \cos\phi = \frac{1}{\sqrt{3}} \rho \sin\phi,$$

so $\tan\phi = \sqrt{3}$ and therefore $\phi = \pi/3$. The solid is above the cone and inside the sphere, hence

$$0 \le \theta \le 2\pi, \qquad 0 \le \phi \le \frac{\pi}{3}, \qquad 0 \le \rho \le 4.$$

Thus

$$V = \int_0^{2\pi} \int_0^{\pi/3} \int_0^4 \rho^2 \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta = 2\pi \left(1 - \frac{1}{2}\right) \frac{64}{3} = \boxed{\frac{64\pi}{3}}.$$

→ ②

⑥ ⇒ 答對一個 1分, 2個 2分
3個 4分, 全對 6分

Equivalently, in cylindrical coordinates,

$$x^2 + y^2 = r^2, \qquad \mathrm{d}V = r \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta.$$

The cone gives the lower surface

$$z = \frac{r}{\sqrt{3}},$$

and the sphere gives the upper surface

$$z = \sqrt{16 - r^2}.$$

To find the outer radius, solve the intersection:

$$\frac{r}{\sqrt{3}} = \sqrt{16 - r^2} \quad \Longrightarrow \quad \frac{r^2}{3} = 16 - r^2 \quad \Longrightarrow \quad r = 2\sqrt{3}.$$

Thus

$$0 \le \theta \le 2\pi, \qquad 0 \le r \le 2\sqrt{3}, \qquad \frac{r}{\sqrt{3}} \le z \le \sqrt{16 - r^2},$$

and

$$V = \int_0^{2\pi} \int_0^{2\sqrt{3}} \int_{r/\sqrt{3}}^{\sqrt{16-r^2}} r \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta = \boxed{\frac{64\pi}{3}}.$$

→ ②

⑥ 評分同上