1. (24%) Examine the series to determine whether it converges absolutely, converges conditionally, or diverges, and clearly indicate which convergence test you applied

   (a) $\sum_{n=1}^{\infty} \frac{\sin[\frac{(2n-1)\pi}{2}]}{n+1}$

   (b) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{4n}{3n+1}\right)^{2n}$

   (c) $\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n+1}$

   (d) $\sum_{n=1}^{\infty} (-1)^n \frac{2\times4\times6\times\ldots\times2n}{2\times5\times8\times\ldots\times(3n-1)}$

Ans:

(a) $\sum_{n=1}^{\infty} \frac{\sin[\frac{(2n-1)\pi}{2}]}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$ is converge by the alternating series test, since

$\lim\limits_{n\to\infty} \frac{1}{n+1} = 0$ and $\frac{1}{(n+2)} < \frac{1}{n+1}$ for $n > 1$.

Considering $\sum_{n=1}^{\infty} \left|\frac{\sin[\frac{(2n-1)\pi}{2}]}{n+1}\right| = \sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges by a limit comparison test to the divergent p-series (harmonic series) $\sum_{n=1}^{\infty} \frac{1}{n}$. Therefore, the series converges conditionally.

(b) $\lim\limits_{n\to\infty} \sqrt[n]{|(-1)^n \left(\frac{4n}{3n+1}\right)^{2n}|} = \lim\limits_{n\to\infty} \left(\frac{4n}{3n+1}\right)^2 = (\frac{4}{3})^2 > 1$

Therefore, by the root test, the series is diverging.

(c) Let $f(\mathrm{x}) = \frac{\ln(x+1)}{x+1}$, $f'(x) = \frac{1-\ln(x+1)}{(x+1)^2} < 0$ for $x \geq 2$. f is positive, continuous and decreasing for $x \geq 2$

$$\int_2^{\infty} \frac{\ln(x+1)}{x+1}\, dx = \lim_{b\to\infty} \frac{[\ln(x+1)]^2}{2} \Big|_2^b = \infty$$

So the series $\sum_{n=2}^{\infty} \frac{\ln(n+1)}{n+1}$ diverges by the integral test. Since $\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n+1} = \sum_{n=2}^{\infty} \frac{\ln(n+1)}{n+1} + \frac{\ln 2}{2}$, so $\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n+1}$ is also diverge.

(d) $\lim\limits_{n\to\infty} |\frac{u_{n+1}}{u_n}| = \lim\limits_{n\to\infty} |\frac{2\times4\times\ldots\times2n\times(2n+2)}{2\times5\times\ldots(3n-1)\times(3n+2)} \times \frac{2\times5\times\ldots(3n-1)}{2\times4\times\ldots\times2n}| = \lim\limits_{n\to\infty} \frac{2n+2}{3n+2} = \frac{2}{3} < 1$

By the ratio test, the series is absolute converges.

2. (12%) Determine the interval of convergence for the power series, including testing the endpoints for convergence

(a) $\sum_{n=1}^{\infty} \frac{3^n+(-2)^n}{n}(x+1)^n$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}3\times7\times11\times\ldots\times(4n-1)(x-3)^n}{3^n}$

Ans:

(a) $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{3^{n+1}+(-2)^{n+1}}{n+1}(x+1)^{n+1}}{\frac{3^n+(-2)^n}{n}(x+1)^n}\right| = \lim_{n\to\infty}\left|\frac{n}{n+1}\frac{3+\frac{(-2)^{n+1}}{3^n}}{1+\frac{(-2)^n}{3^n}}\right||x+1| = 3|x+1|$

By the ratio test, the series converges for $|x+1| < \frac{1}{3}$

When $x = -\frac{4}{3}$: $\sum_{n=1}^{\infty}\frac{3^n+(-2)^n}{n}(\frac{-1}{3})^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n} + \sum_{n=1}^{\infty}\frac{(\frac{2}{3})^n}{n}$. The first term is converge by the alternating series test, for the second term since $\sum_{n=1}^{\infty}(\frac{2}{3})^n$ is a convergent geometric series and $\frac{(\frac{2}{3})^n}{n} < (\frac{2}{3})^n$ for $n > 1$. Therefore, by the direct comparison test, the second term is also converging, the series is thus converging.

When $x = -\frac{2}{3}$: $\sum_{n=1}^{\infty}\frac{3^n+(-2)^n}{n}(\frac{1}{3})^n = \sum_{n=1}^{\infty}\frac{1}{n} + \sum_{n=1}^{\infty}\frac{(-\frac{2}{3})^n}{n}$ since the first term is a diverge p-series and the second term is converge by the same argument as $x = -\frac{4}{3}$, the series is diverge.

So the interval of convergence is $[-\frac{4}{3}, \frac{2}{3})$

- Adding two convergent series always yields a convergent series
- Adding one convergent and one divergent Series results in diverging series
- Adding two divergent series together can be divergent or convergent (for example, when adding $\sum_{i=0}^{\infty} 1$ and $\sum_{i=0}^{\infty} -1$, the results is 0.

(b) $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}3\times7\times11\times\ldots\times(4n-1)(4n+3)(x-3)^{n+1}}{3^{n+1}}\times\right.$

$\left.\frac{3^n}{(-1)^{n+2}3\times7\times11\times\ldots\times(4n-1)(x-3)^n}\right| = \lim_{n\to\infty}\left|\frac{(4n+3)(x-3)}{3}\right| = \infty$

Therefore, $R = 0$ and the series converges only for $x = 3$.

3. (8%) Use a power series to approximate $\sin(1)$ with an error of less than 0.001

Ans: $\sin(1) = 1 - \frac{1}{3!} + \frac{1}{5!} - \cdots . + \frac{(-1)^n}{(2n+1)!} + \cdots$

It is an alternating series, since $\frac{1}{(2*3+1)!} < 0.001$ therefore we know that $\sin(1) \approx$

$1 - \frac{1}{3!} + \frac{1}{5!} \approx \frac{101}{120}$

4. (15%) Evaluate the following expression (For parts (a) and (b), you can first use the basic Taylor series to determine the original functions)

(a) $\frac{\pi}{3} - \frac{\pi^3}{3^3 3!} + \frac{\pi^5}{3^5 5!} - \frac{\pi^7}{3^7 7!} + \cdots$

(b) $\sum_{n=1}^{\infty} \frac{2^n(n+1)}{n!}$

(c) $\lim_{x\to 0} \frac{1}{\ln(1+x)} - \frac{1}{x}$

Ans:

(a) $\frac{\pi}{3} - \frac{\pi^3}{3^3 3!} + \frac{\pi^5}{3^5 5!} - \frac{\pi^7}{3^7 7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{\pi}{3}\right)^{2n+1}}{(2n+1)!} = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

(b) Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$xe^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^{n+1} \to xe^x + e^x = \sum_{n=0}^{\infty} \frac{(n+1)}{n!} x^n \to \sum_{n=0}^{\infty} \frac{2^n(n+1)}{n!} =$

$3e^2$ (Substitute $x = 2$) $\to \sum_{n=1}^{\infty} \frac{2^n(n+1)}{n!} = 3e^2 - 1$

(c) $\lim_{x\to 0} \frac{1}{\ln(1+x)} - \frac{1}{x} = \lim_{x\to 0} \frac{x - \ln(1+x)}{x\ln(1+x)} = \lim_{x\to 0} \frac{x - \left[x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots\right]}{x\left[x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots\right]} = \lim_{x\to 0} \frac{\frac{1}{2}x^2 - \frac{1}{3}x^3 + \cdots}{x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^4 - \cdots} =$

$\lim_{x\to 0} \frac{\frac{1}{2} - \frac{1}{3}x + \cdots}{1 - \frac{1}{2}x + \frac{1}{3}x^2 - \cdots}$(divide by x^2)$= \frac{1}{2}$

5. (12%) Derive the Maclaurin series of $f(x) = arcsin(x)$ and $g(x) = arcsin(3x^2)$. In addition, calculate $g^{(22)}(0)$ (You can use generalized binomial coefficient to represent the final results.)

Ans:

$$\frac{d}{dx}\arcsin(\mathrm{x}) = \frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} \binom{\frac{-1}{2}}{n} (-x^2)^n$$

$$\arcsin(\mathrm{x}) = \sum_{n=0}^{\infty} \binom{\frac{-1}{2}}{n} (-1)^n \frac{x^{2n+1}}{2n+1} + C$$

Substitute 0 into the equation we have $C = 0$. Therefore,

$$\arcsin(\mathrm{x}) = \sum_{n=0}^{\infty} \binom{\frac{-1}{2}}{n} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\arcsin(3x^2) = \sum_{n=0}^{\infty} \binom{\frac{-1}{2}}{n} (-1)^n \frac{3^{2n+1}x^{4n+2}}{2n+1}$$

The definition of Maclaurin series of $g(x)$ is $\sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} x^k$

Comparing the coefficient of x^{22} ($n = 5$ since $5 * 4 + 2 = 22$)

We have $\frac{g^{(22)}(0)}{22!} = \binom{\frac{-1}{2}}{5} (-1)^5 \frac{3^{2*5+1}}{2*5+1} = -\binom{\frac{-1}{2}}{5} \frac{3^{11}}{11}$

$$g^{(22)}(0) = -\binom{\frac{-1}{2}}{5} \frac{3^{11}}{11} 22!$$

6. (10%) Find the arc length of the curve $x = t^2 + 1,\ y = 4t^3 + 3$ over the interval $0 \le t \le 1$

Ans:

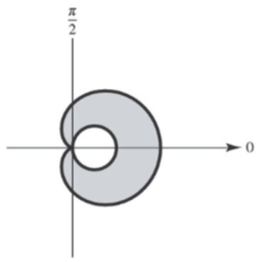
$$\frac{dx}{dt} = 2t, \qquad \frac{dy}{dt} = 12t^2$$

$$s = \int_0^1 \sqrt{(2t)^2 + (12t^2)^2}\, dt = \int_0^1 \sqrt{4t^2 + 144t^4}\, dt$$

$$= \int_0^1 2t\sqrt{1 + 36t^2}\, dt \quad (\text{Let } u = 1 + 36t^2, du = 72tdt)$$

$$= \int_1^{37} \frac{1}{36}\sqrt{u}du = \frac{1}{36} \times \frac{2}{3} u^{\frac{3}{2}}\Big]_1^{37} = \frac{1}{54}[37^{\frac{3}{2}} - 1]$$

7. (10%) Find the area of the shaded region bounded by the curves $r = a(1 + cos(\theta))$ and $r = acos(\theta)$



Ans:

$$A = 2\frac{1}{2}\int_0^{\pi} (a(1 + cos\theta))^2\, d\theta - \frac{a^2\pi}{4} = a^2 \int_0^{\pi} 1 + cos^2\theta + \frac{1 + cos2\theta}{2} d\theta - \frac{a^2\pi}{4}$$

$$= a^2 \left[\frac{3}{2}\theta + 2sin\theta + \frac{\sin(2\theta)}{4}\right]_0^{\pi} - \frac{a^2\pi}{4} = \frac{5a^2\pi}{4}$$

8. (9%) Classify the following surface, if it is quadratic surface you should further classify it into six basic types of surface
   (a) $x^2 + y^2 - z = 0$
   (b) $r^2 = z^2 + 4$ (this representation is in cylindrical coordinates)
   (c) $\rho = 4csc(\Phi)sec(\theta)$ (this representation is in spherical coordinates)

Ans:

(a) $x^2 + y^2 - z = 0 \rightarrow x^2 + y^2 = z$ which is a surface of revolution or elliptic paraboloid

(b) $r^2 = z^2 + 4 \rightarrow x^2 + y^2 - z^2 = 4 \rightarrow \frac{x^2}{(2)^2} + \frac{y^2}{(2)^2} - \frac{z^2}{(2)^2} = 1$ which is a hyperboloid of one sheet

(c) $\rho = 4cs\,c(\Phi)\,sec(\theta) = \frac{4}{\sin(\Phi)\cos(\theta)} \rightarrow x = \rho\sin(\Phi)\cos(\theta) = 4$ which is a plane

Function	Taylor series	Interval of convergence
$\frac{1}{x}$	$1-(x-1)+(x-1)^2-(x-1)^3+\cdots.+(-1)^n(x-1)^n+\cdots.$	$0<x<2$
$\frac{1}{1+x}$	$1-x+x^2-x^3+\cdots.+(-1)^n x^n+\cdots.$	$-1<x<1$
$\ln x$	$(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\cdots.+\frac{(-1)^{n-1}(x-1)^n}{n}+\cdots.$	$0<x\leq 2$
e^x	$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots.+\frac{x^n}{n!}+\cdots.$	$-\infty<x<\infty$
$sin(x)$	$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots.+\frac{(-1)^n x^{2n+1}}{(2n+1)!}+\cdots.$	$-\infty<x<\infty$
$cos(x)$	$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots.+\frac{(-1)^n x^{2n}}{(2n)!}+\cdots.$	$-\infty<x<\infty$
$arctan(x)$	$x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots.+\frac{(-1)^n x^{2n+1}}{2n+1}+\cdots.$	$-1\leq \mathrm{x}\leq 1$
$arcsin(x)$	$x+\frac{x^3}{2\times 3}+\frac{1\times 3x^5}{2\times 4\times 5}+\cdots.+\frac{(2n)!\,x^{2n+1}}{(2^n n!)^2(2n+1)}+\cdots.$	$-1\leq \mathrm{x}\leq 1$
$(1+x)^k$	$1+kx+\frac{k(k-1)x^2}{2!}+\frac{k(k-1)(k-2)x^3}{3!}$ $+\cdots.+\frac{k(k-1)\ldots(k-n+1)x^n}{n!}+\cdots.$	$-1<x<1$

Derivative	Integrals
$\frac{d\sin^{-1}u}{dx}=\frac{u'}{\sqrt{1-u^2}}$	$\int\frac{du}{\sqrt{a^2-u^2}}=\sin^{-1}\frac{u}{a}+C$
$\frac{d\,cos^{-1}u}{dx}=\frac{-u'}{\sqrt{1-u^2}}$	$\int\frac{du}{a^2+u^2}=\frac{1}{a}tan^{-1}\frac{u}{a}+C$
$\frac{d\tan^{-1}u}{dx}=\frac{u'}{1+u^2}$	$\int\frac{du}{u\sqrt{u^2-a^2}}=\frac{1}{a}sec^{-1}\frac{\lvert u\rvert}{a}+C$
$\frac{d\,cot^{-1}u}{dx}=\frac{-u'}{1+u^2}$	
$\frac{d\sec^{-1}u}{dx}=\frac{u'}{\lvert u\rvert\sqrt{u^2-1}}$	
$\frac{d\,csc^{-1}u}{dx}=\frac{-u'}{\lvert u\rvert\sqrt{u^2-1}}$	