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Chapter 14

Multiple Integration

14.1 Summary

1. **Iterated integrals**

(1) $\int_{h_1(y)}^{h_2(y)} f_x(x,y)\,\mathrm{d}x = f(x,y)\Big|_{h_1(y)}^{h_2(y)} = f(h_2(y),y) - f(h_1(y),y)$ **With resp**

(2) $\int_{g_1(x)}^{g_2(x)} f_y(x,y)\,\mathrm{d}y = f(x,y)\Big|_{g_1(x)}^{g_2(x)} = f(x,g_2(x)) - f(x,g_1(x))$ **With resp**

2. **Area of a region in the plane**

1. If R is defined by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, where g_1 and g_2 are continuous on $[a, b]$, then the area of R is given by

$$A = \int_a^b \int_{g_1(x)}^{g_2(x)} \mathrm{d}y \, \mathrm{d}x. \qquad \text{Figure ?? (vertically simple)}$$

2. If R is defined by $c \leq y \leq d$ and $h_1(y) \leq x \leq h_2(y)$, where h_1 and h_2 are continuous on $[c, d]$, then the area of R is given by

$$A = \int_c^d \int_{h_1(y)}^{h_2(y)} \mathrm{d}x \, \mathrm{d}y. \qquad \text{Figure ?? (horizontally simple)}$$

3. **Double integral** If f defined on a closed, bounded region R in the xy-plane, then the **double integral** (二重積分) of f over R is given by

$$\iint_R f(x,y)\,\mathrm{d}A = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i,y_i)\Delta A_i$$

provided the limit exists. If the limit exists, then f is over R. 9

4. **Volume of a solid region** (立體區域體積) If f is integrable over a plane region R and $f(x,y)\geq 0$ for all (x,y) in R, then the volume of the solid region that lies above R and below the graph of f defined as

$$V = \iint_R f(x,y)\,\mathrm{d}A.$$

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5. **Properties of double integrals** (二重積分性質) Let f and g be

continuous over a closed, bounded plane region R, and let c be a constant.

(a) $\iint_R cf(x,y)\,\mathrm{d}A = c\iint_R f(x,y)\,\mathrm{d}A$

(b) $\iint_R [f(x,y) \pm g(x,y)]\,\mathrm{d}A = \iint_R f(x,y)\,\mathrm{d}A \pm \iint_R g(x,y)\,\mathrm{d}A$

(c) $\iint_R f(x,y)\,\mathrm{d}A \geq 0$, if $f(x,y) \geq 0$

(d) $\iint_R f(x,y)\,\mathrm{d}A \geq \iint_R g(x,y)\,\mathrm{d}A$, if $f(x,y) \geq g(x,y)$

(e) $\iint_R f(x,y)\,\mathrm{d}A = \iint_{R_1} f(x,y)\,\mathrm{d}A + \iint_{R_2} f(x,y)\,\mathrm{d}A$, where R is the union of two nonoverlapping subregions R_1 and R_2.

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6. **Fubini's Theorem** (富比尼定理) Let f be continuous on a plane region R.

(a) If R is defined by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, where g_1 and

g_2 are continuous on $[a, b]$, then

$$\iint_R f(x, y)\,\mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,\mathrm{d}y\,\mathrm{d}x.$$

(b) If R is defined by $c \le y \le d$ and $h_1(y) \le x \le h_2(y)$, where h_1 and h_2 are continuous on $[c, d]$, then

$$\iint_R f(x, y)\,\mathrm{d}A = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

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7. **The average value of a function over a region** If f is integrable over the plane region R, then the **average value** (平均值) of f over R is

$$\frac{1}{A}\iint_R f(x, y)\,\mathrm{d}A$$

the mass m of the **lamina** (薄膜) is given by

$$m = \iint_R \rho(x, y)\,\mathrm{d}A. \qquad \textbf{Variable density}$$

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10. **Moments and center of mass of a variable density planar lamina**

Let ρ be a continuous density function on the planar lamina R. The **moments of mass** (質矩) with respect to the x- and y-axes are

$$M_x = \iint_R y\rho(x, y)\,\mathrm{d}A \quad \text{and} \quad M_y = \iint_R x\rho(x, y)\,\mathrm{d}A.$$

If m is the mass of the lamina, then the **center of mass** (質心) is

$$(\overline{x}, \overline{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right).$$

If R represents a simple plane region rather than a lamina, the point

$(\overline{x}, \overline{y})$ is called the **centroid** (**形心**) of the region. 22

11. Suppose a planar lamina is revolving about a line with an **angular speed** (**角速度**) of ω radians per second. The kinetic energy E of the revolving lamina is

$$E = \frac{1}{2} I \omega^2.$$ **Kinetic energy for rotational motion**

12. On the other hand, the kinetic energy E of a mass m moving in a straight line at a velocity v is

$$E = \frac{1}{2} m v^2.$$ **Kinetic energy for linear motion**

The **radius of gyration** (**旋轉半徑**) $\overline{\overline{r}}$ of a revolving mass m with mo-

14. **Triple integral** If f is continuous over a bounded solid region Q, then the **triple integral** (三重積分) of f over Q is defined as

$$\iiint_Q f(x,y,z)\,\mathrm{d}V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i,y_i,z_i)\Delta V_i$$

provided the limit exists. The **volume** (體積) of the solid region Q is given by

$$\text{Volume of } Q = \iiint_Q \mathrm{d}V.$$

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15. **Evaluation by iterated integrals** Let f be continuous on a solid region Q defined by

$$a \le x \le b, \quad h_1(x) \le y \le h_2(x), \quad g_1(x,y) \le z \le g_2(x,y)$$

where h_1, h_2, g_1, and g_2 are continuous functions. Then,

$$\iiint_Q f(x,y,z)\,\mathrm{d}V = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$$

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16. The **center of mass** (**質心**) of a solid region Q of mass m is given by $(\bar{x}, \bar{y}, \bar{z})$, where

$$m = \iiint_Q \rho(x,y,z)\,\mathrm{d}V,\ M_{yz} = \iiint_Q x\rho(x,y,z)\,\mathrm{d}V,\ M_{xz} = \iiint_Q y\rho(x,$$

and

$$\bar{x} = \frac{M_{yz}}{m}, \quad \bar{y} = \frac{M_{xz}}{m}, \quad \bar{z} = \frac{M_{xy}}{m}.$$

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17. The **second moments** (**第二矩**) (or **moments of inertia** (**慣性矩**))

are

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z.$$

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19. **Triple integral in cylindrical coordinates**: If R is r-simple, the iterated form of the triple integral in cylindrical form is

$$\iiint_Q f(x,y,z)\,\mathrm{d}V = \int_{\theta_1}^{\theta_2}\int_{g_1(\theta)}^{g_2(\theta)}\int_{h_1(r\cos\theta, r\sin\theta)}^{h_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)r\,\mathrm{d}z\,\mathrm{d}$$

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20. The rectangular conversion equations for spherical coordinates are

$$x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta \qquad z = \rho\cos\phi.$$

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21. **Triple integral in spherical coordinates**: Using the usual process involving an inner partition, summation, and a limit, you can develop the following version of a triple integral in spherical coordinates for a continuous function f defined on the solid region Q.

$$\iiint_Q f(x,y,z)\,\mathrm{d}V = \int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_1}^{\rho_2} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\,\mathrm{si}$$

22. **Jacobian** If $x = g(u,v)$ and $y = h(u,v)$, then the **Jacobian** (雅可比) of x and y with respect to u and v, denoted by $\partial(x,y)/\partial(u,v)$,

is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}.$$

23. **Change of variables for double integrals** Let R be a vertically or horizontally simple region in the xy-plane, and let S be a vertically or horizontally simple region in the uv-plane. Let T from S to R be given by $T(u,v) = (x,y) = (g(u,v), h(u,v))$, where g and h have continuous first partial derivatives. Assume that T is one-to-one except possibly on the boundary of S. If f is continuous on R, and $\partial(x,y)/\partial(u,v)$ is

nonzero on S, then

$$\iint_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \iint_S f(g(u, v), h(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \mathrm{d}u\,\mathrm{d}v.$$

24. **Gram determinant**: If $x_i = x_i(t_1, \ldots, t_m)$, $i = 1, 2, \ldots, n$, then the **Gram determinant** (**格拉姆行列式**) of $(x_1, x_2, \ldots, x_n)$ with respect to $(t_1, \ldots, t_m)$ is

$$G = \det X^T X$$

where $X = \partial(x_1, x_2, \ldots, x_n)/\partial(t_1, t_2, \ldots, t_m) = \begin{bmatrix} \frac{\partial x_1}{\partial t_1} & \frac{\partial x_1}{\partial t_2} & \cdots & \frac{\partial x_1}{\partial t_m} \\ \frac{\partial x_2}{\partial t_1} & \frac{\partial x_2}{\partial t_2} & \cdots & \frac{\partial x_2}{\partial t_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial t_1} & \frac{\partial x_n}{\partial t_2} & \cdots & \frac{\partial x_n}{\partial t_m} \end{bmatrix}$.

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25. **Cauchy-Binet Formula** (**柯西-比内公式**): Let A be an $m \times n$ matrix and B an $n \times m$ matrix. Then the determinant of their product $C = AB$ can be written as a sum of products of minors of A and B:

$$|C| = \sum_{1 \le k_1 < k_2 < \cdots < k_m \le n} A \begin{pmatrix} 1 & 2 & \cdots & m \\ k_1 & k_2 & \cdots & k_m \end{pmatrix} B \begin{pmatrix} k_1 & k_2 & \cdots & k_m \\ 1 & 2 & \cdots & m \end{pmatrix}.$$

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26. **Change of variables for multiple integrals**: Let Ω be an m-dimensional region in $\mathbb{R}^n$, $m \leq n$, and let $\mathcal{T}$ be an m-dimensional region in $\mathbb{R}^m$. Let T from $\mathcal{T}$ to Ω be given by $T(t_1, t_2, \ldots, t_m) = (x_1(t_1, t_2, \ldots, t_m), x_2(t_1, t_2, \ldots, t_m), \ldots, x_n(t_1, t_2, \ldots, t_m))$, where g_i's have continuous first partial derivatives. Assume that T is one-to-one except possibly on the boundary of $\mathcal{T}$. If f is continuous on Ω, and G is nonzero on $\mathcal{T}$, then

$$\int \cdots \int_{\Omega} f(x_1, \ldots, x_n)\, \mathrm{d}V = \int \cdots \int_{\mathcal{T}} f(x_1(t_1, \ldots, t_m), \ldots, x_n(t_1, \ldots, t_m)$$

27. **Volume of an m-dimensional region in $\mathbb{R}^n$, $m \leq n$**: Let Ω be an m-dimensional region in $\mathbb{R}^n$, $m \leq n$, and let $\mathcal{T}$ be an m-dimensional region in $\mathbb{R}^m$. Let T from $\mathcal{T}$ to Ω be given by $T(t_1, t_2, \ldots, t_m) =$

$(x_1(t_1, t_2, \ldots, t_m), x_2(t_1, t_2, \ldots, t_m), \ldots, x_n(t_1, t_2, \ldots, t_m))$, where g_i's have continuous first partial derivatives. Assume that T is one-to-one except possibly on the boundary of $\mathcal{T}$ and G is nonzero on $\mathcal{T}$. Then the volume of Ω is

$$\int \cdots \int_\Omega \mathrm{d}V = \int \cdots \int_{\mathcal{T}} \sqrt{G}\, \mathrm{d}t_m \cdots \mathrm{d}t_1.$$

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