

# CONTENTS

<b>14 Multiple Integration</b>	<b>1</b>
14.1 Summary . . . . .	1
<b>Index</b>	<b>20</b>

# LIST OF TABLES

# LIST OF FIGURES



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 Chapter **14**
**MULTIPLE INTEGRATION**

## 14.1 Summary

**Section 14.1 Iterated integrals and area in the plane . . . . . 1**
**1. Iterated integrals**

$$(1) \int_{h_1(y)}^{h_2(y)} f_x(x, y) \, dx = f(x, y) \Big|_{h_1(y)}^{h_2(y)} = f(h_2(y), y) - f(h_1(y), y) \quad \text{With resp}$$

$$(2) \int_{g_1(x)}^{g_2(x)} f_y(x, y) \, dy = f(x, y) \Big|_{g_1(x)}^{g_2(x)} = f(x, g_2(x)) - f(x, g_1(x)) \quad \text{With resp}$$

.....2

## 2. Area of a region in the plane

1. If  $R$  is defined by  $a \leq x \leq b$  and  $g_1(x) \leq y \leq g_2(x)$ , where  $g_1$  and  $g_2$  are continuous on  $[a, b]$ , then the area of  $R$  is given by

$$A = \int_a^b \int_{g_1(x)}^{g_2(x)} dy \, dx. \quad \text{Figure ?? (vertically simple)}$$

2. If  $R$  is defined by  $c \leq y \leq d$  and  $h_1(y) \leq x \leq h_2(y)$ , where  $h_1$  and  $h_2$  are continuous on  $[c, d]$ , then the area of  $R$  is given by

$$A = \int_c^d \int_{h_1(y)}^{h_2(y)} dx \, dy. \quad \text{Figure ?? (horizontally simple)}$$

.....4

**Section 14.2 Double integrals and volume** ..... 6

3. **Double integral** If  $f$  defined on a closed, bounded region  $R$  in the  $xy$ -plane, then the **double integral** (二重積分) of  $f$  over  $R$  is given by

$$\iint_R f(x, y) \, dA = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(x_i, y_i) \Delta A_i$$

provided the limit exists. If the limit exists, then  $f$  is over  $R$ .....9

4. **Volume of a solid region** (立體區域體積) If  $f$  is integrable over a plane region  $R$  and  $f(x, y) \geq 0$  for all  $(x, y)$  in  $R$ , then the volume of the solid region that lies above  $R$  and below the graph of  $f$  defined as

$$V = \iint_R f(x, y) \, dA.$$

.....9

5. **Properties of double integrals** (二重積分性質) Let  $f$  and  $g$  be

continuous over a closed, bounded plane region  $R$ , and let  $c$  be a constant.

$$(a) \iint_R cf(x, y) \, dA = c \iint_R f(x, y) \, dA$$

$$(b) \iint_R [f(x, y) \pm g(x, y)] \, dA = \iint_R f(x, y) \, dA \pm \iint_R g(x, y) \, dA$$

$$(c) \iint_R f(x, y) \, dA \geq 0, \quad \text{if } f(x, y) \geq 0$$

$$(d) \iint_R f(x, y) \, dA \geq \iint_R g(x, y) \, dA, \quad \text{if } f(x, y) \geq g(x, y)$$

$$(e) \iint_R f(x, y) \, dA = \iint_{R_1} f(x, y) \, dA + \iint_{R_2} f(x, y) \, dA, \text{ where } R \text{ is the union of two nonoverlapping subregions } R_1 \text{ and } R_2.$$

..... 10

6. **Fubini's Theorem** (富比尼定理) Let  $f$  be continuous on a plane region  $R$ .

(a) If  $R$  is defined by  $a \leq x \leq b$  and  $g_1(x) \leq y \leq g_2(x)$ , where  $g_1$  and



$g_2$  are continuous on  $[a, b]$ , then

$$\iint_R f(x, y) \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx.$$

(b) If  $R$  is defined by  $c \leq y \leq d$  and  $h_1(y) \leq x \leq h_2(y)$ , where  $h_1$  and  $h_2$  are continuous on  $[c, d]$ , then

$$\iint_R f(x, y) \, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy.$$

..... 11

7. **The average value of a function over a region** If  $f$  is integrable over the plane region  $R$ , then the **average value** (平均值) of  $f$  over  $R$  is

$$\frac{1}{A} \iint_R f(x, y) \, dA$$

where  $A$  is the area of  $R$ . ..... 14

## Section 14.3 Change of variables: Polar coordinates ..... 15

8. **Change of variables to polar form** Let  $R$  be a plane region consisting of all points  $(x, y) = (r \cos \theta, r \sin \theta)$  satisfying the conditions  $0 \leq g_1(\theta) \leq r \leq g_2(\theta)$ ,  $\alpha \leq \theta \leq \beta$ , where  $0 \leq (\beta - \alpha) \leq 2\pi$ . If  $g_1$  and  $g_2$  are continuous on  $[\alpha, \beta]$  and  $f$  is continuous on  $R$ , then

$$\iint_R f(x, y) \, dA = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} f(r \cos \theta, r \sin \theta) r \, dr \, d\theta.$$

..... 17

## Section 14.4 Center of mass and moments of inertia ..... 19

9. **Mass of a planar lamina of variable density** If  $\rho$  is a continuous density function on the lamina corresponding to a plane region  $R$ , then

the mass  $m$  of the **lamina** (**薄膜**) is given by

$$m = \iint_R \rho(x, y) \, dA. \quad \text{Variable density}$$

..... 20

### 10. **Moments and center of mass of a variable density planar lamina**

Let  $\rho$  be a continuous density function on the planar lamina  $R$ . The **moments of mass** (**質矩**) with respect to the  $x$ - and  $y$ -axes are

$$M_x = \iint_R y\rho(x, y) \, dA \quad \text{and} \quad M_y = \iint_R x\rho(x, y) \, dA.$$

If  $m$  is the mass of the lamina, then the **center of mass** (**質心**) is

$$(\bar{x}, \bar{y}) = \left( \frac{M_y}{m}, \frac{M_x}{m} \right).$$

If  $R$  represents a simple plane region rather than a lamina, the point

$(\bar{x}, \bar{y})$  is called the **centroid** (形心) of the region.....22

11. Suppose a planar lamina is revolving about a line with an **angular speed** (角速度) of  $\omega$  radians per second. The kinetic energy  $E$  of the revolving lamina is

$$E = \frac{1}{2}I\omega^2. \quad \text{Kinetic energy for rotational motion}$$

12. On the other hand, the kinetic energy  $E$  of a mass  $m$  moving in a straight line at a velocity  $v$  is

$$E = \frac{1}{2}mv^2. \quad \text{Kinetic energy for linear motion}$$

The **radius of gyration** (旋轉半徑)  $\bar{r}$  of a revolving mass  $m$  with mo-

ment of inertia  $I$  is defined as

$$\bar{r} = \sqrt{\frac{I}{m}}. \quad \text{Radius of gyration}$$

..... 25

**Section 14.5 Surface area** ..... 25

13. **Surface area** (曲面面積) If  $f$  and its partial derivatives are continuous on the closed region  $R$  in the  $xy$ -plane, then the area of the surface  $S$  given by  $z = f(x, y)$  over  $R$  is defined as

$$\text{Surface area} = \iint_R dS = \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} dA.$$

..... 27

**Section 14.6 Triple integrals and applications** ..... 30

14. **Triple integral** If  $f$  is continuous over a bounded solid region  $Q$ , then the **triple integral** (三重積分) of  $f$  over  $Q$  is defined as

$$\iiint_Q f(x, y, z) \, dV = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(x_i, y_i, z_i) \Delta V_i$$

provided the limit exists. The **volume** (體積) of the solid region  $Q$  is given by

$$\text{Volume of } Q = \iiint_Q dV.$$

..... 31

15. **Evaluation by iterated integrals** Let  $f$  be continuous on a solid region  $Q$  defined by

$$a \leq x \leq b, \quad h_1(x) \leq y \leq h_2(x), \quad g_1(x, y) \leq z \leq g_2(x, y)$$

where  $h_1$ ,  $h_2$ ,  $g_1$ , and  $g_2$  are continuous functions. Then,

$$\iiint_Q f(x, y, z) \, dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} f(x, y, z) \, dz \, dy \, dx.$$

..... 31

16. The **center of mass** (質心) of a solid region  $Q$  of mass  $m$  is given by  $(\bar{x}, \bar{y}, \bar{z})$ , where

$$m = \iiint_Q \rho(x, y, z) \, dV, \quad M_{yz} = \iiint_Q x\rho(x, y, z) \, dV, \quad M_{xz} = \iiint_Q y\rho(x, y, z) \, dV,$$

and

$$\bar{x} = \frac{M_{yz}}{m}, \quad \bar{y} = \frac{M_{xz}}{m}, \quad \bar{z} = \frac{M_{xy}}{m}.$$

..... 36

17. The **second moments** (第二矩) (or **moments of inertia** (慣性矩))

about the  $x$ -,  $y$ -, and  $z$ -axes are as follows.

$$I_x = \iiint_Q (y^2 + z^2)\rho(x, y, z) \, dV, \quad I_y = \iiint_Q (x^2 + z^2)\rho(x, y, z) \, dV, \quad I_z = \iiint_Q (x^2 + y^2)\rho(x, y, z) \, dV.$$

$$I_x = I_{xz} + I_{xy}, \quad I_y = I_{yz} + I_{xy}, \quad \text{and} \quad I_z = I_{yz} + I_{xz}$$

where  $I_{xy}$ ,  $I_{xz}$  and  $I_{yz}$  are as follows.

$$I_{xy} = \iiint_Q z^2\rho(x, y, z) \, dV, \quad I_{xz} = \iiint_Q y^2\rho(x, y, z) \, dV, \quad I_{yz} = \iiint_Q x^2\rho(x, y, z) \, dV.$$

..... 36

**Section 14.7 Triple integrals in cylindrical and spherical coordinates** ..... 38

18. The rectangular conversion equations for cylindrical coordinates (圓柱坐標)



are

$$x = r \cos \theta \qquad y = r \sin \theta \qquad z = z.$$

..... 38

19. **Triple integral in cylindrical coordinates:** If  $R$  is  $r$ -simple, the iterated form of the triple integral in cylindrical form is

$$\iiint_Q f(x, y, z) \, dV = \int_{\theta_1}^{\theta_2} \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(r \cos \theta, r \sin \theta)}^{h_2(r \cos \theta, r \sin \theta)} f(r \cos \theta, r \sin \theta, z) r \, dz \, dr \, d\theta$$

..... 38

20. The rectangular conversion equations for spherical coordinates are

$$x = \rho \sin \phi \cos \theta \qquad y = \rho \sin \phi \sin \theta \qquad z = \rho \cos \phi.$$

..... 40

21. **Triple integral in spherical coordinates:** Using the usual process involving an inner partition, summation, and a limit, you can develop the following version of a triple integral in spherical coordinates for a continuous function  $f$  defined on the solid region  $Q$ .

$$\iiint_Q f(x, y, z) \, dV = \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} \int_{\rho_1}^{\rho_2} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

..... 41

**Section 14.8 Change of variables: Jacobians** ..... 43

22. **Jacobian** If  $x = g(u, v)$  and  $y = h(u, v)$ , then the **Jacobian** (雅可比) of  $x$  and  $y$  with respect to  $u$  and  $v$ , denoted by  $\partial(x, y)/\partial(u, v)$ ,

is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v}.$$

..... 44

23. **Change of variables for double integrals** Let  $R$  be a vertically or horizontally simple region in the  $xy$ -plane, and let  $S$  be a vertically or horizontally simple region in the  $uv$ -plane. Let  $T$  from  $S$  to  $R$  be given by  $T(u, v) = (x, y) = (g(u, v), h(u, v))$ , where  $g$  and  $h$  have continuous first partial derivatives. Assume that  $T$  is one-to-one except possibly on the boundary of  $S$ . If  $f$  is continuous on  $R$ , and  $\partial(x, y)/\partial(u, v)$  is

nonzero on  $S$ , then

$$\iint_R f(x, y) \, dx \, dy = \iint_S f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du \, dv.$$

..... 45

**Section 14.9 Change of variables: an  $m$ -dimensional region in**

$\mathbb{R}^n$ ,  $m \leq n$  ..... 48

24. **Gram determinant:** If  $x_i = x_i(t_1, \dots, t_m)$ ,  $i = 1, 2, \dots, n$ , then the **Gram determinant** (格拉姆行列式) of  $(x_1, x_2, \dots, x_n)$  with respect to  $(t_1, \dots, t_m)$  is

$$G = \det X^T X$$

$$\text{where } X = \partial(x_1, x_2, \dots, x_n) / \partial(t_1, t_2, \dots, t_m) = \begin{bmatrix} \frac{\partial x_1}{\partial t_1} & \frac{\partial x_1}{\partial t_2} & \cdots & \frac{\partial x_1}{\partial t_m} \\ \frac{\partial x_2}{\partial t_1} & \frac{\partial x_2}{\partial t_2} & \cdots & \frac{\partial x_2}{\partial t_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial t_1} & \frac{\partial x_n}{\partial t_2} & \cdots & \frac{\partial x_n}{\partial t_m} \end{bmatrix}.$$

48

25. **Cauchy-Binet Formula** (柯西-比内公式): Let  $A$  be an  $m \times n$  matrix and  $B$  an  $n \times m$  matrix. Then the determinant of their product  $C = AB$  can be written as a sum of products of minors of  $A$  and  $B$ :

$$|C| = \sum_{1 \leq k_1 < k_2 < \cdots < k_m \leq n} A \begin{pmatrix} 1 & 2 & \cdots & m \\ k_1 & k_2 & \cdots & k_m \end{pmatrix} B \begin{pmatrix} k_1 & k_2 & \cdots & k_m \\ 1 & 2 & \cdots & m \end{pmatrix}.$$

..... 49

26. **Change of variables for multiple integrals:** Let  $\Omega$  be an  $m$ -dimensional region in  $\mathbb{R}^n$ ,  $m \leq n$ , and let  $\mathcal{T}$  be an  $m$ -dimensional region in  $\mathbb{R}^m$ . Let  $T$  from  $\mathcal{T}$  to  $\Omega$  be given by  $T(t_1, t_2, \dots, t_m) = (x_1(t_1, t_2, \dots, t_m), x_2(t_1, t_2, \dots, t_m), \dots, x_n(t_1, t_2, \dots, t_m))$ , where  $g_i$ 's have continuous first partial derivatives. Assume that  $T$  is one-to-one except possibly on the boundary of  $\mathcal{T}$ . If  $f$  is continuous on  $\Omega$ , and  $G$  is nonzero on  $\mathcal{T}$ , then

$$\int \cdots \int_{\Omega} f(x_1, \dots, x_n) dV = \int \cdots \int_{\mathcal{T}} f(x_1(t_1, \dots, t_m), \dots, x_n(t_1, \dots, t_m)) \dots$$

..... 49

27. **Volume of an  $m$ -dimensional region in  $\mathbb{R}^n$ ,  $m \leq n$ :** Let  $\Omega$  be an  $m$ -dimensional region in  $\mathbb{R}^n$ ,  $m \leq n$ , and let  $\mathcal{T}$  be an  $m$ -dimensional region in  $\mathbb{R}^m$ . Let  $T$  from  $\mathcal{T}$  to  $\Omega$  be given by  $T(t_1, t_2, \dots, t_m) =$

$(x_1(t_1, t_2, \dots, t_m), x_2(t_1, t_2, \dots, t_m), \dots, x_n(t_1, t_2, \dots, t_m))$ , where  $g_i$ 's have continuous first partial derivatives. Assume that  $T$  is one-to-one except possibly on the boundary of  $\mathcal{T}$  and  $G$  is nonzero on  $\mathcal{T}$ . Then the volume of  $\Omega$  is

$$\int \cdots \int_{\Omega} dV = \int \cdots \int_{\mathcal{T}} \sqrt{G} dt_m \cdots dt_1.$$

..... 49

**INDEX**

angular speed 角速度, 8

area 面積

of the surface  $S$  曲面  $S$ , 9

average value of a function 函數的平  
均值

over a region  $R$  在一區域  $R$ , 5

Cauchy-Binet Formula 柯西-比內公  
式, 17

center 中心

of mass 質量

of a solid region  $Q$  立體區域  $Q$ ,  
11

centroid 形心

of a simple region 單純區域, 7

change of variables 變數變換

for double integrals 二重積分, 15

for multiple integrals 多重積分, 18

to polar form 極坐標, 6



- using a Gram determinant 格拉姆 function(s) 函數  
行列式, 16
- using a Jacobian 雅可比, 14
- double integral 二重積分, 3
- change of variables for 變數變換,  
15
- of  $f$  over  $R$   $f$  在區域  $R$ , 3
- properties of 性質, 3
- evaluation 計算
- by iterated integrals 逐次積分, 10
- Fubini's Theorem 富比尼定理, 4
- for a triple integral 三重積分, 10
- average value of 平均值, 5
- density 密度, 6
- Gram determinant 格拉姆行列式, 16
- gyration, radius of 旋轉, 半徑, 9
- integrable function 可積函數, 3
- integrable 可積的, 3
- integral(s) 積分
- double 二重, 3
- triple 三重, 10
- iterated integral 逐次積分
- evaluation by 計算, 10

Jacobian 雅可比, 14

$m$ -dimensional region in  $n$ -dimensional space  $n$  維中  $m$  維子區域

volume of 體積, 18

mass 質量

center of 中心

of a solid region  $Q$  立體區域  $Q$ ,  
11

moments of 質矩, 7

of a planar lamina of variable density 非均勻平面狀薄膜, 6

moment(s) 力矩

of mass 質量, 7

moment(s) 矩

of inertia 慣性, 12

second 第二, 12

moments of inertia 慣性矩, 11

multiple integral 多重積分

change of variables for 變數變換,  
18

$n$  維中  $m$  維子區域  $m$ -dimensional region in  $n$ -dimensional space

體積 volume of, 18

properties 性質

of double integrals 二重積分, 3

radius 半徑

of gyration 旋轉, 9

region in the plane 平面區域

area of 面積, 2

second moments 第二矩, 11

surface area 曲面面積

of a solid 立體, 9

triple integral 三重積分, 10

in cylindrical coordinates 圓柱坐標, 13

in spherical coordinates 球坐標, 14

volume of a solid region 立體區域體 力矩 moment(s)

積, 3

volume of a solid region 立體體積,  
10

中心 center

質量 of mass

立體區域  $Q$  of a solid region  $Q$ ,  
11

二重積分 double integral, 3

$f$  在區域  $R$  of  $f$  over  $R$ , 3

性質 properties of, 3

變數變換 change of variables for,  
15

- 質量 of mass, 7
- 三重積分 triple integral, 10
- 球坐標 in spherical coordinates, 14
- 圓柱坐標 in cylindrical coordinates, 13
- 可積函數 integrable function, 3
- 可積的 integrable, 3
- 平面區域 region in the plane
- 面積 area of, 2
- 立體區域體積 volume of a solid region, 3
- 立體體積 volume of a solid region, 10
- 半徑 radius
- 旋轉 of gyration, 9
- 多重積分 multiple integral
- 變數變換 change of variables for, 18
- 曲面面積 surface area
- 立體 of a solid, 9
- 形心 centroid
- 單純區域 of a simple region, 7
- 角速度 angular speed, 8
- 函數 function(s)
- 平均值 average value of, 5
- 密度 density, 6

- 函數的平均值 average value of a function  
    在一區域  $R$  over a region  $R$ , 5
- 性質 properties  
    二重積分 of double integrals, 3
- 柯西-比內公式 Cauchy-Binet Formula, 17
- 計算 evaluation  
    逐次積分 by iterated integrals, 10
- 面積 area  
    曲面  $S$  of the surface  $S$ , 9
- 格拉姆行列式 Gram determinant, 16
- 矩 moment(s)  
    第二 second, 12  
    慣性 of inertia, 12  
    旋轉，半徑 gyration, radius of, 9  
    第二矩 second moments, 11  
    逐次積分 iterated integral  
        計算 evaluation by, 10  
    富比尼定理 Fubini's Theorem, 4  
    三重積分 for a triple integral, 10
- 雅可比 Jacobian, 14
- 慣性矩 moments of inertia, 11
- 質量 mass  
    中心 center of  
        立體區域  $Q$  of a solid region  $Q$ ,

11

極坐標 to polar form, 6

非均勻平面狀薄膜 of a planar  
lamina of variable density, 6

質矩 moments of, 7

積分 integral(s)

二重 double, 3

三重 triple, 10

變數變換 change of variables

二重積分 for double integrals, 15

多重積分 for multiple integrals, 18

格拉姆行列式 using a Gram deter-  
minant, 16

雅可比 using a Jacobian, 14