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Chapter 12

Vector-Valued Functions

12.1 Summary

1. **Vector-valued function** A function of the form

$$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} \quad \textbf{(Plane)}$$

or

$$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k} \quad \textbf{(Space)}$$

is a **vector-valued function** (**向量函數**), where the **component functions** (**分量函數**) f, g, and h are real-valued functions of the parameter t. Vector-valued functions are sometimes denoted as $\mathbf{r}(t) = \langle f(t), g(t)\rangle$ or $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$. 4

2. **The limit of a vector-valued function**

1. If $\mathbf{r}$ is a vector-valued function such that $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$, then

$$\lim_{t\to a} \mathbf{r}(t) = \left[\lim_{t\to a} f(t)\right]\mathbf{i} + \left[\lim_{t\to a} g(t)\right]\mathbf{j} \qquad \textbf{Plane}$$

provided f and g have limits as $t \to a$.

2. If $\mathbf{r}$ is a vector-valued function such that $\mathbf{r}(t) = f(t)\,\mathbf{i}+g(t)\,\mathbf{j}+h(t)\,\mathbf{k}$,

then

$$\lim_{t\to a}\mathbf{r}(t)=\left[\lim_{t\to a}f(t)\right]\mathbf{i}+\left[\lim_{t\to a}g(t)\right]\mathbf{j}+\left[\lim_{t\to a}h(t)\right]\mathbf{k}$$ **Space**

provided f, g, and h have limits as $t\to a$.

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3. **Continuity of a vector-valued function** A vector-valued function $\mathbf{r}$ is **continuous at a point** (在一點連續) given by $t=a$ if the limit of $\mathbf{r}(t)$ exists as $t\to a$ and

$$\lim_{t\to a}\mathbf{r}(t)=\mathbf{r}(a).$$

A vector-valued function $\mathbf{r}$ is **continuous on an interval** (在一區間上連續) I if it is continuous at every point in the interval. 21

Section 12.2 Differentiation and integration of vector-valued

(a) If $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$, where f and g are differentiable functions of

t, then

$$\mathbf{r}'(t) = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j}.$$

(b) If $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where f, g, and h are differentiable functions of t, then

$$\mathbf{r}'(t) = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}.$$

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6. The parametrization of the curve represented by the vector-valued function

$$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$$

is **smooth on an open interval** (**在一開區間平滑**) if f', g', and h' are continuous on I and $\mathbf{r}'(t) \neq \mathbf{0}$ for any value of t in the interval I.

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7. **Properties of the derivative** Let $\mathbf{r}$ and $\mathbf{u}$ be differentiable vector-valued functions of t, let w be a differentiable real-valued function of t, and let c be scalar.

(a) $D_t\left[c\,\mathbf{r}(t)\right] = c\,\mathbf{r}'(t)$

(b) $D_t\left[\mathbf{r}(t) \pm \mathbf{u}(t)\right] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$

(c) $D_t\left[w(t)\,\mathbf{r}(t)\right] = w(t)\,\mathbf{r}'(t) + w'(t)\,\mathbf{r}(t)$

(d) $D_t\left[\mathbf{r}(t) \cdot \mathbf{u}(t)\right] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$

(e) $D_t\left[\mathbf{r}(t) \times \mathbf{u}(t)\right] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$

(f) $D_t\left[\mathbf{r}\left(w(t)\right)\right] = \mathbf{r}'\left(w(t)\right) w'(t)$

(g) If $\mathbf{r}(t) \cdot \mathbf{r}(t) = c$, then $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.

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8. Integration of vector-valued functions

(a) If $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$, where f and g are continuous on $[a, b]$, then the **indefinite integral** (不定積分) (**antiderivative**) of $\mathbf{r}$ is

$$\int \mathbf{r}(t)\,\mathrm{d}t = \left[\int f(t)\,\mathrm{d}t\right]\mathbf{i} + \left[\int g(t)\,\mathrm{d}t\right]\mathbf{j}$$

and its **definite integral** (定積分) over the interval $a \le t \le b$ is

$$\int_a^b \mathbf{r}(t)\,\mathrm{d}t = \left[\int_a^b f(t)\,\mathrm{d}t\right]\mathbf{i} + \left[\int_a^b g(t)\,\mathrm{d}t\right]\mathbf{j}.$$

(b) If $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where f, g, and h are continuous on $[a, b]$, then the **indefinite integral** (**antiderivative**) of $\mathbf{r}$ is

$$\int \mathbf{r}(t)\,\mathrm{d}t = \left[\int f(t)\,\mathrm{d}t\right]\mathbf{i} + \left[\int g(t)\,\mathrm{d}t\right]\mathbf{j} + \left[\int h(t)\,\mathrm{d}t\right]\mathbf{k}$$

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and its **definite integral** over the interval $a \le t \le b$ is

$$\int_a^b \mathbf{r}(t)\,\mathrm{d}t = \left[\int_a^b f(t)\,\mathrm{d}t\right]\mathbf{i} + \left[\int_a^b g(t)\,\mathrm{d}t\right]\mathbf{j} + \left[\int_a^b h(t)\,\mathrm{d}t\right]\mathbf{k}.$$

9. **Velocity and acceleration** If x and y are twice -differentiable functions of t, and $\mathbf{r}$ is a vector-valued function given by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$, then the velocity vector, acceleration vector, and speed at time t are as follows.

$$\underline{\text{Velocity}}\ (\underline{速度}) = \mathbf{v}(t) = \mathbf{r}'(t) = x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j}$$

$$\underline{\text{Acceleration}}\ (\underline{加速度}) = \mathbf{a}(t) = \mathbf{r}''(t) = x''(t)\,\mathbf{i} + y''(t)\,\mathbf{j}$$

$$\textbf{Speed (速率)} = \|\mathbf{v}(t)\| = \left\|\mathbf{r}'(t)\right\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

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10. For motion along a space curve, $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$, you have

$$\textbf{Velocity} = \mathbf{v}(t) = \mathbf{r}'(t) = x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} + z'(t)\,\mathbf{k}$$
$$\textbf{Acceleration} = \mathbf{a}(t) = \mathbf{r}''(t) = x''(t)\,\mathbf{i} + y''(t)\,\mathbf{j} + z''(t)\,\mathbf{k}$$
$$\textbf{Speed} = \|\mathbf{v}(t)\| = \left\|\mathbf{r}'(t)\right\| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}.$$

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11. **Position function for a projectile** Neglecting air resistance, the path of a projectile launched from an initial height h with initial speed

v_0 and angle of elevation θ is described by the vector function

$$\mathbf{r}(t) = (v_0 \cos\theta)t\,\mathbf{i} + \left[h + (v_0 \sin\theta)t - \frac{1}{2}gt^2\right]\mathbf{j}$$

where g is the acceleration due to gravity. 72

Section 12.4 Tangent vectors and normal vectors 75

12. **Tangent vectors** Let C be a **smooth curve** (平滑曲線) represented by $\mathbf{r}$ on an open interval I. The **unit tangent vector** (單位切向量) $\mathbf{T}(t)$ at t is defined as

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}, \quad \mathbf{r}'(t) \neq \mathbf{0}.$$

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13. The **tangent line to a curve** (曲線的切線) at a point is the line that

passes through the point and is parallel to the unit tangent vector. 79

14. There are infinitely many vectors that are orthogonal to the tangent vector $\mathbf{T}(t)$. One of these is the vector $\mathbf{T}'(t)$. This follows the property

$$\mathbf{T}(t) \cdot \mathbf{T}(t) = \|\mathbf{T}(t)\|^2 = 1 \implies \mathbf{T}(t) \cdot \mathbf{T}'(t) = 0.$$

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15. **Principal unit normal vector** Let C be a smooth curve represented by $\mathbf{r}$ on an open interval I. If $\mathbf{T}'(t) \neq \mathbf{0}$, then the **principal unit normal vect** (主單位法向量) at t is defined as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}.$$

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16. **Acceleration vector** If $\mathbf{r}(t)$ is the position vector for a smooth curve C and $\mathbf{N}(t)$ exists, then the acceleration vector $\mathbf{a}(t)$ lies in the plane determined by $\mathbf{T}(t)$ and $\mathbf{N}(t)$. 90

17. **Tangential and normal components of acceleration** If $\mathbf{r}(t)$ is the position vector for a smooth curve C [for which $\mathbf{N}(t)$ exists], then the tangential and normal components of acceleration are as follows.

$$a_{\mathbf{T}} = D_t[\|\mathbf{v}\|] = \mathbf{a} \cdot \mathbf{T} = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|}$$

$$a_{\mathbf{N}} = \|\mathbf{v}\| \ \|\mathbf{T}'\| = \mathbf{a} \cdot \mathbf{N} = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|} = \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^2}$$

Note that $a_{\mathbf{N}} \geq 0$. The normal component of acceleration is also called the **centripetal component of acceleration** (向心加速度分量). 93

18. **Arc length of a space curve** If C is a smooth curve given by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$, on an interval $[a, b]$, then the **arc length** (弧長) of C on the interval is

$$s = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,\mathrm{d}t = \int_a^b \left\|\mathbf{r}'(t)\right\|\,\mathrm{d}t.$$

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19. **Arc length function** Let C be a smooth curve given by $\mathbf{r}(t)$ defined on the closed interval $[a, b]$. For $a \leq t \leq b$, the **arc length function** (弧長函數) is given by

$$s(t) = \int_a^t \left\|\mathbf{r}'(u)\right\|\,\mathrm{d}u = \int_a^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2}\,\mathrm{d}u.$$

The arc length s is called the **arc length parameter** (弧長參數). 111

20. **Arc length parameter** If C is a smooth curve given by

$$\mathbf{r}(s) = x(s)\,\mathbf{i} + y(s)\,\mathbf{j} \quad \text{or} \quad \mathbf{r}(s) = x(s)\,\mathbf{i} + y(s)\,\mathbf{j} + z(s)\,\mathbf{k}$$

where s is the arc length parameter, then

$$\left\|\mathbf{r}'(s)\right\| = 1.$$

Moreover, if t is any parameter for the vector-valued function $\mathbf{r}$ such that $\left\|\mathbf{r}'(t)\right\| = 1$, then t must be the arc length parameter. 115

21. **Curvature** Let C be a smooth curve (in the plane or in space) given by $\mathbf{r}(s)$, where s is the arc length parameter. The **curvature** (曲率) K at s is given by

$$K = \left\|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right\| = \left\|\mathbf{T}'(s)\right\|.$$

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22. **Formulas for curvature** If C is a smooth curve by $\mathbf{r}(t)$, then the curvature K of C at t is given by

$$K = \frac{\left\|\mathbf{T}'(t)\right\|}{\left\|\mathbf{r}'(t)\right\|} = \frac{\left\|\mathbf{r}'(t) \times \mathbf{r}''(t)\right\|}{\left\|\mathbf{r}'(t)\right\|^3}.$$

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23. **Curvature in rectangular coordinates** If C is the graph of a twice-differentiable function given by $y = f(x)$, then the curvature K at the point (x, y) is given by

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}}.$$

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24. **Circle of curvature** Let C be a curve with curvature K at point P.

The circle passing through point P with radius $r = 1/K$ is called the **circle of curvature** (**曲率圓**) if the circle lies on the concave side of the curve and shares a common tangent line with the curve at point. The radius is called the **radius of curvature** (**曲率半徑**) at P, and the center of the circle is called the **center of curvature** (**曲率中心**).
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25. **Acceleration, speed, and curvature** If $\mathbf{r}(t)$ is the position vector for a smooth curve C, then the acceleration vector is given by

$$\mathbf{a}(t) = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\mathbf{T} + K\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2\mathbf{N}$$

where K is the curvature of C and $\mathrm{d}s/\mathrm{d}t$ is the speed. 132

26. A moving object with mass m is in contact with a stationary object. The

total force required to produce an acceleration $\mathbf{a}$ along a given path is

$$\mathbf{F} = m\,\mathbf{a} = m\left(\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\right)\mathbf{T} + mK\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2\mathbf{N} = ma_{\mathbf{T}}\,\mathbf{T} + ma_{\mathbf{N}}\,\mathbf{N}.$$

27. Summary of velocity, acceleration, and curvature

Let C be a curve (in the plane or in space) given by the position function:

$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$ **Curve in the plane**

$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ **Curve in the space**

Velocity vector, speed, and acceleration

vector:

$\mathbf{v}(t) = \mathbf{r}'(t)$ **Velocity vector**

$\|\mathbf{v}(t)\| = \dfrac{\mathrm{d}s}{\mathrm{d}t} = \|\mathbf{r}'(t)\|$ **Speed**

$$\mathbf{a}(t) = \mathbf{r}''(t) = a_{\mathbf{T}}\,\mathbf{T}(t) + a_{\mathbf{N}}\,\mathbf{N}(t)$$ **Acceleration vector**

Unit tangent vector and principal unit normal vector:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \text{ and } \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

Components of acceleration:

$$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = \frac{\mathbf{v}\cdot\mathbf{a}}{\|\mathbf{v}\|} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$$

$$a_{\mathbf{N}} = \mathbf{a}\cdot\mathbf{N} = \frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|} = \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^2} = K\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2$$

Formulas for curvature in the plane:

$$K = \frac{|y''|}{[1+(y')^2]^{3/2}}$$ **C given by $y = f(x)$**

$K = \dfrac{\lvert x'y'' - y'x'' \rvert}{[(x')^2 + (y')^2]^{3/2}}$	C **given by** $x = x(t), y = y(t)$

Formulas for curvature in the plane or in space:

$K = \left\lVert \mathbf{T}'(s) \right\rVert = \left\lVert \mathbf{r}''(s) \right\rVert$	s **is arc length parameter.**
$K = \dfrac{\left\lVert \mathbf{T}'(t) \right\rVert}{\lVert \mathbf{r}'(t) \rVert} = \dfrac{\left\lVert \mathbf{r}'(t) \times \mathbf{r}''(t) \right\rVert}{\lVert \mathbf{r}'(t) \rVert^3}$	t **is general parameter.**
$K = \dfrac{\mathbf{a}(t) \cdot \mathbf{N}(t)}{\lVert \mathbf{v}(t) \rVert^2}$	

Cross product formulas apply only to curves in space.

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