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Chapter 11

Vectors and the Geometry of Space

11.1 Summary

1. **Component form of a vector in the plane** If $\mathbf{v}$ is a vector in the plane whose initial point is the origin and whose terminal point is (v_1, v_2), then the **component form** (分量形式) of $\mathbf{v}$ is given by

$$\mathbf{v} = \langle v_1, v_2 \rangle .$$

The coordinates v_1 and v_2 are called the **components** (分量) of $\mathbf{v}$. If both the initial point and the terminal point lie at the origin, then $\mathbf{v}$ is called the **zero vector** (零向量) and is denoted by $\mathbf{0} = \langle 0, 0 \rangle$. 11

2. **Vector addition and scalar multiplication** Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ be vectors and let c be a scalar.

   (a) The **vector sum** (向量和) of $\mathbf{u}$ and $\mathbf{v}$ is the vector $\mathbf{u}+\mathbf{v} = \langle u_1 + v_1, u_2 + v$

   (b) The **scalar multiple** (純數倍) of c and $\mathbf{u}$ is the vector $c\mathbf{u} = \langle cu_1, cu_2 \rangle$.

   (c) The **negative** (負) of $\mathbf{v}$ is the vector $-\mathbf{v} = (-1)\mathbf{v} = \langle -v_1, -v_2 \rangle$.

   (d) The **difference** (差) of $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{u}-\mathbf{v} = \mathbf{u}+(-\mathbf{v}) = \langle u_1 - v_1, u_2 - v_2 \rangle$.

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3. **Properties of vector operations** Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be

vectors in the plane, and let c and d be scalars.

1. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ **Commutative Property** (交換性質)
2. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$ **Associative Property** (結合性質)
3. $\mathbf{u}+\mathbf{0}=\mathbf{u}$ **Additive Identity Property** (加法單位元
4. $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$ **Additive Inverse Property** (加法反元素
5. $c(d\,\mathbf{u})=(cd)\,\mathbf{u}$
6. $(c+d)\,\mathbf{u}=c\,\mathbf{u}+d\,\mathbf{u}$ **Distributive Property** (分配性質)
7. $c(\mathbf{u}+\mathbf{v})=c\,\mathbf{u}+c\,\mathbf{v}$ **Distributive Property** (分配性質)
8. $1(\mathbf{u})=\mathbf{u}, \quad 0(\mathbf{u})=\mathbf{0}$

4. **Length of a scalar multiple** Let $\mathbf{v}$ be a vector and let c be a scalar.

Then

$$\|c\,\mathbf{v}\| = |c|\,\|\mathbf{v}\|\,. \qquad |c| \text{ is the absolute value of } c.$$

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5. **Unit vector in the direction of $\mathbf{v}$** If $\mathbf{v}$ is a nonzero vector in the plane, then the vector

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

has length 1 and the same direction as $\mathbf{v}$. 24

6. Unit vector on the unit circle: $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$

$\mathbf{v} = \|\mathbf{v}\|\,\langle \cos\theta, \sin\theta \rangle = \|\mathbf{v}\|\cos\theta\,\mathbf{i} + \|\mathbf{v}\|\sin\theta\,\mathbf{j}$ 30

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7. **Vectors in space (空間中的向量)** Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors in space and let c be a scalar.

(a) Equality of Vectors: $\mathbf{u} = \mathbf{v}$ if and only if $u_1 = v_1$, $u_2 = v_2$, and $u_3 = v_3$.

(b) Component Form: If $\mathbf{v}$ is represented by the directed line segment from $P(p_1, p_2, p_3)$ to $Q(q_1, q_2, q_3)$, then

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle .$$

(c) Length: $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$.

(d) Unit Vector in the Direction of $\mathbf{v}$: $\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \left(\dfrac{1}{\|\mathbf{v}\|}\right) \langle v_1, v_2, v_3 \rangle , \quad \mathbf{v} \neq \mathbf{0}$

(e) Vector Addition: $\mathbf{v} + \mathbf{u} = \langle v_1 + u_1, v_2 + u_2, v_3 + u_3 \rangle$

(f) Scalar Multiplication: $c\,\mathbf{v} = \langle cv_1, cv_2, cv_3 \rangle$

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8. **Parallel vectors (平行向量)** Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if there is some scalar c such that $\mathbf{u} = c\,\mathbf{v}$. 53

Section 11.3 The dot product of two vectors 62

9. **Dot product** The **dot product** (**内積**) of $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

The **dot product** (**内積**) of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

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10. **Properties of the dot product** Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in the plane or in space and let c be a scalar.

(a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ **Commutative property**

(b) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ **Distributive property**

(c) $c(\mathbf{u} \cdot \mathbf{v}) = c\,\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot c\,\mathbf{v}$

(d) $\mathbf{0} \cdot \mathbf{v} = 0$

(e) $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$

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11. **Angle between two vectors (兩向量間的夾角)** If θ is the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, then

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}. \tag{11.1}$$

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12. **Orthogonal vectors (正交向量)** The vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** (正交) if $\mathbf{u} \cdot \mathbf{v} = 0$. 70

13. **Direction cosines** (方向餘弦) The angles α, β, and γ are the **direction an** (方向角) of $\mathbf{v}$, and $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are the **direction cosines** (方向餘弦) of $\mathbf{v}$.

- $\cos\alpha = \dfrac{v_1}{\|\mathbf{v}\|}$ **α is the angle between $\mathbf{v}$ and $\mathbf{i}$**
- $\cos\beta = \dfrac{v_2}{\|\mathbf{v}\|}$ **β is the angle between $\mathbf{v}$ and $\mathbf{j}$**
- $\cos\gamma = \dfrac{v_3}{\|\mathbf{v}\|}$. **$\gamma$ is the angle between $\mathbf{v}$ and $\mathbf{k}$**

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

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14. **Projection** (**投影**) and **vector components** (**向量分量**) Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors. Moreover, let $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is parallel to $\mathbf{v}$, and $\mathbf{w}_2$ is orthogonal to $\mathbf{v}$, as shown in Figure ??.

- $\mathbf{w}_1$ is called the projection of $\mathbf{u}$ onto $\mathbf{v}$ or the vector component of $\mathbf{u}$ along $\mathbf{v}$, and is denoted by $\mathbf{w}_1 = \operatorname{proj}_{\mathbf{v}} \mathbf{u}$.
- $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$ is called the vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$.

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15. **Projection using the dot product** If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, then the projection of $\mathbf{u}$ onto $\mathbf{v}$ is given by

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}.$$

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16. **Work** (**功**) The work W done by a constant force $\mathbf{F}$ as its point of application moves along the vector $\overrightarrow{PQ}$ is given by either of the following.

(a) $W = \left\| \operatorname{proj}_{\overrightarrow{PQ}} \mathbf{F} \right\| \; \left\| \overrightarrow{PQ} \right\|$ (Projection form)

(b) $W = \mathbf{F} \cdot \overrightarrow{PQ}$ (Dot form)

17. **Cross product of two vectors in space** Let

$$\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j} + u_3\,\mathbf{k} \quad \text{and} \quad \mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$$

be vectors in space. The **cross product** (**外積**) of $\mathbf{u}$ and $\mathbf{v}$ is the vector

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\,\mathbf{i} - (u_1 v_3 - u_3 v_1)\,\mathbf{j} + (u_1 v_2 - u_2 v_1)\,\mathbf{k}.$$

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18. Determinant form of $\mathbf{u} \times \mathbf{v}$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

$$= (u_2v_3 - u_3v_2)\,\mathbf{i} - (u_1v_3 - u_3v_1)\,\mathbf{j} + (u_1v_2 - u_2v_1)\,\mathbf{k}$$

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19. **Algebraic property of the cross product** (**外積的代數性質**)

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in space, and let c be a scalar.

(a) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

(b) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

(c) $c(\mathbf{u} \times \mathbf{v}) = (c\,\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\,\mathbf{v})$

(d) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

(e) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

(f) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

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20. **Geometric properties of the cross product** (外積幾何性質) Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in space, and let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$.

(a) $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

(b) $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\ \|\mathbf{v}\| \sin\theta$

(c) $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are scalar multiple of each other.

(d) $\|\mathbf{u} \times \mathbf{v}\| =$ area of parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides.

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21. **The triple scalar product** For $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j} + u_3\,\mathbf{k}$, $\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$, and $\mathbf{w} = w_1\,\mathbf{i} + w_2\,\mathbf{j} + w_3\,\mathbf{k}$, the **triple scalar product** (純量三重積) is given by

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

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22. **Geometric property of triple scalar product** (三重純量積的幾何性質)

The volume V of a paralleled with vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ as adjacent edges is given by

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|.$$

23. **Parametric equations of a line in space** A line L paralleled to the vector $\mathbf{v} = \langle a, b, c \rangle$ and passing through the point $P(x_1, y_1, z_1)$ is represented by the **parametric equations** (**參數式**)

$$x = x_1 + at, \quad y = y_1 + bt, \quad \text{and} \quad z = z_1 + ct.$$

24. **Symmetric equations (對稱方程式) of the line**

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \qquad \textbf{Symmetric equations}$$

25. **Standard equation of a plane in space** (**空間中平面的標準式**)

The plane containing the point (x_1, y_1, z_1) and having normal vector $\mathbf{n} = \langle a, b, c\rangle$ can be represented by the standard form of the equation of a plane

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

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26. **General form** (**一般型**) **of the equation of a plane in space**

$$ax + by + cz + d = 0$$ **General form of equation**

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27. **Distances between a point and a plane** (**平面與點的距離**) The

distance between a plane and a point Q (not in the plane) is

$$D = \left\| \operatorname{proj}_{\mathbf{n}} \overrightarrow{PQ} \right\| = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$$

where P is a point in the plane and $\mathbf{n}$ is normal to the plane. 140

28. The distance between the point $Q(x_0, y_0, z_0)$ and the plane given by $ax + by + cz + d = 0$ is

$$D = \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2 + b^2 + c^2}} \quad \text{or} \quad D = \frac{|ax_0 + by_0 + cz_0 + \cdots}{\sqrt{a^2 + b^2 + c^2}}$$

where $P(x_1, y_1, z_1)$ is a point in the plane and $d = -(ax_1 + by_1 + cz_1)$. 143

29. **Distances between a point and a line in space** (空間中線與點的距離)

33. **Surface of revolution** If the graph of a radius function r is revolved about one of the coordinate axes, the equation of the resulting **surface of revolution** (旋轉曲面) has one of the following forms.

(a) Revolved about the x-axis: $y^2 + z^2 = [r(x)]^2$

(b) Revolved about the y-axis: $x^2 + z^2 = [r(y)]^2$

(c) Revolved about the z-axis: $x^2 + y^2 = [r(z)]^2$

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34. **The cylindrical coordinate system** In a **cylindrical coordinate syster** (**圓柱座標系統**), a point P in space is represented by an ordered triple (r, θ, z).

(a) (r, θ) is a polar representation of the projection of P in the xy-plane.

(b) z is the directed distance from (r, θ) to P.

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35. **Cylindrical to rectangular** (**圓柱到直角**):

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

Rectangular to cylindrical (**直角到圓柱**):

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}, \quad z = z$$

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36. **The spherical coordinate system** In a **spherical coordinate system** (**球面座標系統**), a point P in space is represented by an ordered triple (ρ, θ, ϕ).

1. ρ is the distance between P and the origin, $\rho \geq 0$.
2. θ is the same angle used in cylindrical coordinates for $r \geq 0$.
3. ϕ is the angle between the positive z-axis and the line segment $\overrightarrow{OP}$,

$0 \leq \phi \leq \pi$.

Note that the first and third coordinates, ρ and ϕ, are nonnegative. ρ is the lowercase Greek letter rho, and ϕ is the lowercase Greek letter phi. 189

37.(a) **Spherical to rectangular:**

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi.$$

(b) **Rectangular to spherical:**

$$\rho^2 = x^2 + y^2 + z^2, \quad \tan\theta = \frac{y}{x}, \quad \phi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right).$$

(c) **Spherical to cylindrical ($r \geq 0$):**

$$r^2 = \rho^2 \sin^2\phi, \quad \theta = \theta, \quad z = \rho \cos\phi.$$

(d) **Cylindrical to spherical ($r \geq 0$):**

$$\rho = \sqrt{r^2 + z^2}, \quad \theta = \theta, \quad \phi = \arccos\left(\frac{z}{\sqrt{r^2 + z^2}}\right).$$

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