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Chapter 10

Conics, Parametric Equations, and Polar Coordinates

10.1 Summary

1. **Three ways to define the conics**

   (a) the intersections of planes and cones

   (b) **general second-degree equation** (**一般二次方程式**)

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

(c) **locus** (**軌跡**) (collection) of points satisfying a certain geometric property

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2. **Standard equation of a parabola** The **standard form** (**標準式**) of the equation of a parabola with vertex (h, k) and directrix $y = k - p$ is

$$(x-h)^2 = 4p(y-k). \qquad \text{Vertical axis}$$

For directrix $x = h - p$, the equation is

$$(y-k)^2 = 4p(x-h). \qquad \text{Horizontal axis}$$

The focus lies on the axis p units (*directed distance*) from the vertex.

The coordinates of the focus are as follows.

$(h, k+p)$ Vertical axis $(h+p, k)$ Horizontal axis

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3. **Reflective property of a parabola** Let P be a point on a parabola. The tangent line to the parabola at the point P makes equal angles with the following two lines.

(a) The line passing though P and the focus.

(b) The line passing through P parallel to the axis of the parabola.

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4. **Standard equation of an ellipse** The standard form of the equation of an ellipse with center (h, k) and major and minor axes of

lengths $2a$ and $2b$, where $a > b$, is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$ **Major axis is horizontal**

or

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1.$$ **Major axis is vertical**

The foci lie on the major axis, c units from the center, with $c^2 = a^2 - b^2$.

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5. **Reflecting property of an ellipse** Let P be a point on an ellipse. The tangent line to the ellipse at point P makes equal angles with the lines through P and the foci. 26

6. **Eccentricity of an ellipse** The **eccentricity** (**離心率**) e of an

ellipse is given by the ratio

$$e = \frac{c}{a}.$$

7. **Standard equation of a hyperbola** The standard form of the equation of a hyperbola with center at (h, k) is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \quad \textbf{Transverse axis is horizontal}$$

or

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1. \quad \textbf{Transverse axis is vertical}$$

The vertices are a units from the center, and the foci are c units from the center, where $c^2 = a^2 + b^2$.

8. **Asymptotes of a hyperbola** For a **horizontal transverse axis**

(水平貫軸), the equation of the asymptotes are

$$y = k + \frac{b}{a}(x - h) \qquad \text{and} \qquad y = k - \frac{b}{a}(x - h).$$

For a **vertical transverse axis** (垂直貫軸), the equation of the asymptotes are

$$y = k + \frac{a}{b}(x - h) \quad \text{and} \quad y = k - \frac{a}{b}(x - h).$$

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9. **Eccentricity of a hyperbola** The **eccentricity** (離心率) e of a hyperbola is given by the ratio

$$e = \frac{c}{a}.$$

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10. **Plane curve** If f and g are continuous functions of t on an interval I, then the equations

$$x = f(t) \quad \text{and} \quad y = g(t)$$

are called **parametric equations** (參數方程式) and t is called the **parameter** (參數). The set of points (x, y) obtained as t varies over the interval I is called the **graph** (圖) of the parametric equations. Taken together, the parametric equations and the graph are called a **plane curve** (平面曲線), denoted by C. 54

11. **Smooth curve (平滑曲線)** A curve C represented by $x = f(t)$ and $y = g(t)$ on an interval I is called **smooth** (平滑) if f' and g' are continuous on I and not simultaneously 0, except possibly at the

14. **Arc length in parametric form** If a smooth curve C is given by $x = f(t)$ and $y = g(t)$ such that C does not intersect itself on the interval $a \le t \le b$ (except possibly at the endpoints), then the arc length of C over the interval is given by

$$s = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,\mathrm{d}t.$$

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15. **Area of a surface of revolution** If a smooth curve C given by $x = f(t)$ and $y = g(t)$ does not cross itself on an interval $a \le t \le b$, then the area S of the surface of revolution formed by revolving C about the coordinate axes is given by the following.

19. **Area in polar coordinates** If f is continuous and nonnegative on the interval $[\alpha, \beta]$, $0 < \beta - \alpha \leq 2\pi$, then the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is

given by

$$A = \frac{1}{2}\int_\alpha^\beta [f(\theta)]^2 \,\mathrm{d}\theta = \frac{1}{2}\int_\alpha^\beta r^2 \,\mathrm{d}\theta, \quad 0 < \beta - \alpha \leq 2\pi.$$

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20. **Arc length of a polar curve** Let f be a function whose derivative is continuous on an interval $\alpha \leq \theta \leq \beta$. The length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$s = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,\mathrm{d}\theta = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta.$$

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21. **Area of a surface of revolution** Let f be a function whose derivative is continuous on an interval $\alpha \leq \theta \leq \beta$. The area of the

(a) The conic is an ellipse if $0 < e < 1$.

(b) The conic is a parabola if $e = 1$.

(c) The conic is a hyperbola if $e > 1$.

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23. **Polar equations of conics** The graph of a polar equation of the form

$$r = \frac{ed}{1 \pm e\cos\theta} \quad \text{or} \quad r = \frac{ed}{1 \pm e\sin\theta}$$

is a conic, where $e > 0$ is the eccentricity and $|d|$ is the distance between the focus at the pole and its corresponding directrix. 162

24. The four types of equations indicated in Theorem 10.17 can be classified as follows, where $d > 0$.

a. Horizontal directrix above the pole: $r = \dfrac{ed}{1 + e\sin\theta}$

b. Horizontal directrix below the pole: $r = \dfrac{ed}{1 - e\sin\theta}$

c. Vertical directrix to the right of the pole: $r = \dfrac{ed}{1 + e\cos\theta}$

d. Vertical directrix to the left of the pole: $r = \dfrac{ed}{1 - e\cos\theta}$

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25. Ellipse: $b^2 = a^2(1 - e^2)$ Hyperbola: $b^2 = a^2(e^2 - 1)$ 169

26. **Kepler's Laws** (克卜勒定律)

(a) Each planet moves in an elliptical orbit with the sun as a focus.

(b) A ray from the sun to the planet sweeps out equal areas of the ellipse in equal times.

(c) The square of the period is proportional to the cube of the mean distance between the planet and the sun.

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