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 Chapter **9**

## INFINITE SERIES

### 9.1 Summary

#### Section 9.1 Sequences ..... 1

1. **The limit of a sequence**      Let  $L$  be a real number. The **limit** (極限) of a sequence  $\{a_n\}$  is  $L$ , written as

$$\lim_{n \rightarrow \infty} a_n = L$$

if for each  $\varepsilon > 0$ , there exists  $M > 0$  such that  $|a_n - L| < \varepsilon$  whenever

$n > M$ . If the limit  $L$  of a sequence exists, then the sequence **converges** (**收斂**) to  $L$ . If the limit of a sequence does not exist, then the sequence **diverges** (**發散**)..... 6

2. **Limit of a sequence**      Let  $L$  be a real number. Let  $f$  be a function of a real variable such that

$$\lim_{x \rightarrow \infty} f(x) = L.$$

If  $\{a_n\}$  is a sequence such that  $f(n) = a_n$  for every positive integer  $n$ , then

$$\lim_{n \rightarrow \infty} a_n = L.$$

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3. **Properties of limits of sequences**      Let  $\lim_{n \rightarrow \infty} a_n = L$  and  $\lim_{n \rightarrow \infty} b_n = K$ .

$$\mathbf{1.} \lim_{n \rightarrow \infty} (a_n \pm b_n) = L \pm K \quad \mathbf{2.} \lim_{n \rightarrow \infty} ca_n = cL, \text{ } c \text{ is any real number}$$

$$\mathbf{3.} \lim_{n \rightarrow \infty} (a_n b_n) = LK \quad \mathbf{4.} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{L}{K}, \text{ } b_n \neq 0 \text{ and } K \neq 0$$

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4. **Commonly used ordering** If  $a > 0$  and  $b > 1$ , then

$$\ln n < n^a < b^n < n!$$

where  $a_n < b_n$  denotes that  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0 \dots \dots \dots 12$

5. **Squeeze Theorem for sequences** (數列夾擠定理)] If

$$\lim_{n \rightarrow \infty} a_n = L = \lim_{n \rightarrow \infty} b_n$$

and there exists an integer  $N$  such that  $a_n \leq c_n \leq b_n$  for all  $n > N$ ,

then

$$\lim_{n \rightarrow \infty} c_n = L.$$

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6. **Absolute Value Theorem** (絕對值定理) For the sequence  $\{a_n\}$ ,  
if

$$\lim_{n \rightarrow \infty} |a_n| = 0 \quad \text{then} \quad \lim_{n \rightarrow \infty} a_n = 0.$$

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7. **Monotone sequence** (單調數列) A sequence  $\{a_n\}$  is **monotonic**  
(單調) if its terms are nondecreasing

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots$$



or if its terms are nonincreasing

$$a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots .$$

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## 8. Bounded sequence (有界數列)

- (a) A sequence  $\{a_n\}$  is **bounded above** if there is a real number  $M$  such that  $a_n \leq M$  for all  $n$ . The number  $M$  is called an upper bound (上界) of the sequence.
- (b) A sequence  $a_n$  is **bounded below** if there is a real number  $N$  such that  $N \leq a_n$  for all  $n$ . The number  $N$  is called a lower bound (下界) of the sequence.
- (c) A sequence  $\{a_n\}$  is bounded (有界) if it is bounded above and bounded below.

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9. **Bounded monotonic sequences** (**單調有界數列**)      If a sequence  $\{a_n\}$  is bounded and monotonic; then it converges. .... 26

**Section 9.2 Series and convergence** ..... 30

10. **Convergent and divergent series**      For the infinite series  $\sum_{n=1}^{\infty} a_n$  the  **$n$ th partial sum** (**第  $n$  項部分和**) is given by

$$S_n = a_1 + a_2 + \cdots + a_n.$$

If the sequence of partial sums  $\{S_n\}$  converges to  $S$ , then the series  $\sum_{n=1}^{\infty} a_n$  **converges** (**收斂**). The limit  $S$  is called the **sum of the series**

(級數和).

$$S = a_1 + a_2 + \cdots + a_n + \cdots \qquad S = \sum_{n=1}^{\infty} a_n$$

If  $\{S_n\}$  diverges, then the series diverges (發散)..... 33

11. Telescoping series (對消級數)      If  $a_n = b_{n+1} - b_n$ , then  $\sum_{i=1}^n a_i = \sum_{i=1}^n (b_{i+1} - b_i) = b_{n+1} - b_1$ . Moreover, if the series converges, its sum is  $S = \lim_{n \rightarrow \infty} b_{n+1} - b_1$ ..... 37

12. Convergence of a geometric series      A geometric series (幾何級數) with ratio  $r$  diverges if  $|r| \geq 1$ . If  $0 < |r| < 1$ , then the series converges to the sum

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad 0 < |r| < 1.$$

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13. **Properties of infinite series**      Let  $\sum a_n$  and  $\sum b_n$  be convergent series, and let  $A$ ,  $B$ , and  $c$  be real numbers. If  $\sum_{n=1}^{\infty} a_n = A$  and  $\sum_{n=1}^{\infty} b_n = B$ , then the following series converge to the indicated sums.
- (a)  $\sum_{n=1}^{\infty} ca_n = cA$
- (b)  $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$
- (c)  $\sum_{n=1}^{\infty} (a_n - b_n) = A - B$
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14. **Limit of the  $n$ th term of a convergent series**      If  $\sum_{n=1}^{\infty} a_n$  is convergent, then  $\lim_{n \rightarrow \infty} a_n = 0$ . ..... 43
15.  **$n$ th-term test for divergence**      If  $\lim_{n \rightarrow \infty} a_n \neq 0$ , then  $\sum_{n=1}^{\infty} a_n$  diverges. .... 45

## Section 9.3 The Integral Test and $p$ -series ..... 49

16. **The Integral Test (積分檢定)** If  $f$  is positive, continuous, and decreasing for  $x \geq 1$  and  $a_n = f(n)$ , then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x) \, dx$$

either both converge or both diverge. .... 50

17.  **$p$ -series:** A series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

is a  **$p$ -series** ( **$p$  級數**), where  $p$  is a positive constant. For  $p = 1$ , the

series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

is the **harmonic series** (**調和級數**). A **general harmonic series** (**一般調和**

of the form  $\sum \frac{1}{(an + b)}$ . ..... 58

18. **Euler-Mascheroni constant**  $\gamma$  ( $C$ ) (**尤拉常數**  $\gamma$  ( $C$ ))

[http://en.wikipedia.org/wiki/Euler%E2%80%93Mascheroni\\_constant](http://en.wikipedia.org/wiki/Euler%E2%80%93Mascheroni_constant)

$$\gamma = \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n \frac{1}{k} - \ln n \right) \approx 0.5772156649$$

is a mathematical constant recurring in analysis and number theory. . 58

19. **Riemann zeta function**  $\zeta(s)$  (**黎曼  $\zeta$  函數**)

[http://en.wikipedia.org/wiki/Riemann\\_zeta\\_function](http://en.wikipedia.org/wiki/Riemann_zeta_function)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is a function of a complex variable  $s$  that analytically continues the sum of the infinite series which converges when the real part of  $s$  is greater than 1. The Riemann zeta function plays a pivotal role in analytic number theory and has applications in physics, probability theory, and applied statistics.....59

## 20. **Convergence of $p$ series**

The  $p$ -series ( $p$  級數)

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$$

**1.** converges if  $p > 1$ , and      **2.** diverges if  $0 < p \leq 1$ . ..... 60

## Section 9.4 Comparisons of series ..... 63

21. **Direct Comparison Test** (直接互比檢定)      Let  $0 < a_n \leq b_n$  for all  $n$ .

1. If  $\sum_{n=1}^{\infty} b_n$  converges, then  $\sum_{n=1}^{\infty} a_n$  converges.

2. If  $\sum_{n=1}^{\infty} a_n$  diverges, then  $\sum_{n=1}^{\infty} b_n$  diverges. .... 65

22. **Limit Comparison Test** (極限互比檢定)      Suppose  $a_n > 0$ ,  $b_n > 0$ ,

$$\lim_{n \rightarrow \infty} \left( \frac{a_n}{b_n} \right) = L$$

where  $L$  is finite and positive. Then the two series  $\sum a_n$  and  $\sum b_n$  either both converge both diverge. .... 71

## Section 9.5 Alternating series ..... 76

23. **Alternating Series Test** (交錯級數檢定)      Let  $a_n > 0$ . The



alternating series

$$\sum_{n=1}^{\infty} (-1)^n a_n \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converge if the following two conditions are met.

- 1.**  $\lim_{n \rightarrow \infty} a_n = 0$     **2.**  $a_{n+1} \leq a_n$ , for all  $n$  ..... 78

24. **Alternating Series Remainder** (交錯級數餘項)    If a convergent alternating series satisfies the condition  $a_{n+1} \leq a_n$ , then the absolute value of the remainder  $R_N$  involved in approximating the sum  $S$  by  $S_N$  is less than (or equal to) the first neglected term. That is,

$$|S - S_N| = |R_N| \leq a_{N+1}.$$

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25. **Absolute convergence** (絕對收斂)      If the series  $\sum |a_n|$  converges, then the series  $\sum a_n$  also converges. .... 88
26. **Absolute and conditional convergence**
- (a)  $\sum a_n$  is **absolutely convergent** (絕對收斂) if  $\sum |a_n|$  converges.
- (b)  $\sum a_n$  is **conditionally convergent** (條件收斂) if  $\sum a_n$  converges but  $\sum |a_n|$  diverges.
- ..... 90
27. If  $\sum a_n$  is conditionally convergent and  $S$  is any real number, the terms of the series can be rearranged to converge to  $S$ ..... 95
- Section 9.6 The ratio and root test** ..... 95
28. **Ratio Test** (比例檢定)      Let  $\sum a_n$  be a series with nonzero terms.

(a)  $\sum a_n$  converges absolutely if  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ .

(b)  $\sum a_n$  diverges if  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$  or  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$ .

(c) The Ratio Test is inconclusive if  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$ .

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29. **Root Test (根式検定)** Let  $\sum a_n$  be a series.

(a)  $\sum a_n$  converges absolutely if  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$ .

(b)  $\sum a_n$  diverges if  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} > 1$  or  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \infty$ .

(c) The Root Test is inconclusive if  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1$ .

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## 30. Guidelines for testing a series for convergence or divergence

- Does the  $n$ th term approach 0? If not, the series diverges.
- Is the series one of the special types-geometric,  $p$ -series, telescoping, or alternating?
- Can the Integral Test, the Root Test, or the Ratio Test be applied?
- Can the series be compared favorably to one of the special types?

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**Section 9.7 Taylor polynomials and approximations** ..... 110

31. **Taylor polynomial and Maclaurin polynomial**      If  $f$  has  $n$  deriv-

atives at  $c$ , then the polynomial

$$P_n(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!} (x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!} (x - c)^n$$

is called the  **$n$ th Taylor polynomial for  $f$  at  $c$**  ( $f$  在  $c$  的  $n$  階泰勒多項式)

If  $c = 0$ , then

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots + \frac{f^{(n)}(0)}{n!} x^n$$

is also called the  **$n$ th Maclaurin polynomial for  $f$  at  $c$**  ( $f$  在  $c$  的  $n$  階馬)

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32. **Taylor's Theorem** (泰勒定理)      If a function  $f$  is differentiable through order  $n + 1$  in an interval  $I$  containing  $c$ , then, for each  $x$  in  $I$ ,

there exists  $z$  between  $x$  and  $c$  such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!} (x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!} (x-c)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} (x-c)^{n+1}.$$

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**Section 9.8 Power series** ..... 138

33. **Power series** If  $x$  is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

is called a **power series** (**冪級數**). More generally, an infinite series of

the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots + a_n(x-c)^n + \cdots$$

is called a **power series centered at  $c$**  (以  $c$  為中心的冪級數), where  $c$  is a constant.....140

34. **Convergence of a power series** For a power series centered at  $c$ , precisely one of the following is true.

1. The series converges only at  $c$ .
2. There exists a real number  $R > 0$  such that the series converges absolutely for  $|x - c| < R$ , and diverges for  $|x - c| > R$ .
3. The series converges absolutely for all  $x$ .

The number  $R$  is the **radius of convergence** (收斂半徑) of the power

series. If the series converges only at  $c$ , the radius of convergence is  $R = 0$ , and if the series converges for all  $x$ , the radius of convergence is  $R = \infty$ . The set of all values of  $x$  for which the power series converges is the **interval of convergence** (收斂區間) of the power series... 144

### 35. **Properties of functions defined by power series**

If the

function given by

$$f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n = a_0 + a_1(x - c) + a_2(x - c)^2 + a_3(x - c)^3 + \dots$$

has a radius of convergence of  $R > 0$ , then, on the interval  $(c - R, c + R)$ ,  $f$  is differentiable (and therefore continuous). Moreover, the derivative and antiderivative of  $f$  are as follows.

**1.**  $f'(x) = \sum_{n=1}^{\infty} n a_n (x - c)^{n-1} = a_1 + 2a_2(x - c) + 3a_3(x - c)^2 + \dots$



$$2. \int f(x) dx = C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1} = C + a_0(x-c) + a_1 \frac{(x-c)^2}{2} + a_2 \frac{(x-c)^3}{3} + \dots$$

The radius of convergence of the series obtained by differentiating or integrating a power series is the same as that of the original power series. The interval of convergence, however, may differ as a result of the behavior at the endpoints. .... 158

36. The interval of convergence of the series obtained by differentiating a power series may get worse but cannot get improved. However, the interval of convergence of the series obtained by integrating a power series may get improve but cannot get worse. .... 158

**Section 9.9 Representation of functions by power series ... 162**

37. **Operations with power series**      Let  $f(x) = \sum a_n x^n$  and  $g(x) = \sum b_n x^n$ .

1.  $f(kx) = \sum_{n=0}^{\infty} a_n k^n x^n$       2.  $f(x^N) = \sum_{n=0}^{\infty} a_n x^{nN}$       3.  $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$  ..... 168

**Section 9.10 Taylor and Maclaurin series** ..... 174

38. **The form of a convergent power series** (收斂冪級數的型式)

If  $f$  is represented by a power series  $f(x) = \sum a_n (x - c)^n$  for all  $x$  in an open interval  $I$  containing  $c$ , then  $a_n = f^{(n)}(c)/n!$  and

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!} (x - c)^2 + \dots + \frac{f^{(n)}(c)}{n!} (x - c)^n + \dots$$

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39. **Taylor and Maclaurin series**      If a function  $f$  has derivatives of all

orders at  $x = c$ , then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x - c)^n = f(c) + f'(c)(x - c) + \cdots + \frac{f^{(n)}(c)}{n!} (x - c)^n + \cdots$$

is called the **Taylor series** (泰勒級數) for  $f(x)$  at  $c$ . Moreover, if  $c = 0$ , then the series is the **Maclaurin series** (馬克勞林級數) for  $f$ .....178

40. **Convergence of Taylor series** If  $\lim_{n \rightarrow \infty} R_n = 0$  for all  $x$  in the interval  $I$ , then the Taylor series for  $f$  converges and equals  $f(x)$ ,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x - c)^n.$$

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41. **Guidelines for finding a Taylor series**

(a) Differentiate  $f(x)$  several times and evaluate each derivative at  $c$ .

$$f(c), \quad f'(c), \quad f''(c), \quad f'''(c), \quad \dots, \quad f^{(n)}(c), \quad \dots$$

Try to recognize a pattern in these numbers.

(b) Use the sequence developed in the first step to form the **Taylor coefficients** (**泰勒係數**)  $a_n = f^{(n)}(c)/n!$ , and determine the interval of convergence for the resulting power series

$$f(c) + f'(c)(x - c) + \frac{f''(c)}{2!} (x - c)^2 + \dots + \frac{f^{(n)}(c)}{n!} (x - c)^n + \dots$$

(c) Within this interval of convergence, determine whether or not the series converges to  $f(x)$ .

## 42. Power series for elementary functions

Function

$$\frac{1}{x} = 1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + (x - 1)^4 - \dots + (-1)^n (x - 1)^n - \dots$$

$$\frac{1}{1 + x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots + (-1)^n x^n + \dots$$

$$\ln x = (x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \frac{(x - 1)^4}{4} + \dots + \frac{(-1)^{(n-1)}(x - 1)^n}{n} - \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots + \frac{x^n}{n!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots + \frac{(-1)^n x^{2n+1}}{2n+1} + \dots$$

$$\arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \dots + \frac{(2n)!x^{2n+1}}{(2^n n!)^2(2n+1)} + \dots$$

$$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \frac{k(k-1)(k-2)(k-3)x^4}{4!} + \dots$$

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43. **Euler's Formula:**  $e^{ix} = \cos x + i \sin x$  ..... 194

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