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Chapter 9

INFINITE SERIES

9.1 Summary

1. **The limit of a sequence** Let L be a real number. The **limit** (**極限**) of a sequence $\{a_n\}$ is L, written as

$$\lim_{n\to\infty} a_n = L$$

if for each $\varepsilon > 0$, there exists $M > 0$ such that $|a_n - L| < \varepsilon$ whenever

$n > M$. If the limit L of a sequence exists, then the sequence **converges** (**收斂**) to L. If the limit of a sequence does not exist, then the sequence **diverges** (**發散**). 6

2. **Limit of a sequence** Let L be a real number. Let f be a function of a real variable such that

$$\lim_{x \to \infty} f(x) = L.$$

If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n, then

$$\lim_{n \to \infty} a_n = L.$$

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3. **Properties of limits of sequences** Let $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = K$.

1. $\lim_{n\to\infty}(a_n \pm b_n) = L \pm K$ **2.** $\lim_{n\to\infty} ca_n = cL$, c is any real numbe

3. $\lim_{n\to\infty}(a_n b_n) = LK$ **4.** $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{K}$, $b_n \neq 0$ and $K \neq 0$

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4. **Commonly used ordering** If $a > 0$ and $b > 1$, then

$$\ln n \prec n^a \prec b^n \prec n!$$

where $a_n \prec b_n$ denotes that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$. 12

5. **Squeeze Theorem for sequences** (數列夾擠定理)] If

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$$

and there exists an integer N such that $a_n \leq c_n \leq b_n$ for all $n > N$, then

$$\lim_{n\to\infty} c_n = L.$$

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6. **Absolute Value Theorem** (**絕對值定理**) For the sequence $\{a_n\}$, if

$$\lim_{n\to\infty} |a_n| = 0 \quad \text{then} \quad \lim_{n\to\infty} a_n = 0.$$

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7. **Monotone sequence** (**單調數列**) A sequence $\{a_n\}$ is **monotonic** (**單調**) if its terms are nondecreasing

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots$$

or if its terms are nonincreasing

$$a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots .$$

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8. **Bounded sequence** (有界數列)

(a) A sequence $\{a_n\}$ is **bounded above** if there is a real number M such that $a_n \leq M$ for all n. The number M is called an **upper bound** (上界) of the sequence.

(b) A sequence a_n is **bounded below** if there is a real number N such that $N \leq a_n$ for all n. The number N is called a **lower bound** (下界) of the sequence.

(c) A sequence $\{a_n\}$ is **bounded** (有界) if it is bounded above and bounded below.

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9. **Bounded monotonic sequences (單調有界數列)** If a sequence $\{a_n\}$ is bounded and monotonic; then it converges. 26

Section 9.2 Series and convergence 30

10. **Convergent and divergent series** For the infinite series $\sum_{n=1}^{\infty} a_n$ the **nth partial sum** (**第 n 項部分和**) is given by

$$S_n = a_1 + a_2 + \cdots + a_n.$$

If the sequence of partial sums $\{S_n\}$ converges to S, then the series $\sum_{n=1}^{\infty} a_n$ **converges** (**收斂**). The limit S is called the **sum of the series**

(**級數和**).

$$S = a_1 + a_2 + \cdots + a_n + \cdots \qquad S = \sum_{n=1}^{\infty} a_n$$

If $\{S_n\}$ diverges, then the series **diverges** (**發散**). 33

11. **Telescoping series** (**對消級數**) If $a_n = b_{n+1} - b_n$, then $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} (b_{i+1} - b_i) = b_{n+1} - b_1$. Moreover, if the series converges, its sum is $S = \lim_{n\to\infty} b_{n+1} - b_1$. 37

12. **Convergence of a geometric series** A **geometric series** (**幾何級數**) with ratio r diverges if $|r| \geq 1$. If $0 < |r| < 1$, then the series converges to the sum

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad 0 < |r| < 1.$$

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13. **Propertiest of infinite series** Let $\sum a_n$ and $\sum b_n$ be convergent series, and let A, B, and c be real numbers. If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, then the following series converge to the indicated sums.

   (a) $\sum_{n=1}^{\infty} ca_n = cA$

   (b) $\sum_{n=1}^{\infty}(a_n + b_n) = A + B$

   (c) $\sum_{n=1}^{\infty}(a_n - b_n) = A - B$

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14. **Limit of the nth term of a convergent series** If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$. 43

15. **nth-term test for divergent** If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges. 45

16. **The Integral Test (積分檢定)** If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)\,\mathrm{d}x$$

either both converge or both diverge. 50

17. **p-series**: A series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

is a **p-series** (p 級數), where p is a positive constant. For $p = 1$, the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

is the **harmonic series** (**調和級數**). A **general harmonic series** (**一般調和**

of the form $\sum \frac{1}{(an+b)}$. 58

18. **Euler-Mascheroni constant γ (C)** (**尤拉常數 γ (C)**)

`http://en.wikipedia.org/wiki/Euler%E2%80%93Mascheroni_consta`

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{1}{k} - \ln n\right) \approx 0.5772156649$$

is a mathematical constant recurring in analysis and number theory. . 58

19. **Riemann zeta function $\zeta(s)$** (**黎曼 ζ 函數**)

`http://en.wikipedia.org/wiki/Riemann_zeta_function`

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is a function of a complex variable s that analytically continues the sum of the infinite series which converges when the real part of s is greater than 1. The Riemann zeta function plays a pivotal role in analytic number theory and has applications in physics, probability theory, and applied statistics. 59

20. **Convergence of p series** The **p-series** (p 級數)

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

1. converges if $p > 1$, and **2.** diverges if $0 < p \leq 1$. 60

21. **Direct Comparison Test** (直接互比檢定) Let $0 < a_n \leq b_n$ for all n.

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges. 65

22. **Limit Comparison Test** (極限互比檢定) Suppose $a_n > 0$, $b_n > 0$,

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L$$

where L is finite and positive. Then the two series $\sum a_n$ and $\sum b_n$ either both converge both diverge. 71

23. **Alternating Series Test** (交錯級數檢定) Let $a_n > 0$. The

alternating series

$$\sum_{n=1}^{\infty}(-1)^n a_n \quad \text{and} \quad \sum_{n=1}^{\infty}(-1)^{n+1} a_n$$

converge if the following two conditions are met.

1. $\lim_{n\to\infty} a_n = 0$ **2.** $a_{n+1} \le a_n$, for all n 78

24. **Alternating Series Remainder** (交錯級數餘項) If a convergent alternating series satisfies the condition $a_{n+1} \le a_n$, then the absolute value of the remainder R_N involved in approximating the sum S by S_N is less than (or equal to) the first neglected term. That is,

$$|S - S_N| = |R_N| \le a_{N+1}.$$

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25. **Absolute convergence** (**絕對收斂**) If the series $\sum |a_n|$ converges, then the series $\sum a_n$ also converges. 88

26. **Absolute and conditional convergence**

    (a) $\sum a_n$ is **absolutely convergent** (**絕對收斂**) if $\sum |a_n|$ converges.

    (b) $\sum a_n$ is **conditionally convergent** (**條件收斂**) if $\sum a_n$ converges but $\sum |a_n|$ diverges.

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27. If $\sum a_n$ is conditionally convergent and S is any real number, the terms of the series can be rearranged to converge to S. 95

    Section 9.6 The ratio and root test 95

28. **Ratio Test** (**比例檢定**) Let $\sum a_n$ be a series with nonzero terms.

(a) $\sum a_n$ converges absolutely if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1.$

(b) $\sum a_n$ diverges if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty.$

(c) The Ratio Test is inconclusive if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1.$

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29. **Root Test (根式檢定)** Let $\sum a_n$ be a series.

(a) $\sum a_n$ converges absolutely if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1.$

(b) $\sum a_n$ diverges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty.$

(c) The Root Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1.$

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30. **Guidelines for testing a series for convergence or divergence**

- ⊡ Does the nth term approach 0? If not, the series diverges.
- ⊡ Is the series one of the special types-geometric, p-series, telescoping, or alternating?
- ⊡ Can the Integral Test, the Root Test, or the Ratio Test be applied?
- ⊡ Can the series be compared favorably to one of the special types?

31. **Taylor polynomial and Maclaurin polynomial** If f has n deriv-

atives at c, then the polynomial

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the **nth Taylor polynomial for f at c** (f 在 c 的 n 階泰勒多項式

If $c = 0$, then

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

is also called the **nth Maclaurin polynomial for f at c** (f 在 c 的 n 階馬

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32. **Taylor's Theorem** (泰勒定理) If a function f is differentiable through order $n+1$ in an interval I containing c, then, for each x in I,

there exists z between x and c such that

$$f(x) = f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n+R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}.$$

33. **Power series** If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

is called a **power series** (冪級數). More generally, an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0+a_1(x-c)+a_2(x-c)^2+a_3(x-c)^3+\cdots+a_n(x-c)^n+\cdots$$

is called a **power series centered at c** (以 c 爲中心的冪級數), where c is a constant. 140

34. **Convergence of a power series** For a power series centered at c, precisely one of the following is true.

1. The series converges only at c.
2. There exists a real number $R > 0$ such that the series converges absolutely for $|x-c| < R$, and diverges for $|x-c| > R$.
3. The series converges absolutely for all x.

The number R is the **radius of convergence** (收斂半徑) of the power series. If the series converges only at c, the radius of convergence is $R = 0$, and if the series converges for all x, the radius of convergence is $R = \infty$. The set of all values of x for which the power series converges is the **interval of convergence** (**收斂區間**) of the power series. . . 144

35. **Properties of functions defined by power series** If the function given by

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

has a radius of convergence of $R > 0$, then, on the interval $(c-R,\ c+R)$, f is differentiable (and therefore continuous). Moreover, the derivative and antiderivative of f are as follows.

1. $f'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1} = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots$

orders at $x = c$, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

is called the **Taylor series** (泰勒級數) for $f(x)$ at c. Moreover, if $c = 0$, then the series is the **Maclaurin series** (馬克勞林級數) for f. 178

40. **Convergence of Taylor series** If $\lim_{n\to\infty} R_n = 0$ for all x in the interval I, then the Taylor series for f converges and equals $f(x)$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

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41. **Guidelines for finding a Taylor series**

(a) Differentiate $f(x)$ several times and evaluate each derivative at c.

$$f(c), \quad f'(c), \quad f''(c), \quad f'''(c), \quad \dots, \quad f^{(n)}(c), \quad \dots$$

Try to recognize a pattern in these numbers.

(b) Use the sequence developed in the first step to form the **Taylor coefficients** (**泰勒係數**) $a_n = f^{(n)}(c)/n!$, and determine the interval of convergence for the resulting power series

$$f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots .$$

(c) Within this interval of convergence, determine whether or not the series converges to $f(x)$.

42. Power series for elementary functions

Function

$$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \cdots + (-1)^n(x -$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots$$

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + \frac{(-1)^{(n-1)}(x -}{n}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$$

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