Chapter 13 Functions of Several Variables

Szu-Chi Chung

Department of Applied Mathematics, National Sun Yat-sen University

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Table of Contents

Table of Contents

Functions of several variables

- The work done by a force ($W = FD$) and the volume of a right circular cylinder ($V = \pi r^2 h$) are both functions of two variables.
- The volume of a rectangular solid ($V = lwh$) is a function of three variables.
- The notation for a function of two or more variables is similar to that for a function of a single variable. Here are two examples:

$$z = \underbrace{f(x, y)}_{\text{2 variables}} = x^2 + xy \quad \text{and} \quad w = \underbrace{f(x, y, z)}_{\text{3 variables}} = x + 2y - 3z.$$

Definition 13.1 (A function of two variables)

Let D be a set of ordered pairs of real numbers. If to each ordered pair (x, y) in D there corresponds a unique real number $f(x, y)$, then f is called a function of x and y. The set D is the domain of f, and the corresponding set of values for $f(x, y)$ is the range of f.

- For the function given by $z = f(x, y)$, x and y are called the independent variables and z is called the dependent variable.
- As with functions of one variable, the most common way to describe a function of several variables is with an equation. In addition, the domain is the set of points for which the equation is defined.

Example 1 (Domains of functions of several variables)

Find the domain of each function.

a. $f(x, y) = \frac{\sqrt{x^2+y^2-9}}{x}$ **b.** $g(x, y, z) = \frac{x}{\sqrt{9-x^2-y^2-z^2}}$

- Functions of several variables can be combined in the same ways as functions of single variables.

$$
\begin{aligned}
(f \pm g)(x, y) &= f(x, y) \pm g(x, y) && \text{Sum or difference} \\
(fg)(x, y) &= f(x, y)g(x, y) && \text{Product} \\
\frac{f}{g}(x, y) &= \frac{f(x, y)}{g(x, y)}, \quad g(x, y) \neq 0 && \text{Quotient}
\end{aligned}
$$

- You cannot form the composite of two functions of several variables!
- However, if h is a function of several variables and g is a function of a single variable, you can form the composition function $(g \circ h)(x, y)$ as follows:

$$(g \circ h)(x, y) = g(h(x, y)) \qquad \text{Composition}$$

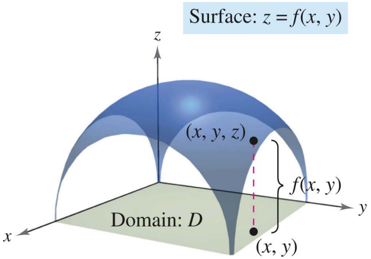
- A function that can be written as a sum of functions of the form $cx^m y^n$ (where c is a real number m and n are nonnegative integers) is called a polynomial function of two variables.
- For instance, the functions given by

$$f(x, y) = x^2 + y^2 - 2xy + x + 2 \quad \text{and} \quad g(x, y) = 3xy^2 + x - 2$$

  are polynomial functions of two variables.
- A rational function is the quotient of two polynomial functions. Similar terminology is used for functions of more than two variables.

The graph of a function of two variables

- The graph of a function f of two variables is the set of all points (x, y, z) for which $z = f(x, y)$ and (x, y) is in the domain of f.
- This graph can be interpreted geometrically as a surface in space. In figure below, note that the graph of $z = f(x, y)$ is a surface whose projection onto the xy-plane is the D, the domain of f.
- To each point (x, y) in D there corresponds a point (x, y, z) on the surface, and, conversely, to each point (x, y, z) on the surface there corresponds a point (x, y) in D.



Example 2 (Describing the graph of a function of two variables)

Considering the function given by $f(x, y) = \sqrt{16 - 4x^2 - y^2}$. a. Find the domain and range of the function. b. Describe the graph of f.

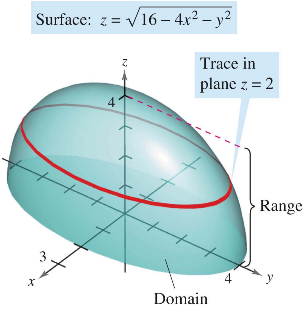
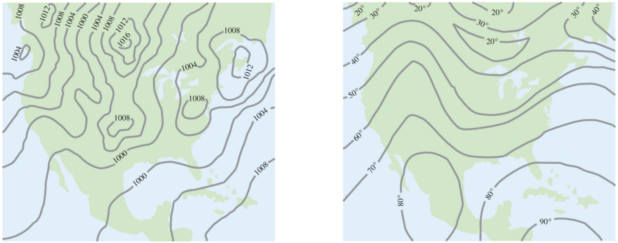


Figure 1: The graph of $f(x, y) = \sqrt{16 - 4x^2 - y^2}$ is the upper half of an ellipsoid.

Level curves

- A second way to visualize a function of two variables is to use a scalar field in which the scalar $z = f(x, y)$ is assigned to the point (x, y).
- A scalar field can be characterized by level curves (or contour lines) along which the value of $f(x, y)$ is constant.
- For instance, the weather map in Figure 2(a) shows level curves of equal pressure called isobars. In weather maps for which the level curves represent points of equal temperature, the level curves are called isotherms, as shown in Figure 2(b).



(a) Level curves show the lines of equal pressure (isobars) measured in milliards.

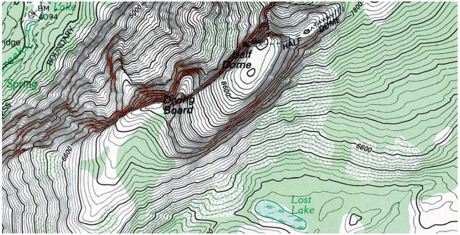
(b) Level curves show the lines of equal temperature (isotherms) measured in degrees Fahrenheit.

Figure 2: Level curves of weather map.

- Another common use of level curves is in representing electric potential fields. In this type of map, the level curves are called equipotential lines.
- Contour maps are commonly used to show regions on Earth's surface, with the level curves representing the height above sea level. This type of map is called a topographic map.

- For example, the mountain shown in Figure 3(a) is represented by the topographic map in Figure 3(b). A contour map depicts the variation of z with respect to x and y by the spacing between level curves.
- Much space between level curves indicates that z is changing slowly, whereas little space indicates a rapid change in z. Furthermore, to produce a good three-dimensional illusion in a contour map, it is important to choose c-values that are evenly spaced.

(a) Mountain.

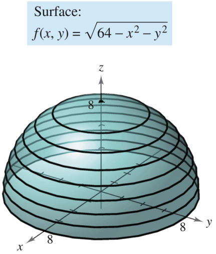


(b) The topographic map of the mountain.

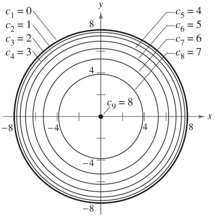
Figure 3: Mountain.

Example 3 (Sketching a contour map)

The hemisphere given by $f(x, y) = \sqrt{64 - x^2 - y^2}$ is shown in Figure 4(a). Sketch a contour map of this surface using level curves corresponding to $c = 0, 1, 2, \ldots, 8$.



(a) Hemisphere: $f(x, y) = \sqrt{64 - x^2 - y^2}$.



(b) Contour map of $f(x, y) = \sqrt{64 - x^2 - y^2}$.

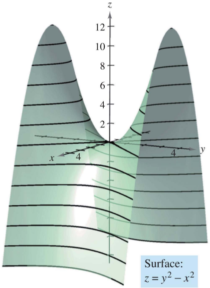
Figure 4: Contour map of hemisphere.

Example 4 (Sketching a contour map)

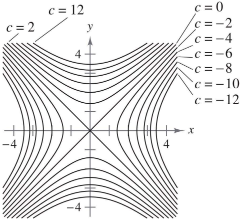
The hyperbolic paraboloid given by

$$z = y^2 - x^2$$

is shown in Figure 5(a). Sketch a contour map of this surface.



(a) Hyperbolic paraboloid.



(b) Hyperbolic level curves (at increments of 2).

Figure 5: Level curves of hyperbolic paraboloid.

Level surfaces

- The concept of a level curve can be extended by one dimension to define a level surface.
- If f is a function of three variables and c is a constant, the graph of the equation $f(x, y, z) = c$ is a level surface of the function f, as shown in Figure 6.

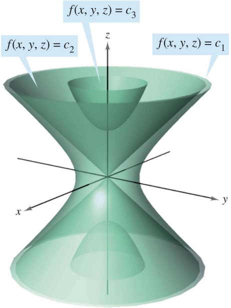


Figure 6: Level surfaces of $f(x, y, z) = c$.

Example 6 (Level surfaces)

Describe the level surfaces of the function

$$f(x, y, z) = 4x^2 + y^2 + z^2.$$

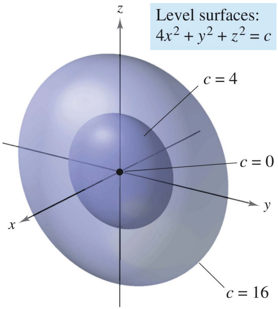
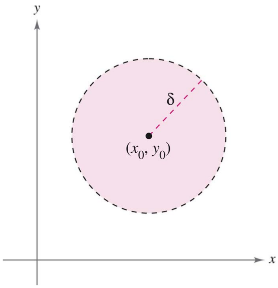

z
Level surfaces:
$4x^2 + y^2 + z^2 = c$
$c = 4$
$c = 0$
y
x
$c = 16$

Table of Contents

Neighborhoods in the plane

- Using the formula for the distance between two points (x, y) and (x_0, y_0) in the plane, you can define the $\underline{\delta\text{-neighborhood}}$ about (x_0, y_0) to be the disk centered at (x_0, y_0) with radius $\delta > 0$

$$\left\{(x, y) : \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta\right\} \qquad \text{Open disk}$$



- When this formula contains the less than inequality sign, $<$, the disk is called open, and when it contains the less than or equal to inequality sign, $\leq$, the disk is called closed. This corresponds to the use of $<$ and $\leq$ to define open and closed intervals.
- A point (x_0, y_0) in a plane region R is an interior point of R if there exists a δ-neighborhood about (x_0, y_0) that lies entirely in R, as shown in Figure 7.

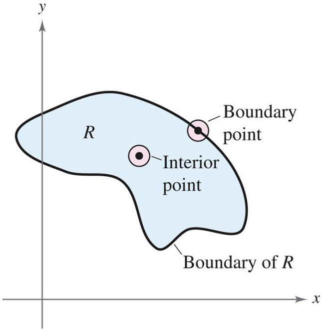


Figure 7: The boundary and interior points of a region R.

- If every point in R is an interior point, then R is an open region. A point (x_0, y_0) is a boundary point of R if every open disk centered at (x_0, y_0) contains points inside R and points outside R.
- By definition, a region must contain its interior points, but it need not contain its boundary points. If a region contains all its boundary points, the region is closed region.
- A region that contains some but not all of its boundary points is neither open nor closed!

Definition 13.2 (Limit of a function of two variables)

Let f be a function of two variables defined, except possible at (x_0, y_0), on an open disk centered at (x_0, y_0), and let L be a real number. Then

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if for each ε there corresponds a $\delta > 0$ such that

$$|f(x,y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta.$$

- Graphically, this definition of a limit implies that for any point $(x, y) \neq (x_0, y_0)$ in the disk of radius δ, the value $f(x, y)$ lies between $L + \varepsilon$ and $L - \varepsilon$, as shown in Figure 8.

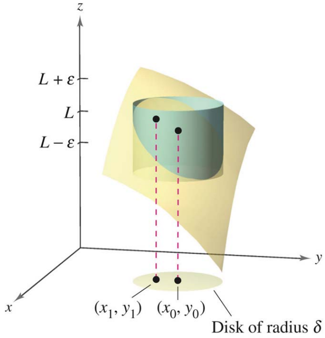


Figure 8: For any (x, y) in the disk of radius δ, the value $f(x, y)$ lies between $L + \varepsilon$ and $L - \varepsilon$.

- The definition of the limit of a function of two variables is similar to the definition of the limit of a function of a single variable, yet there is a critical difference!

- To determine whether a function of a single variable has a limit, you need only test the approach from **two** directions—from the right and from the left. If the function approaches the same limit from the right and from the left, you can conclude that the limit exists.
- However, for a function of two variables, the statement

$$(x, y) \to (x_0, y_0)$$

means that the point (x, y) is allowed to approach (x_0, y_0) from **any** direction.
- If the value of

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y)$$

is not the same for all possible approaches, or paths, to (x_0, y_0), the limit does not exist!

Example 1 (Verifying a limit by the definition)

Show that $\lim_{(x,y)\to(a,b)} x = a$.

Example 2 (Verifying a limit)

Evaluate $\lim_{(x,y)\to(1,2)} \frac{5x^2y}{x^2+y^2}$.

Example 3 (Finding a limit)

Evaluate $\lim_{(x,y)\to(0,0)} \frac{5x^2y}{x^2+y^2}$.

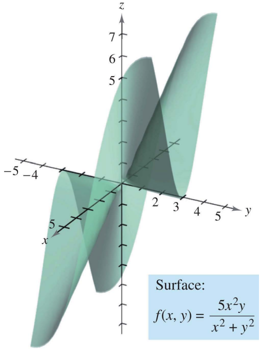


Figure 9: $\lim_{(x,y)\to(0,0)} \frac{5x^2y}{x^2+y^2} = 0.$

- For some functions, it is easy to recognize that a limit does not exist. For instance, it is clear that the limit $\lim_{(x,y)\to(0,0)} \frac{1}{x^2+y^2}$ does not exist because the values of $f(x,y)$ increase without bound as (x,y) approaches $(0,0)$ along any path (see Figure 10).

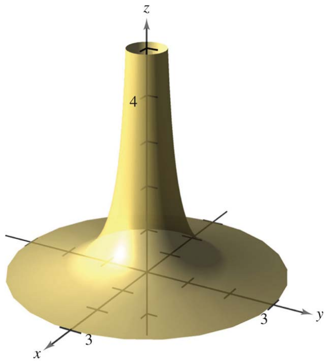


Figure 10: $\lim_{(x,y)\to(0,0)} \frac{1}{x^2+y^2}$ does not exist.

Example 4 (A limit that does not exist)

Show that the following limit does not exist.

$$\lim_{(x,y)\to(0,0)} \left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2$$

Continuity of a function of two variables

- The limit of $f(x, y) = 5x^2y/(x^2 + y^2)$ as $(x, y) \to (1, 2)$ can be evaluated by direct substitution. That is, the limit is $f(1, 2) = 2$.
- In such cases the function f is said to be continuous at the point $(1, 2)$.

Definition 13.3 (Continuity of a function of two variables)

A function f of two variables is continuous at a point (x_0, y_0) in an open region R if $f(x_0, y_0)$ is equal to the limit of $f(x, y)$ as (x, y) approaches (x_0, y_0). That is,

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0).$$

The function f is continuous in the open region R if it is continuous at every point in R.

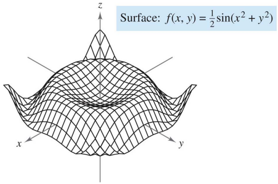
- The function $f(x, y) = \frac{5x^2y}{x^2+y^2}$ is not continuous at $(0, 0)$. However, because the limit at this point exists, you can remove the discontinuity by defining f at $(0, 0)$ as being equal to its limit there. Such a discontinuity is call <u>removable</u>.
- The function $f(x, y) = \left[\frac{x^2-y^2}{x^2+y^2}\right]^2$ is not continuous at $(0, 0)$, and this discontinuity is <u>nonremovable</u>.

Theorem 13.1 (Continuity of a function of two variables)

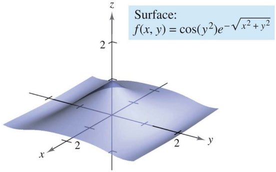
If k is a real number and f and g are continuous at (x_0, y_0), then the following function are continuous at (x_0, y_0).

1. *Scalar multiple: kf*
2. *Product: fg*
3. *Sum and difference: $f \pm g$*
4. *Quotient: f/g, if $g(x_0, y_0) \neq 0$.*

- Theorem 13.1 establishes the continuity of polynomial and rational functions at every point in their domains. Furthermore, the continuity of other types of functions can be extended naturally from one to two variables.
- For instance, the functions whose graphs are shown in Figures 11(a) and 11(b) are continuous at every point in the plane.



(a) The function $f(x, y) = \frac{1}{2}\sin(x^2 + y^2)$ is continuous at every point in the plane.



(b) The function $f(x, y) = \cos(y^2)e^{-\sqrt{x^2+y^2}}$ is continuous at every point in the plane.

Figure 11: Surfaces about continuity.

Theorem 13.2 (Continuity of a composite function)

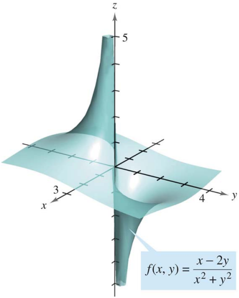
If h is continuous at (x_0, y_0) and g is continuous at $h(x_0, y_0)$, then the composite function given by $(g \circ h)(x, y) = g(h(x, y))$ is continuous at (x_0, y_0). That is,

$$\lim_{(x,y)\to(x_0,y_0)} g(h(x, y)) = g(h(x_0, y_0)).$$

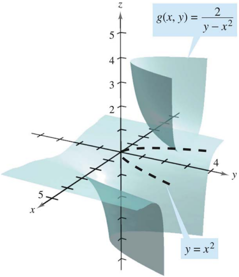
Example 5 (Testing for continuity)

Discuss the continuity of each function.

a. $f(x, y) = \frac{x-2y}{x^2+y^2}$ **b.** $g(x, y) = \frac{2}{y-x^2}$.



(a) The function $f(x, y) = \frac{x-2y}{x^2+y^2}$ is not continuous at $(0, 0)$.



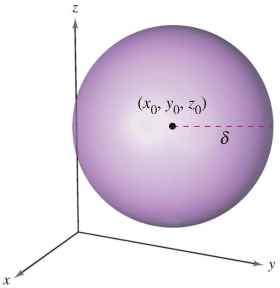
(b) The function $g(x, y) = \frac{2}{y-x^2}$ is not continuous on the parabola $y = x^2$.

Continuity of a function of three variables

- The definitions of limits and continuity can be extended to functions of three variables by considering points (x, y, z) within the open sphere

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < \delta^2. \qquad \text{Open sphere}$$

- The radius of this sphere is δ, and the sphere is centered at (x_0, y_0, z_0), as shown below:



- A point (x_0, y_0, z_0) in a region R in space is an interior point of R if there exists a δ-sphere about (x_0, y_0, z_0) that lies entirely in R. If every point in R is an interior point, then R is called open.

Definition 13.4 (Continuity of a function of three variables)

A function f of three variables is continuous at a point (x_0, y_0, z_0) in an open region R if $f(x_0, y_0, z_0)$ is defined and is equal to the limit of $f(x, y, z)$ as (x, y, z) approaches (x_0, y_0, z_0). That is,

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z) = f(x_0, y_0, z_0).$$

The function f is continuous in the open region R if it is continuous at every point in R.

Example 6 (Testing continuity of a function of three variables)

Discuss the continuity of $f(x, y, z) = \frac{1}{x^2+y^2-z}$.

Table of Contents

Partial derivatives of a function of two variables

Definition 13.5 (Partial derivatives of a function of two variables)

If $z = f(x, y)$, then the first partial derivatives of f with respect to x and y are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

Partial derivative with respect to x and

$$f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

Partial derivative with respect to y , provided the limits exist.

- You can determine the rate of change of f with respect to one of its several independent variables. This process is called partial differentiation.
- This definition indicates that if $z = f(x, y)$, then to find f_x you consider y constant and differentiate with respect to x.
- Similarly, to find f_y, you consider x constant and differentiate with respect to y.

Example 1 (Finding partial derivatives)

Find the partial derivatives f_x and f_y for the function
a. $f(x, y) = 3x - x^2y^2 + 2x^3y$. **b.** $f(x, y) = (\ln x)(\sin x^2 y)$.

(Notation for first partial derivatives)
For $z = f(x, y)$, the partial derivatives f_x and f_y are denoted by

$$\frac{\partial}{\partial x} f(x, y) = f_x(x, y) = z_x = \frac{\partial z}{\partial x}$$

and

$$\frac{\partial}{\partial y} f(x, y) = f_y(x, y) = z_y = \frac{\partial z}{\partial y}.$$

The first partials evaluated at the point (a, b) are denoted by

$$\left.\frac{\partial z}{\partial x}\right|_{(a,b)} = f_x(a, b) \quad \text{and} \quad \left.\frac{\partial z}{\partial y}\right|_{(a,b)} = f_y(a, b).$$

Example 2 (Finding and evaluating partial derivatives)

For $f(x, y) = xe^{x^2 y}$, find f_x and f_y, and evaluate each at the point $(1, \ln 2)$.

- The partial derivatives of a function of two variables, $z = f(x, y)$, have a useful geometric interpretation. If $y = y_0$, then $z = f(x, y_0)$ represents the curve formed by intersecting the surface $z = f(x, y)$ with the plane $y = y_0$, as shown in Figure 13(a).
- Therefore,

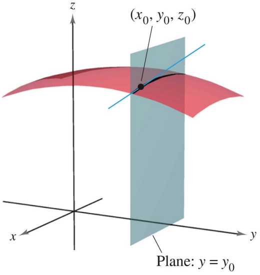
$$f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

  represents the slope of this curve at the point $(x_0, y_0, f(x_0, y_0))$.
- Note that both the curve and the tangent line lie in the plane $y = y_0$. Similarly,

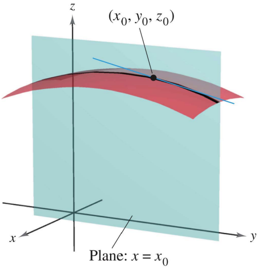
$$f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

  represents the slope of the curve given by the intersection of $z = f(x, y)$ and the plane $x = x_0$ at $(x_0, y_0, f(x_0, y_0))$, as shown in Figure 13(b).

- Informally, the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point (x_0, y_0, z_0) denote the slopes of the surface in the x- and y-directions , respectively.



(a) $\frac{\partial f}{\partial x}$ = slope in x-direction.



(b) $\frac{\partial f}{\partial y}$ = slope in y-direction.

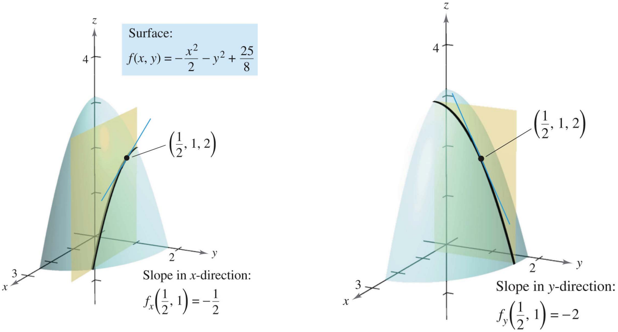
Figure 13: Partial derivatives of a function of two variables.

Example 3 (Finding the slopes of a surface in the x- and y-directions)

Find the slopes in the x-direction and in the y-direction of the surface given by

$$f(x,y) = -\frac{x^2}{2} - y^2 + \frac{25}{8}$$

at the point $\left(\frac{1}{2}, 1, 2\right)$.



(a) Slope in x-direction: $f_x\left(\frac{1}{2}, 1\right) = -\frac{1}{2}$.

(b) Slope in y-direction: $f_y\left(\frac{1}{2}, 1\right) = -2$.

Figure 14: Partial derivatives of $f(x, y) = -\frac{x^2}{2} - y^2 + \frac{25}{8}$ at $\left(\frac{1}{2}, 1, 2\right)$.

Example 4 (Finding the slopes of a surface in the x- and y-directions)

Find the slopes of the surface given by

$$f(x,y) = 1 - (x-1)^2 - (y-2)^2$$

at the point $(1, 2, 1)$ in the x-direction and in the y-direction.

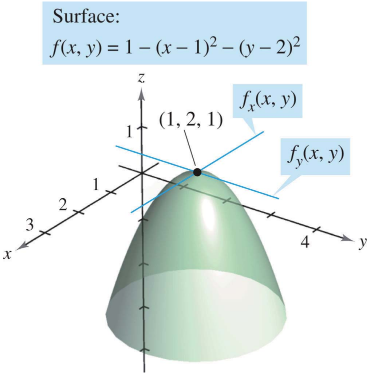


Figure 15: Finding the slopes of a surface in the x- and y-directions.

Partial derivatives of a function of three or more variables

- The concept of a partial derivative can be extended naturally to functions of three or more variables. For instance, if $w = f(x, y, z)$, there are three partial derivatives, each of which is formed by holding two of the variables constant.
- That is, to define the partial derivative of w with respect to x, consider y and z to be constant and differentiate with respect to x.
- $$\frac{\partial w}{\partial x} = f_x(x, y, z) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$$
  $$\frac{\partial w}{\partial y} = f_y(x, y, z) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}$$
  $$\frac{\partial w}{\partial z} = f_z(x, y, z) = \lim_{\Delta z \to 0} \frac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}$$

- In general, if $w = f(x_1, x_2, \ldots, x_n)$, there are n partial derivatives denoted by

$$\frac{\partial w}{\partial x_k} = f_{x_k}(x_1, x_2, \ldots, x_n), \quad k = 1,\ 2,\ \ldots,\ n.$$

- To find the partial derivative with respect to one of the variables, hold the other variables constant and differentiate with respect to the given variable.

Example 6 (Finding partial derivatives)

a. $f(x, y, z) = xy + yz^2 + xz$ with respect to z
b. $f(x, y, z) = z\sin(xy^2 + 2z)$ with respect to z
c. $f(x, y, z, w) = (x + y + z)/w$ with respect to w

Higher-order partial derivatives

- The function $z = f(x, y)$ has the following second partial derivatives.
  1. Differentiate twice with respect to x:
  $$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}.$$
  2. Differentiate twice with respect to y:
  $$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}.$$
  3. Differentiate first with respect to x and then with respect to y:
  $$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}.$$
  4. Differentiate first with respect to y and then with respect to x:
  $$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}.$$

Example 7 (Finding second partial derivatives)

Find the second partial derivatives of $f(x, y) = 3xy^2 - 2y + 5x^2y^2$, and determine the value of $f_{xy}(-1, 2)$.

Theorem 13.3 (Equality of mixed partial derivatives)

If f is a function of x and y such that f_{xy} and f_{yx} are continuous on an open disk R, then, for every (x, y) in R,

$$f_{xy}(x, y) = f_{yx}(x, y).$$

Example 8 (Finding higher-order partial derivatives)

Show that $f_{xz} = f_{zx}$ and $f_{xzz} = f_{zxz} = f_{zzx}$ for the function given by $f(x, y, z) = ye^x + x \ln z$.

Table of Contents

Increments and differentials

- For $y = f(x)$, the differential of y was defined as $\mathrm{d}y = f'(x)\,\mathrm{d}x$.
- Similar terminology is used for a function of two variables, $z = f(x, y)$. That is, Δx and Δy are the increments of x and y, and the increment of z is given by

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y). \qquad \text{Increment of } z$$

Definition 13.6 (Total differential)

If $z = f(x, y)$ and Δx and Δy are increments of x and y, then the differentials of the independent variables x and y are

$$\mathrm{d}x = \Delta x \quad \text{and} \quad \mathrm{d}y = \Delta y$$

and the total differential of the dependent variable z is

$$\mathrm{d}z = \frac{\partial z}{\partial x}\,\mathrm{d}x + \frac{\partial z}{\partial y}\,\mathrm{d}y = f_x(x, y)\,\mathrm{d}x + f_y(x, y)\,\mathrm{d}y.$$

- This definition can be extended to a function of more variables. For instance, if $w = f(x, y, z, u)$, then $\mathrm{d}x = \Delta x$, $\mathrm{d}y = \Delta y$, $\mathrm{d}z = \Delta z$, $\mathrm{d}u = \Delta u$, and the total differential of w is

$$\mathrm{d}w = \frac{\partial w}{\partial x}\,\mathrm{d}x + \frac{\partial w}{\partial y}\,\mathrm{d}y + \frac{\partial w}{\partial z}\,\mathrm{d}z + \frac{\partial w}{\partial u}\,\mathrm{d}u.$$

Example 1 (Finding the total differential)

Find the total differential for each function.

a. $z = 2x \sin y - 3x^2y^2$ **b.** $w = x^2 + y^2 + z^2$

Differentiability

- For a differentiable function given by $y = f(x)$, you can use the differential $\mathrm{d}y = f'(x)\,\mathrm{d}x$ as an approximation (for small Δx) to the value $\Delta y = f(x + \Delta x) - f(x)$.
- When a similar approximation is possible for a function of two variables, the function is said to be differentiable.

Definition 13.7 (Differentiability)

A function f given by $z = f(x, y)$ is differentiable at (x_0, y_0) if Δz can be written in the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where both ϵ_1 and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. The function f is differentiable in a region R if it is differentiable at each point in R.

Example 2 (Showing that a function is differentiable)

Show that the function given by

$$f(x, y) = x^2 + 3y$$

is differentiable at every point in the plane.

Theorem 13.4 (Sufficient condition for differentiability)

If f is a function of x and y, where f_x and f_y are continuous in an open region R, then f is differentiable on R.

Approximation by differentials

- Theorem 13.4 tells you that you can choose $(x + \Delta x, y + \Delta y)$ close enough to (x, y) to make $\epsilon_1 \Delta x$ and $\epsilon_2 \Delta y$ insignificant. In other words, for small Δx and Δy, you can use the approximation $\Delta z \approx \mathrm{d}z$.

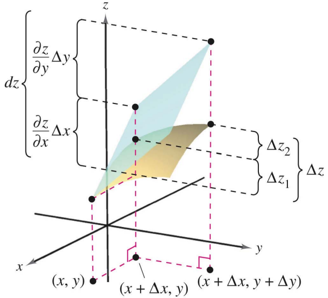


Figure 16: The exact change in z is Δz. This change can be approximated by the differential $\mathrm{d}z$.

- The partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ can be interpreted as the slopes of the surface in the x- and y-directions.
- This means that
$$\mathrm{d}z = \frac{\partial z}{\partial x}\,\Delta x + \frac{\partial z}{\partial y}\,\Delta y$$
represents the change in height of a plane that is tangent to the surface at the point $(x, y, f(x, y))$.
- Because a plane in space is represented by a linear equation in the variables x, y, and z, the approximation of Δz by $\mathrm{d}z$ is called a **linear approximation**.

Example 3 (Using a differential as an approximation)

Use the differential $\mathrm{d}z$ to approximate the change in $z = \sqrt{4 - x^2 - y^2}$ as (x, y) moves from the point $(1, 1)$ to the point $(1.01, 0.97)$. Compare this approximation with the exact change in z.

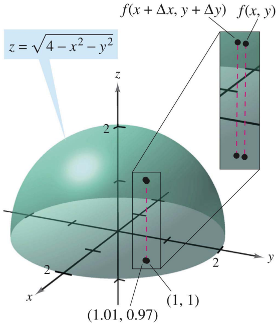


Figure 17: As (x, y) moves from $(1, 1)$ to the point $(1.01, 0.97)$, the value of $f(x, y)$ changes by about 0.0137.

- A function of three variables $w = f(x, y, z)$ is called differentiable at (x, y, z) provided that

$$\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$$

can be written in the form

$$\Delta w = f_x\, \Delta x + f_y\, \Delta y + f_z\, \Delta z + \epsilon_1\, \Delta x + \epsilon_2\, \Delta y + \epsilon_3\, \Delta z$$

where ϵ_1, ϵ_2, and $\epsilon_3 \to 0$ as $(\Delta x, \Delta y, \Delta z) \to (0, 0, 0)$.
- With this definition of differentiability, Theorem 13.4 has the following extension for functions of three variables:

Sufficient condition for differentiability: If f is a function of x, y, and z, where f, f_x, f_y, and f_z are continuous in an open region R, then f is differentiable on R.

Theorem 13.5 (Differentiability implies continuity)

If a function of x and y is differentiable at (x_0, y_0), then it is continuous at (x_0, y_0).

Example 5 (A function that is not differentiable)

Show that $f_x(0,0)$ and $f_y(0,0)$ both exist, but that f is not differentiable at $(0,0)$ where f is defined as

$$f(x,y) = \begin{cases} \frac{-3xy}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}.$$

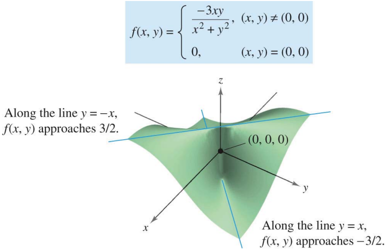


Figure 18: A function not differentiable but partial differential derivatives exist.

Table of Contents

Chain Rules for functions of several variables

Theorem 13.6 (Chain Rule: one independent variable)

Let $w = f(x, y)$, where f is a differentiable function of x and y. If $x = g(t)$ and $y = h(t)$, where g and h are differentiable functions of t, then w is a differentiable function of t, and

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial w}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial w}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}. \qquad \textit{See Figure 19}$$

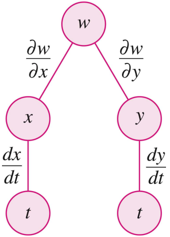


Figure 19: Chain Rule: one independent variable w is a function of x and y, which are each functions of t. It represents the derivative of w with respect to t.

Example 1 (Using the Chain Rule with one independent variable)

Let $w = x^2 y - y^2$, where $x = \sin t$ and $y = e^t$. Find $\mathrm{d}w/\,\mathrm{d}t$ when $t = 0$.

- The Chain Rule in Theorem 13.6 can be extended to any number of variables. For example, if each x_i is a differentiable function of a single variable t, then for

$$w = f(x_1, x_2, \ldots, x_n)$$

you have

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial w}{\partial x_1}\frac{\mathrm{d}x_1}{\mathrm{d}t} + \frac{\partial w}{\partial x_2}\frac{\mathrm{d}x_2}{\mathrm{d}t} + \cdots + \frac{\partial w}{\partial x_n}\frac{\mathrm{d}x_n}{\mathrm{d}t}.$$

Example 3 (Finding partial derivatives by substitution)

Find $\partial w/\partial s$ and $\partial w/\partial t$ for $w = 2xy$, where $x = s^2 + t^2$ and $y = s/t$.

Theorem 13.7 (Chain Rule: two independent variables)

Let $w = f(x, y)$, where f is a differentiable function of x and y. If $x = g(s, t)$ and $y = h(s, t)$ such that the first partial $\frac{\partial x}{\partial s}$, $\frac{\partial x}{\partial t}$, $\frac{\partial y}{\partial s}$, and $\frac{\partial y}{\partial t}$ all exist, then $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ exist and are given by

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}.$$

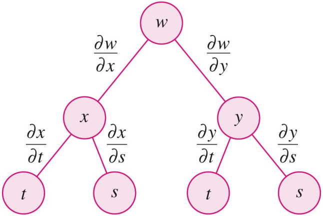


Figure 20: Chain Rule: two independent variables.

Example 4 (The Chain Rule with two independent variables)

Use the Chain Rule to find $\partial w/\partial s$ and $\partial w/\partial t$ for

$$w = 2xy$$

where $x = s^2 + t^2$ and $y = s/t$.

- The Chain Rule in Theorem 13.7 can also be extended to any number of variables.
- For example, if w is a differentiable function of the n variables x_1, x_2, …, x_n, where each x_i is a differentiable function of the m variables t_1, t_2, …, t_m, then for $w = f(x_1, x_2, \ldots, x_n)$ you obtain the following.

$$
\begin{aligned}
\frac{\partial w}{\partial t_1} &= \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_1} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_1} \\
\frac{\partial w}{\partial t_2} &= \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_2} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_2} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_2} \\
&\vdots \\
\frac{\partial w}{\partial t_m} &= \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_m} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_m} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_m}
\end{aligned}
$$

Example 5 (The Chain Rule for a function of three variables)

Find $\partial w/\partial s$ and $\partial w/\partial t$ when $s = 1$ and $t = 2\pi$ for the function given by $w = xy + yz + xz$ where $x = s\cos t$, $y = s\sin t$, and $z = t$.

Implicit partial differentiation

- An application of the Chain Rule to determine the derivative of a function defined implicitly.
- Suppose that x and y are related by the equation $F(x, y) = 0$, where it is assumed that $y = f(x)$ is a differentiable function of x. To find $\mathrm{d}y/\,\mathrm{d}x$ use Chain Rule. You consider the function given by

$$w = F(x, y) = F(x, f(x))$$

  you can apply Theorem 13.6 to obtain

$$\frac{\mathrm{d}w}{\mathrm{d}x} = F_x(x, y)\,\frac{\mathrm{d}x}{\mathrm{d}x} + F_y(x, y)\,\frac{\mathrm{d}y}{\mathrm{d}x}.$$

- Because $w = F(x, y) = 0$ for all x in the domain of f, you know that $\mathrm{d}w/\,\mathrm{d}x = 0$ and you have

$$F_x(x, y)\,\frac{\mathrm{d}x}{\mathrm{d}x} + F_y(x, y)\,\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

- Now, if $F_y(x, y) \neq 0$, you can use the fact that $\mathrm{d}x/\,\mathrm{d}x = 1$ to conclude that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x(x, y)}{F_y(x, y)}.$$

- A similar procedure can be used to find the partial derivatives of functions of several variables that are defined implicitly.

Theorem 13.8 (Chain Rule: implicit differentiation)

If the equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x(x, y)}{F_y(x, y)}, \quad F_y(x, y) \neq 0.$$

If the equation $F(x, y, z) = 0$ defines z implicitly as a differentiable function of x and y, then

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}, \quad F_z(x, y, z) \neq 0.$$

Example 6 (Finding a derivative implicitly)

Find $\mathrm{d}y/\mathrm{d}x$, given $y^3 + y^2 - 5y - x^2 + 4 = 0$.

Example 7 (Finding partial derivatives implicitly)

Find $\partial z/\partial x$ and $\partial z/\partial y$, given $3x^2z - x^2y^2 + 2z^3 + 3yz - 5 = 0$.

Table of Contents

Directional derivative

- You are standing on the hillside pictured in Figure 21 and want to determine the hill's incline toward the z-axis.
- If the hill were represented by $z = f(x, y)$, you already know how to determine the slopes in two different directions—the slope in the y-direction would be given by the partial derivative $f_y(x, y)$, and the slope in the x-direction would be given by the partial derivative $f_x(x, y)$.

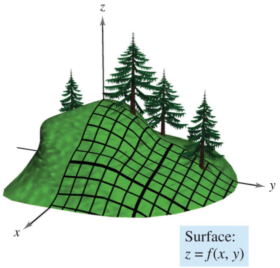
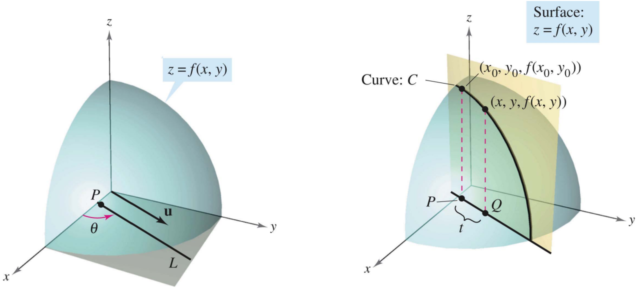


Figure 21: Hill's incline toward the z-axis: Surface $z = f(x, y)$.

- You will see that these two partial derivatives can be used to find the slope in any direction. To determine the slope at a point on a surface, you will define a new type of derivative called a directional derivative.
- Begin by letting $z = f(x, y)$ be a surface and $P(x_0, y_0)$ be a point in the domain of f, as shown in Figure 22(a). The "direction" of the directional derivative is given by a unit vector

$$\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

  where θ is the angle the vector makes with the positive x-axis.
- To find the slope, reduce the problem to two dimensions by intersecting the surface with a vertical plane passing through the point P and parallel to $\mathbf{u}$, as shown in Figure 22(b). This vertical plane intersects the surface to form a curve C.



(a) Line L with direction of $\mathbf{u}$ in xy-plane and surface $z = f(x, y)$.

(b) Curve C on surface $z = f(x, y)$ with projection line L on xy-plane.

Figure 22: Line and curve on surface $z = f(x, y)$.

- The slope of the surface at $(x_0, y_0, f(x_0, y_0))$ in the direction of $\mathbf{u}$ is defined as the slope of the curve C at that point. You can write the slope of the curve C as a limit that looks much like those used in single-variable calculus.

- The vertical plane used to form C intersects the xy-plane in a line L, represented by the parametric equations

$$x = x_0 + t\cos\theta \quad \text{and} \quad y = y_0 + t\sin\theta$$

so that for any value of t, the point $Q(x, y)$ lies on the line L.
- For each of the points P and Q, there is a corresponding point on the surface.

$$\begin{array}{ll} (x_0, y_0, f(x_0, y_0)) & \text{Point above } P \\ (x, y, f(x, y)) & \text{Point above } Q \end{array}$$

- Moreover, because the distance between P and Q is

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = \sqrt{(t\cos\theta)^2 + (t\sin\theta)^2} = |t|$$

you can write the slope of the secant line through $(x_0, y_0, f(x_0, y_0))$ and $(x, y, f(x, y))$ as

$$\frac{f(x, y) - f(x_0, y_0)}{t} = \frac{f(x_0 + t\cos\theta, y_0 + t\sin\theta) - f(x_0, y_0)}{t}.$$

- Finally, by letting t approach 0, you arrive at the following definition.

Definition 13.8 (Directional derivative)

Let f be a function of two variables x and y and let $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ be a unit vector. Then the directional derivative of f in the direction of $\mathbf{u}$, denoted by $D_{\mathbf{u}}f$, is

$$D_{\mathbf{u}}f(x, y) = \lim_{t\to 0} \frac{f(x + t\cos\theta, y + t\sin\theta) - f(x, y)}{t}$$

provided this limit exists.

Theorem 13.9 (Directional derivative)

If f is a differentiable function of x and y, then the directional derivative of f in the direction of the unit vector $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ is

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta.$$

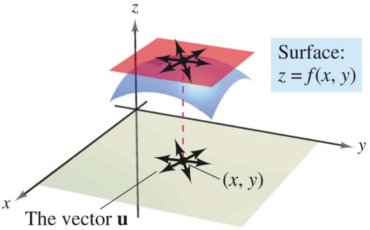
- There are infinitely many directional derivatives of a surface at a given point—one for each direction specified by $\mathbf{u}$!
- Two of these are the partial derivatives f_x and f_y.

1. Direction of positive x-axis ($\theta = 0$): $\mathbf{u} = \cos 0\,\mathbf{i} + \sin 0\,\mathbf{j} = \mathbf{i}$

$$D_{\mathbf{i}} f(x, y) = f_x(x, y)\cos 0 + f_y(x, y)\sin 0 = f_x(x, y).$$

2. Direction of positive y-axis ($\theta = \pi/2$): $\mathbf{u} = \cos\frac{\pi}{2}\,\mathbf{i} + \sin\frac{\pi}{2}\,\mathbf{j} = \mathbf{j}$

$$D_{\mathbf{j}} f(x, y) = f_x(x, y)\cos\frac{\pi}{2} + f_y(x, y)\sin\frac{\pi}{2} = f_y(x, y).$$



Example 1 (Finding a directional derivative)

Find the directional derivative of

$$f(x, y) = 4 - x^2 - \frac{1}{4}y^2 \qquad \text{Surface}$$

at $(1, 2)$ in the direction of

$$\mathbf{u} = \left(\cos\frac{\pi}{3}\right)\mathbf{i} + \left(\sin\frac{\pi}{3}\right)\mathbf{j}. \qquad \text{Direction}$$

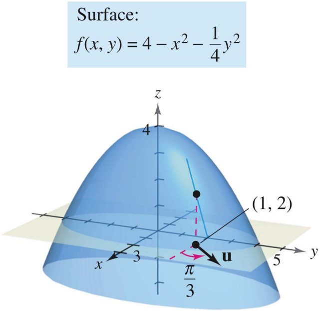


Figure 23: Directional derivative of surface: $f(x, y) = 4 - x^2 - \frac{1}{4}y^2$ at $(1, 2)$ with $\theta = \pi/3$.

Example 2 (Finding a directional derivative)

Find the directional derivative of

$$f(x, y) = x^2 \sin 2y \qquad \text{Surface}$$

at $(1, \pi/2)$ in the direction of

$$\mathbf{v} = 3\,\mathbf{i} - 4\,\mathbf{j}. \qquad \text{Direction}$$

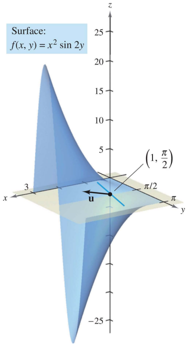


Figure 24: Finding a directional derivative.

The gradient of a function of two variables

- The gradient of a function of two variables is a **vector-valued function** of two variables.

Definition 13.9 (Gradient of a function of two variables)

Let $z = f(x, y)$ be a function of x and y such that f_x and f_y exist. Then the gradient of f, denoted by $\nabla f(x, y)$, is the vector

$$\nabla f(x, y) = f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j}.$$

∇f is read as "del f". Another notation for the gradient is **grad** $f(x, y)$. In Figure 25, note that for each (x, y), the gradient $\nabla f(x, y)$ is a vector in the plane (not a vector in space).

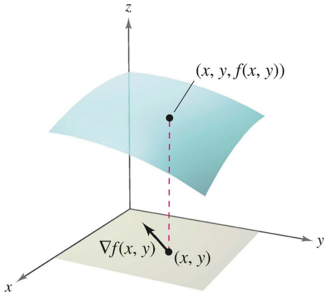


Figure 25: The gradient of f is a vector in the xy-plane.

Example 3 (Finding the gradient of a function)

Find the gradient of $f(x, y) = y \ln x + xy^2$ at the point $(1, 2)$.

Theorem 13.10 (Alternative form of the directional derivative)

If f is a differentiable function of x and y, then the directional derivative of f in the direction of the unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.$$

Example 4 (Using $\nabla f(x, y)$ to find a directional derivative)

Find the directional derivative of

$$f(x, y) = 3x^2 - 2y^2$$

at $\left(-\frac{3}{4}, 0\right)$ in the direction from $P\left(-\frac{3}{4}, 0\right)$ to $Q(0, 1)$.

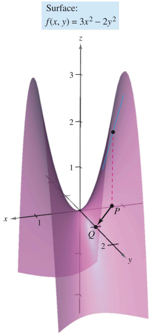


Figure 26: Directional derivative of surface $z = f(x, y) = 3x^2 - 2y^2$.

- In many applications, you may want to know in which direction to move so that $f(x, y)$ increases most rapidly. This direction is called the direction of steepest ascent, and it is given by the gradient.

Theorem 13.11 (Properties of the gradient)

Let f be differentiable at the point (x, y).

1. *If $\nabla f(x, y) = \mathbf{0}$, then $D_{\mathbf{u}} f(x, y) = 0$ for all $\mathbf{u}$.*
2. *The direction of maximum increase of f is given by $\nabla f(x, y)$. The maximum value of $D_{\mathbf{u}} f(x, y)$ is $\|\nabla f(x, y)\|$.*
3. *The direction of minimum increase of f is given by $-\nabla f(x, y)$. The minimum value of $D_{\mathbf{u}} f(x, y)$ is $-\|\nabla f(x, y)\|$.*

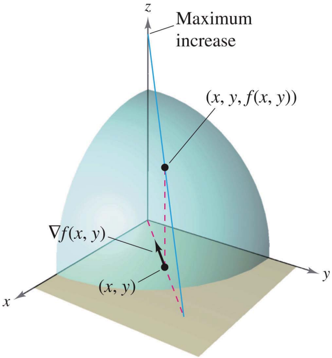


Figure 27: The gradient of f is a vector in the xy-plane that points in the direction of maximum increase on the surface given by $z = f(x, y)$.

Example 5 (Finding the direction of maximum increase)

The temperature in degrees Celsius on the surface of a metal plate is

$$T(x, y) = 20 - 4x^2 - y^2$$

where x and y are measured in centimeters. In what direction from $(2, -3)$ does the temperature increase most rapidly? What is this rate of increase?

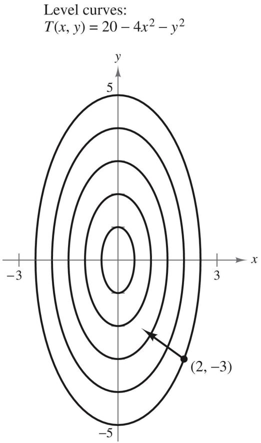


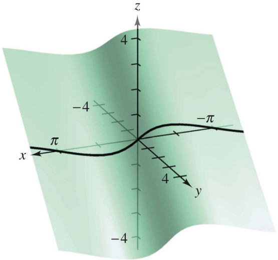
Figure 28: The direction of the most rapid increase in temperature in $(2, -3)$ is given by $-16\,\mathbf{i} + 6\,\mathbf{j}$.

Theorem 13.12 (Gradient is normal to level curves)

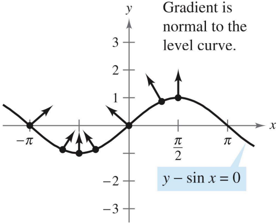
If f is differentiable at (x_0, y_0) and $\nabla f(x_0, y_0) \neq \mathbf{0}$, then $\nabla f(x_0, y_0)$ is normal to the level curve through (x_0, y_0).

Example 7 (Finding a normal vector to a level curve)

Sketch the level curve corresponding to $c = 0$ for the function given by $f(x, y) = y - \sin x$ and find a normal vector at several points on the curve.



(a) The surface is given by $f(x, y) = y - \sin x$.



(b) The level curve is given by $f(x, y) = 0$.

Figure 29: Finding a normal vector to a level curve.

Functions of three variables

Definition 13.10 (Directional derivative and gradient for three variables)

Let f be a function of x, y, and z, with continuous first partial derivatives. The **directional derivative** of f in the direction of a unit vector $\mathbf{u} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ is given by

$$D_{\mathbf{u}} f(x, y, z) = af_x(x, y, z) + bf_y(x, y, z) + cf_z(x, y, z).$$

The **gradient** of f is defined as

$$\nabla f(x, y, z) = f_x(x, y, z)\,\mathbf{i} + f_y(x, y, z)\,\mathbf{j} + f_z(x, y, z)\,\mathbf{k}.$$

Definition 13.10

Properties of the gradient are as follows.

1. $D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$
2. If $\nabla f(x, y, z) = \mathbf{0}$, then $D_{\mathbf{u}} f(x, y, z) = 0$ for all $\mathbf{u}$.
3. The direction of maximum increase of f is given by $\nabla f(x, y, z)$. The maximum value of $D_{\mathbf{u}} f(x, y, z)$ is
$$\|\nabla f(x, y, z)\|. \qquad \text{Maximum value of } D_{\mathbf{u}} f(x, y, z)$$
4. The direction of minimum increase of f is given by $-\nabla f(x, y, z)$. The minimum value of $D_{\mathbf{u}} f(x, y, z)$ is
$$-\|\nabla f(x, y, z)\|. \qquad \text{Minimum value of } D_{\mathbf{u}} f(x, y, z)$$

Example 8 (Finding the gradient for a function of three variables)

Find $\nabla f(x, y, z)$ for the function given by

$$f(x, y, z) = x^2 + y^2 - 4z$$

and find the direction of maximum increase of f at the point $(2, -1, 1)$.

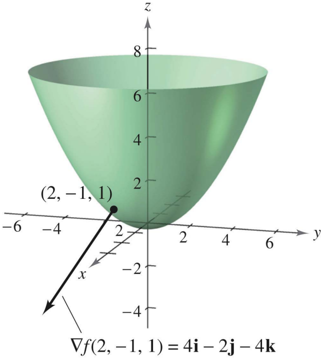


Figure 30: Level surface and gradient vector for $f(x, y, z) = x^2 + y^2 - 4z$.

Table of Contents

Tangent plane and normal line to a surface

- You can represent the surfaces in space primarily by equations of the form
$$z = f(x, y). \qquad \text{Equation of a surface } S$$
In the development to follow, however, it is convenient to use the more general representation $F(x, y, z) = 0$.
- For a surface S given by $z = f(x, y)$, you can convert to the general form by defining F as $F(x, y, z) = f(x, y) - z$.
- Because $f(x, y) - z = 0$, you can consider S to be the level surface of F given by
$$F(x, y, z) = 0. \qquad \text{Alternative equation of surface } S$$

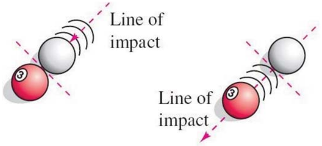
Example 1 (Writing an equation of a surface)

For the function given by

$$F(x, y, z) = x^2 + y^2 + z^2 - 4$$

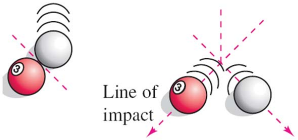
describe the level surface given by $F(x, y, z) = 0$.

- Normal lines are important in analyzing surfaces. For example, consider the collision of two billiard balls. When a stationary ball is struck at a point P on its surface, it moves along the line of impact determined by P and the center of the ball.
- The impact can occur in two ways. If the cue ball is moving along the **line of impact**, it stops dead and imparts all of its momentum to the stationary ball.



- If the cue ball is not moving along the line of impact, it is deflected to one side or the other and retains part of its momentum.

- That part of the momentum that is transferred to the stationary ball occurs along the line of impact, regardless of the direction of the cue ball. This line of impact is called the **normal line** to the surface of the ball at the point P.



- In the process of finding a normal line to a surface, you are also able to solve the problem of finding a **tangent plane** to the surface. Let S be a surface given by

$$F(x, y, z) = 0$$

and let $P(x_0, y_0, z_0)$ be a point on S.

- Let C be a curve on S through P that is defined by the vector-valued function

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}.$$

- Then, for all t,

$$F(x(t), y(t), z(t)) = 0.$$

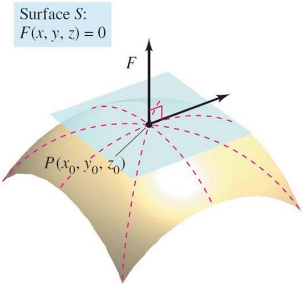
If F is differentiable and $x'(t)$, $y'(t)$, and $z'(t)$ all exist, it follows from the Chain Rule that

$$0 = F'(t) = F_x(x, y, z)x'(t) + F_y(x, y, z)y'(t) + F_z(x, y, z)z'(t).$$

- At (x_0, y_0, z_0), the equivalent vector form is

$$0 = \underbrace{\nabla F(x_0, y_0, z_0)}_{\text{Gradient}} \cdot \underbrace{\mathbf{r}'(t_0)}_{\text{Tangent vector}} .$$

- This result means that the gradient at P is orthogonal to the tangent vector of every curve on S through P. So, all tangent lines on S lie in a plane that is normal to $\nabla F(x_0, y_0, z_0)$ and contains P.



Definition 13.11 (Tangent plane and normal line)

Let F be differentiable at the point $P(x_0, y_0, z_0)$ on the surface S given by $F(x, y, z) = 0$ such that $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$.

1. The plane through P that is normal to $\nabla F(x_0, y_0, z_0)$ is called the tangent plane to S at P.
2. The line through P having the direction of $\nabla F(x_0, y_0, z_0)$ is called the normal line to S at P.

Theorem 13.13 (Equation of tangent plane)

If F is differentiable at (x_0, y_0, z_0), then an equation of the tangent plane to the surface given by $F(x, y, z) = 0$ at (x_0, y_0, z_0) is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

Example 2 (Finding an equation of a tangent plane)

Find an equation of the tangent plane to the hyperboloid given by

$$z^2 - 2x^2 - 2y^2 = 12$$

at the point $(1, -1, 4)$.

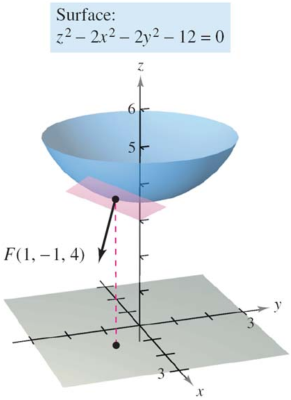


Figure 31: Tangent plane to surface: $z^2 - 2x^2 - 2y^2 - 12 = 0$.

- To find the equation of the tangent plane at a point on a surface given by $z = f(x, y)$, you can define the function F by

$$F(x, y, z) = f(x, y) - z.$$

- Then S is given by the level surface $F(x, y, z) = 0$, and by Theorem 13.13 an equation of the tangent plane to S at the point (x_0, y_0, z_0) is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

Example 3 (Finding an equation of the tangent plane)

Find the equation of the tangent plane to the paraboloid

$$z = 1 - \frac{1}{10}(x^2 + 4y^2)$$

at the point $\left(1, 1, \frac{1}{2}\right)$.

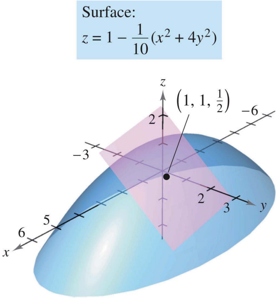


Figure 32: Finding an equation of the tangent plane.

Example 4 (Finding an equation of a normal line to a surface)

Find a set of symmetric equations for the normal line to the surface given by $xyz = 12$ at the point $(2, -2, -3)$.

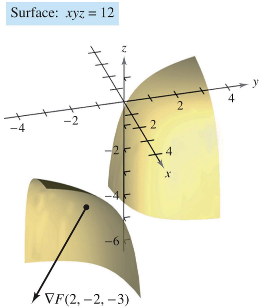


Figure 33: Finding an equation of a normal line to a surface.

Example 5 (Finding the equation of a tangent line to a curve)

Describe the tangent line to the curve of intersection of the surfaces

$$x^2 + 2y^2 + 2z^2 = 20 \qquad \text{Ellipsoid}$$
$$x^2 + y^2 + z = 4 \qquad \text{Paraboloid}$$

at the point $(0, 1, 3)$, as shown in Figure 34.

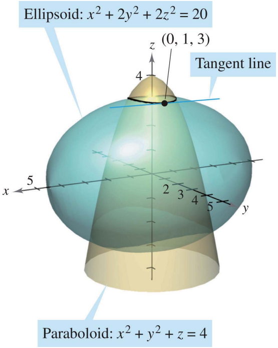


Figure 34: Finding the equation of a tangent line to a curve.

A comparison of the gradients $\nabla f(x, y)$ and $\nabla F(x, y, z)$

- This section concludes with a comparison of the gradients $\nabla f(x, y)$ and $\nabla F(x, y, z)$.
- You know that the gradient of a function f of two variables is normal to the level curves of f. Specifically, Theorem 13.12 states that if f is differentiable at (x_0, y_0) and $\nabla f(x_0, y_0) \neq \mathbf{0}$, then $\nabla f(x_0, y_0)$ is normal to the level curve through (x_0, y_0).
- Having developed normal lines to surfaces, you can now extend this result to a function of three variables.

Theorem 13.14 (Gradient is normal to level surfaces)

If F is differentiable at (x_0, y_0, z_0) and $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, then $\nabla F(x_0, y_0, z_0)$ is normal to the level surface through (x_0, y_0, z_0).

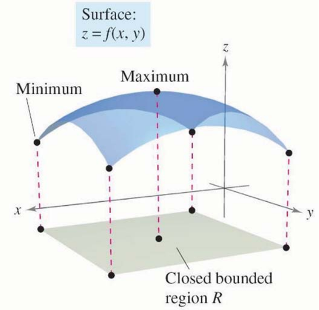
Table of Contents

Absolute extrema and relative extrema

- Consider the continuous function f of two variables, defined on a closed bounded region R. The values $f(a,b)$ and $f(c,d)$ such that

$$f(a,b) \leq f(x,y) \leq f(c,d) \qquad (a,b) \text{ and } (c,d) \text{ are in } R$$

for all (x,y) in R are called the minimum and maximum of f in the region R, as shown below.



- A region in the plane is closed if it contains all of its boundary points. The Extreme Value Theorem deals with a region in the plane that is both closed and bounded.
- A region in the plane is called bounded if it is a subregion of a closed disk in the plane.

Theorem 13.15 (Extreme Value Theorem)

Let f be a continuous function of two variables x and y defined on a closed bounded region R in the xy-plane.

1. *There is at least one point in R at which f takes on a minimum value.*
2. *There is at least one point in R at which f takes on a maximum value.*

- A minimum is also called an **absolute minimum** and a maximum is also called an **absolute maximum**. As in single-variable calculus, there is a distinction made between absolute extrema and **relative extrema**.

Definition 13.12 (Relative extrema)

Let f be a function defined on a region R containing (x_0, y_0).

1. The function f has a relative minimum at (x_0, y_0) if
$$f(x, y) \geq f(x_0, y_0)$$
for all (x, y) in an open disk containing (x_0, y_0).
2. The function f has a relative maximum at (x_0, y_0) if
$$f(x, y) \leq f(x_0, y_0)$$
for all (x, y) in an open disk containing (x_0, y_0).

- To say that f has a relative maximum at (x_0, y_0) means that the point (x_0, y_0, z_0) is at least as high as all nearby points on the graph of $z = f(x, y)$.
- Similarly, f has a relative minimum at (x_0, y_0) if (x_0, y_0, z_0) is at least as low as all nearby points on the graph. (See Figure 35.)
- To locate relative extrema of f, you can investigate the points at which the gradient of f is $\mathbf{0}$ or the points at which one of the partial derivatives does not exist. Such points are called **critical point** of f.

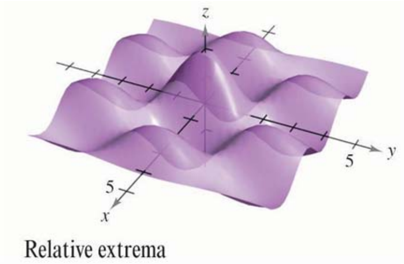


Figure 35: Relative extrema.

Definition 13.13 (Critical point)

Let f be defined on an open region R containing (x_0, y_0). The point (x_0, y_0) is a critical point of f if one of the following is true.

1. $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$
2. $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

- If f is differentiable and

$$\nabla f(x_0, y_0) = f_x(x_0 y_0)\,\mathbf{i} + f_y(x_0, y_0)\,\mathbf{j} = 0\,\mathbf{i} + 0\,\mathbf{j}$$

  then every directional derivative at (x_0, y_0) must be 0. This implies that the function has a horizontal tangent plane at the point (x_0, y_0), as shown in Figure 36.
- It appears that such a point is a likely location of a relative extremum. This is confirmed by Theorem 13.16.

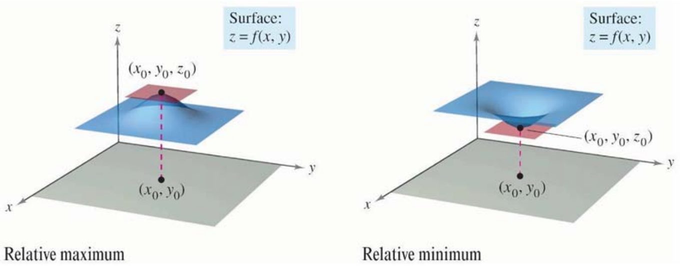


Figure 36: Relative extrema.

Theorem 13.16 (Relative extrema occur only at critical points)

If f has a relative extremum at (x_0, y_0) on an open region R, then (x_0, y_0) is a critical point of f.

Example 1 (Finding a relative extremum)

Determine the relative extrema of

$$f(x,y) = 2x^2 + y^2 + 8x - 6y + 20.$$

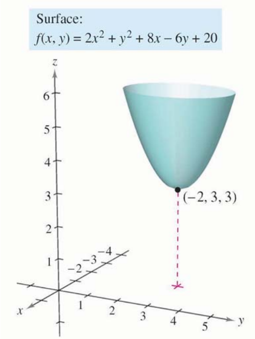


Figure 37: The function $z = f(x, y)$ has a relative minimum at $(-2, 3)$.

Example 2 (Finding a relative extremum)

Determine the relative extrema of $f(x, y) = 1 - (x^2 + y^2)^{1/3}$.

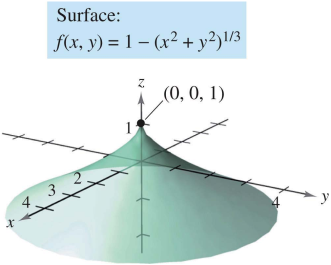


Figure 38: $f_x(x, y)$ and $f_y(x, y)$ are undefined at $(0, 0)$.

The second partials test

- To find relative extrema you need only examine values of $f(x, y)$ at critical points.
- However, as is true for a function of one variable, the critical points of a function of two variables do not always yield relative maxima or minima. Some critical points yield **saddle points**, which are neither relative maxima nor relative minima.
- As an example of a critical point that does not yield a relative extremum, consider the surface given by

$$f(x, y) = y^2 - x^2 \qquad \text{Hyperbolic paraboloid}$$

as shown in Figure 39. At the point $(0, 0)$, both partial derivatives are 0.

- The function f does not, however, have a relative extremum at this point because in any open disk centered at $(0, 0)$ the function takes on both negative values (along the x-axis) and positive values (along the y-axis).
- So, the point $(0, 0, 0)$ is a saddle point of the surface.

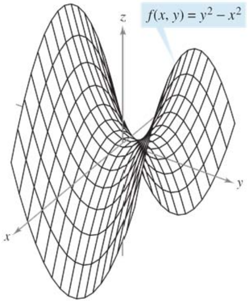


Figure 39: $f_x(0, 0) = f_y(0, 0) = 0$ where $f(x, y) = y^2 - x^2$.

Theorem 13.17 (Second Partials Test)

Let f have continuous second partial derivatives on an open region containing a point (a, b) for which

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0.$$

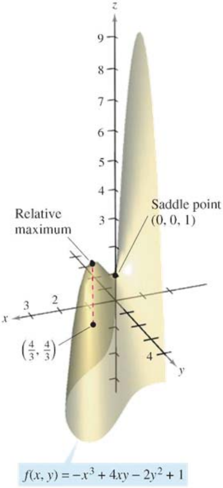
To test for relative extrema of f, consider the quantity $d = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$.

1. *If $d > 0$ and $f_{xx}(a, b) > 0$, then f has a relative minimum at (a, b).*
2. *If $d > 0$ and $f_{xx}(a, b) < 0$, then f has a relative maximum at (a, b).*
3. *If $d < 0$, then $(a, b, f(a, b))$ is a saddle point.*
4. *The test is inconclusive if $d = 0$.*

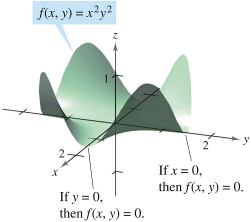
Example 3 (Using the Second Partials Test)

Find the relative extrema of

$$f(x, y) = -x^3 + 4xy - 2y^2 + 1.$$



(a) sing the Second Partials Test.



(b) Failure of the Second Partials Test.

Figure 40: Second Partials Test.

Example 4 (Failure of the Second Partials Test)

Find the relative extrema of $f(x, y) = x^2y^2$.

- The Second Partials Test can fail to find relative extrema in two ways. If either of the first partial derivatives does not exist, you cannot use the test. Also, if

$$d = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2 = 0$$

the test fails. In such cases, you can try a sketch or some other approach.
- Absolute extrema of a function can occur in two ways. First, some relative extrema also happen to be absolute extrema. For instance, in Example 1, $f(-2,3)$ is an absolute minimum of the function. (On the other hand, the relative maximum found in Example 3 is not an absolute maximum of the function.)
- Second, absolute extrema can occur at a boundary point of the domain as illustrated below.

Example 5 (Finding absolute extrema)

Find the absolute extrema of the function

$$f(x, y) = \sin xy$$

on the closed region given by $0 \leq x \leq \pi$ and $0 \leq y \leq 1$.

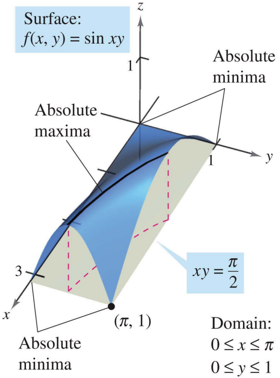


Figure 41: Finding absolute extrema.

Table of Contents

Lagrange multipliers

- Many optimization problems have restrictions, or **constraints**, on the values that can be used to produce the optimal solution. Such constraints tend to complicate optimization problems because the optimal solution can occur at a boundary point of the domain.
- In this section, you will study an ingenious technique for solving such problems. It is called the Method of Lagrange Multipliers.
- To see how this technique works, suppose you want to find the rectangle of maximum area that can be inscribed in the ellipse given by $\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1$. Let (x, y) be the vertex of the rectangle in the first quadrant, as shown in Figure 42.

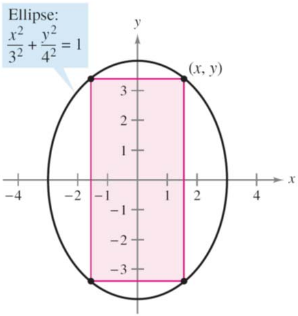


Figure 42: Objective function: $f(x, y) = 4xy$ and constrain: $\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1$.

- Because the rectangle has sides of lengths $2x$ and $2y$, its area is given by

$$f(x, y) = 4xy. \qquad \text{Objective function}$$

- You want to find x and y such that $f(x, y)$ is a maximum.

- Your choice of (x, y) is restricted to first-quadrant points that lie on the ellipse

$$\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1. \qquad \text{Constraint}$$

- Now, consider the constraint equation to be a fixed level curve of

$$g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2}.$$

  The level curves of f represent a family of hyperbolas $f(x, y) = 4xy = k$. In this family, the level curves that meet the given constraint correspond to the hyperbolas that intersect the ellipse.

- Moreover, to maximize $f(x, y)$, you want to find the hyperbola that just barely satisfies the constraint. The level curve that does this is the one that is tangent to the ellipse, as shown in Figure 43.

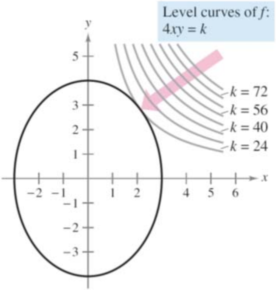


Figure 43: Level curves of f: $4xy = k$; Constraint: $g(x,y) = \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1$.

- If $\nabla f(x,y) = \lambda \nabla g(x,y)$, then scalar λ is called Lagrange Multiplier.

Theorem 13.18 (Lagrange's Theorem)

Let f and g have continuous first partial derivatives such that f has an extremum at a point (x_0, y_0) on the smooth constraint curve $g(x, y) = c$. If $\nabla g(x_0, y_0) \neq \mathbf{0}$, then there is a real number λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

Method of Lagrange Multipliers Let f and g satisfy the hypothesis of Lagrange's Theorem 13.18, and let f have a minimum or maximum subject to the constraint $g(x, y) = c$. To find the minimum or maximum of f, use the following steps.

1. Simultaneously solve the equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = c$ by solving the following system of equations.

$$f_x(x, y) = \lambda g_x(x, y) \qquad f_y(x, y) = \lambda g_y(x, y) \qquad g(x, y) = c$$

2. Evaluate f at each solution point obtained in the first step. The largest value yields the maximum of f subject to the constraint $g(x, y) = c$, and the smallest value yields the minimum of f subject to the constraint $g(x, y) = c$.

Alternative: Let $F(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - c)$. Then solve the free-constrained optimization problem for F.

Constrained optimization problems

Example 1 (Using a Lagrange Multiplier with one constraint)

Find the maximum value of $f(x, y) = 4xy$ where $x > 0$ and $y > 0$, subject to the constraint $(x^2/3^2) + (y^2/4^2) = 1$.

Example 3 (Lagrange multipliers and three variables)

Find the minimum value of

$$f(x, y, z) = 2x^2 + y^2 + 3z^2 \qquad \text{Objective function}$$

subject to the constraint $2x - 3y - 4z = 49$.

Example 4 (Optimization inside a region)

Find the extreme values of

$$f(x, y) = x^2 + 2y^2 - 2x + 3 \qquad \text{Objective function}$$

subject to the constraint $x^2 + y^2 \leq 10$.

The method of Lagrange multipliers with two constraints

- For optimization problems involving two constraint functions g and h, you can introduce a second Lagrange multiplier, μ (the lowercase Greek letter mu), and then solve the equation

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

where the gradient vectors are not parallel, as illustrated in Example 5.

Example 5 (Optimization with two constraints)

Let $T(x, y, z) = 20 + 2x + 2y + z^2$ represent the temperature at each point on the sphere $x^2 + y^2 + z^2 = 11$. Find the extreme temperatures on the curve formed by the intersection of the plane $x + y + z = 3$ and the sphere.