1. (20%) Find the following limit. (If the limit does not exist or has an infinite limit, you should point it out.)

   (a) $\lim_{(x,y)\to(0,0)} \arccos(\frac{x^3+y^3}{x^2+y^2})$

   (b) $\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$

   (c) $\lim_{(x,y,z)\to(0,0,0)} \frac{e^{xyz}-1}{x^2+y^2+z^2}$

   (d) $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$

Ans:

(a) Let $x = rcos(\theta), y = rsin(\theta)$ $\underset{(x,y)\to(0,0)}{lim} \arccos(\frac{x^3+y^3}{x^2+y^2}) =$

$\underset{r\to 0}{lim} \arccos(\frac{r^3(cos^3\theta+sin^3\theta)}{r^2}) = \frac{\pi}{2}$

(b) Let $y = mx^{\frac{1}{3}}$, $\underset{x\to 0}{lim}\frac{xm^3x}{x^2-m^6x^2} = \underset{x\to 0}{lim}\frac{m^3}{1+m^6} = \frac{m^3}{1+m^6}$. which means that if we follow the trajectory of different line $y = mx^{\frac{1}{3}}$ to approach (0,0) we will get different value for different m, therefore, the limit does not exist.

(c) Let $x = \rho\, sin(\Phi)\cos(\theta)\, , y = \rho\, sin(\Phi)\, sin(\theta)\, , z = \rho\, cos(\Phi)$

$$\lim_{(x,y,z)\to(0,0,0)} \frac{e^{xyz}-1}{x^2+y^2+z^2} = \lim_{\rho^+\to 0}\left(\frac{e^{\rho^3 sin^2(\Phi)\cos(\theta)\, sin(\theta)\, cos(\Phi)}-1}{\rho^2}\right) =$$

(L'Hôpital's rule) $\lim_{\rho^+\to 0}\left(\frac{e^{\rho^3\mu}3\rho^2\mu}{2\rho}\right) = \lim_{\rho^+\to 0}\rho\left(\frac{e^{\rho^3\mu}3\mu}{2}\right) = 0$

Where $\mu = sin^2(\Phi)cos(\theta)\, sin(\theta)\, cos(\Phi)$.

(d) Since $\left|\frac{xy^2}{x^2+y^2}\right| = \left|\frac{y^2}{x^2+y^2}\right||x| \le |x|$. Therefore, $0 \le \left|\frac{xy^2}{x^2+y^2}\right| \le |x|$. Furthermore, we know that $\lim_{(x,y)\to(0,0)}|x| = 0$. By the squeeze theorem, $\lim_{(x,y)\to(0,0)}\left|\frac{xy^2}{x^2+y^2}\right| = 0$. It follows that $\lim_{(x,y)\to(0,0)}\frac{xy^2}{x^2+y^2} = 0$.

2. (12%)

(a) Let $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{when } (x,y) \neq (0,0) \\ 0 & \text{when } (x,y) = (0,0) \end{cases}$, evaluate $f_x(0,0)$ and $f_{xy}(0,0)$

(b) Given the equation $\mathrm{w} - \sqrt{x-y} - \sqrt{y-z} = 0$, differentiate implicitly to find the three first partial derivatives of w $(\frac{\partial \mathrm{w}}{\partial \mathrm{x}}, \frac{\partial \mathrm{w}}{\partial \mathrm{y}}, \frac{\partial \mathrm{w}}{\partial \mathrm{z}})$

(c) Find a set of parametric equations for the tangent line to the curve of intersection of the surface $x^2 + y^2 + z^2 = 4$ and $(x-1)^2 + y^2 = 1$ at the point $(1, 1, \sqrt{2})$.

Ans:

(a) $f_x(0,0) = \lim\limits_{\Delta x \to 0} \frac{f(0+\Delta x, 0) - f(0,0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{\frac{\Delta x \times 0}{\Delta x^2} - 0}{\Delta x} = 0$

$f_x(x,y) = \frac{-x^2y+y^3}{(x^2+y^2)^2}$ when $(x,y) \neq (0,0)$

$$f_{xy}(0,0) = \lim_{\Delta y \to 0} \frac{f_x(0,0+\Delta y) - f_x(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\frac{(\Delta y)^3}{(\Delta y)^4} - 0}{\Delta y} = \lim_{\Delta y \to 0} \frac{1}{(\Delta y)^2} = \infty$$

(b) $F(x,y,z,w) = w - \sqrt{x-y} - \sqrt{y-z} = 0$

$$\frac{\partial \mathrm{w}}{\partial \mathrm{x}} = \frac{-F_x}{F_w} = \frac{1}{2}\frac{(x-y)^{\frac{-1}{2}}}{1} = \frac{1}{2\sqrt{x-y}}$$

$$\frac{\partial \mathrm{w}}{\partial \mathrm{y}} = \frac{-F_y}{F_w} = \frac{-1}{2}(x-y)^{\frac{-1}{2}} + \frac{1}{2}(y-z)^{\frac{-1}{2}} = \frac{-1}{2\sqrt{x-y}} + \frac{1}{2\sqrt{y-z}}$$

$$\frac{\partial \mathrm{w}}{\partial \mathrm{z}} = \frac{-F_z}{F_w} = \frac{-1}{2\sqrt{y-z}}$$

(c) Begin by finding the gradients to both surfaces at $(1, 1, \sqrt{2})$

Let $F = x^2 + y^2 + z^2 - 4, G = (x-1)^2 + y^2 - 1$

$\nabla F = 2x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k},\ \nabla F(1,1,\sqrt{2}) = 2\boldsymbol{i} + 2\boldsymbol{j} + 2\sqrt{2}\boldsymbol{k}$

$\nabla G = (2x-2)\boldsymbol{i} + 2y\boldsymbol{j},\ \nabla G(1,1,\sqrt{2}) = 2\boldsymbol{j}$

The cross product of these two gradients is a vector that is tangent to both surfaces at $(1, 1, \sqrt{2})$

$$\nabla F \times \nabla G = -4\sqrt{2}\boldsymbol{i} + 4\boldsymbol{k}$$

So the parametric equation can be written as:

$$x = -\sqrt{2}t + 1, y = 1, z = t + \sqrt{2}$$

3. (10%) Given $f(x,y) = y^2 + \sin(xy)$. Find the directions at the point (0,1) where the directional derivative of $f(x,y)$ in that direction is 1. Express your result as unit vector.

Ans:

$$\nabla f = y\cos(xy)\boldsymbol{i} + \big(2y + x\cos(xy)\big)\boldsymbol{j}$$
$$\nabla f(0,1) = \boldsymbol{i} + 2\boldsymbol{j}$$

Let the unit vector be $u = p\boldsymbol{i} + q\boldsymbol{j}$ where $p^2 + q^2 = 1$

$$D_u f = \nabla f \cdot (p,q) = p + 2q = 1$$

Solve the above equations we get $p = 1, q = 0$ or $p = \frac{-3}{5}, q = \frac{4}{5}$

So $u = \boldsymbol{i}$ or $\mathrm{u} = \frac{-3}{5}\boldsymbol{i} + \frac{4}{5}\boldsymbol{j}$

4. (15%) Let $f(x,y) = x^4 - 2x^2 - 2xy^2 - y^2$
   (a) Find the critical points of $f(x,y)$
   (b) Determine whether they are local maximum, local minimum or saddle points

Ans:

(a) $f_x = 4x^3 - 4x - 2y^2 = 4(x^3 - x) - 2y^2$, $f_y = -(4x+2)y$.
Let $f_x = 0$ and $f_y = 0$,

From $f_y = 0$ we know $x = \frac{-1}{2}$ or $y = 0$. If $x = \frac{-1}{2}, y = \pm\frac{\sqrt{3}}{2}$. When $y = 0, x = 0, \pm 1$

Therefore, the critical points are $(0,0), (1,0), (-1,0), \left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right), \left(\frac{-1}{2}, \frac{-\sqrt{3}}{2}\right)$

(b)
Since $f_{xx} = 4(3x^2 - 1), f_{xy} = f_{yx} = -4y, f_{yy} = -(4x+2)$.

(x,y)	f_{xx}	f_{xy}	f_{yy}	d	
(0,0)	-4	0	-2	8	Local maximum
(1,0)	8	0	-6	-48	Saddle point
$(-1,0)$	8	0	2	16	Local minimum
$\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$	-1	-2$\sqrt{3}$	0	-12	Saddle point
$\left(\frac{-1}{2}, \frac{-\sqrt{3}}{2}\right)$	-1	2$\sqrt{3}$	0	-12	Saddle point

5. (15%) Evaluate the following expression

(a) $\int_0^1 \int_{\sqrt{x}}^1 e^{y^3}\, dydx$

(b) $\int_1^3 \int_0^x \frac{1}{\sqrt{x^2+y^2}} dydx$

(c) $\int_0^{\frac{\pi}{4}} \int_0^6 \int_0^{6-r} rz\ \ dzdrd\theta$

Ans:

(a) $\int_0^1 \int_{\sqrt{x}}^1 e^{y^3}\, dydx = \int_0^1 \int_0^{y^2} e^{y^3}\, dxdy = \int_0^1 \left[xe^{y^3}\right]_0^{y^2} dy = \int_0^1 y^2 e^{y^3}\, dy =$

$\frac{1}{3} e^{y^3} \Big|_0^1 = \frac{1}{3}(e-1)$

(b) $R = \{(x,y) | 1 \le x \le 3, 0 \le y \le x\} = \left\{(r,\theta) \middle| \frac{1}{\cos(\theta)} \le r \le \frac{3}{\cos(\theta)}, 0 \le \theta \le \frac{\pi}{4}\right\}$

$$\int_1^3 \int_0^x \frac{1}{\sqrt{x^2+y^2}} dydx = \int_0^{\frac{\pi}{4}} \int_{\frac{1}{\cos(\theta)}}^{\frac{3}{\cos(\theta)}} \frac{1}{r} r\, drd\theta = \int_0^{\frac{\pi}{4}} \frac{3}{\cos(\theta)} - \frac{1}{\cos(\theta)} d\theta$$

$$= \int_0^{\frac{\pi}{4}} 2\sec(\theta)\, d\theta = 2\ln|\sec(\theta) + \tan(\theta)| \Big|_0^{\frac{\pi}{4}} = 2\ln(\sqrt{2}+1)$$

(c) $\int_0^{\frac{\pi}{4}} \int_0^6 \int_0^{6-r} rz\ \ dzdrd\theta = \int_0^{\frac{\pi}{4}} \int_0^6 \frac{rz^2}{2} \Big|_0^{6-r} drd\theta = \int_0^{\frac{\pi}{4}} \int_0^6 \frac{1}{2}(r^3 - 12r^2 +$

$36r)\, drd\theta = \int_0^{\frac{\pi}{4}} \frac{1}{2}\left[\frac{r^4}{4} - 4r^3 + 18r^2\right]_0^6 d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{2}(108) d\theta = \frac{27\pi}{2}$

6. (10%) Find the area of the surface given by $z = f(x,y) = xy$ that lies above the region R where $R = \{(x,y): x^2 + y^2 \le 9\}$

Ans:

$$f_x = y, f_y = x$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + x^2 + y^2}$$

$$S = \int_0^{2\pi} \int_0^3 \sqrt{1+r^2}\, rdrd\theta = \frac{1}{2} \int_0^{2\pi} \int_1^{10} \sqrt{u}\, dud\theta = \frac{1}{3} \int_0^{2\pi} (10\sqrt{10} - 1) d\theta$$

$$= \frac{2\pi}{3}(10\sqrt{10} - 1)$$

7. (15%) Evaluate the triple integral $\int\int\int_Q x^2+y^2 dV$ where $Q=\{-1\leq \mathrm{x}\leq 1, -\sqrt{1-x^2}\leq \mathrm{y}\leq \sqrt{1-x^2}, \sqrt{x^2+y^2}\leq \mathrm{z}\leq 1\}$

Ans:

Use cylindrical coordinates

$$\int\int\int_Q x^2+y^2\ dV=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1} x^2+y^2\ dzdydx$$

$$=\int_0^{2\pi}\int_0^1\int_r^1 r^2 r\ dzdrd\theta=2\pi\int_0^1 r^3(1-r)\ dr=\frac{1}{10}\pi$$