If the limit does not exist or has an infinite limit, you should point it out. In addition, do not use the L'Hôpital's rule to solve the limit problem.

1. (20%) Determine the following limit

   (a) $\lim_{x\to 3}\frac{1}{x-3}\left(\frac{1}{x-1}+\frac{1}{x-5}\right)$

   (b) $\lim_{x\to 2}\frac{\sqrt{x+2}-2}{x-2}$

   (c) $\lim_{x\to -3}\frac{|x+3|}{x^2+x-6}$

   (d) $\lim_{x\to -\infty}\frac{4x^2}{2x^2+x+cosx}$

   (e) $\lim_{x\to 0^+}\sin(x)\sin(\frac{2}{x^2})$

Ans:

(a) $\lim_{x\to 3}\frac{1}{x-3}\left(\frac{1}{x-1}+\frac{1}{x-5}\right)=\lim_{x\to 3}\frac{1}{x-3}\left(\frac{x-5+x-1}{(x-1)(x-5)}\right)=\lim_{x\to 3}\frac{1}{x-3}\left(\frac{2(x-3)}{(x-1)(x-5)}\right)=$

$\lim_{x\to 3}\frac{2}{(x-1)(x-5)}=-\frac{1}{2}$

(b)

$$\lim_{x\to 2}\left(\frac{\sqrt{x+2}-2}{x-2}\right)=\lim_{x\to 2}\frac{(\sqrt{x+2}-2)(\sqrt{x+2}+2)}{(x-2)(\sqrt{x+2}+2)}=\lim_{x\to 2}\frac{x-2}{(x-2)(\sqrt{x+2}+2)}$$
$$=\lim_{x\to 2}\frac{1}{(\sqrt{x+2}+2)}=\frac{1}{4}$$

(c) $\lim_{x\to -3^+}\frac{|x+3|}{x^2+x-6}=\lim_{x\to -3^+}\frac{x+3}{(x+3)(x-2)}=\lim_{x\to -3^+}\frac{1}{(x-2)}=\frac{-1}{5}$

$$\lim_{x\to -3^-}\frac{|x+3|}{x^2+x-6}=\lim_{x\to -3^-}\frac{-(x+3)}{(x+3)(x-2)}=\lim_{x\to -3^-}\frac{-1}{(x-2)}=\frac{1}{5}$$

Therefore, the limit does not exist.

(d) $\lim_{x\to -\infty}\frac{4x^2}{2x^2+x+cosx}=\lim_{x\to \infty}\frac{4}{2+\frac{1}{x}+\frac{\cos(x)}{x^2}}=2$

(e) To find the limit when approach 0, we consider the interval $(-\pi,\pi)$

For any $x>0,-1\le\sin(\frac{2}{x^2})\le 1\Rightarrow -\sin(x)\le\sin(x)\sin(\frac{2}{x^2})\le\sin(x)$，

In addition, $\lim_{x\to 0^+}-\sin(x)=0=\lim_{x\to 0^+}\sin(x)$

According to Squeeze theorem， $\lim_{x\to 0^+}\sin(x)\sin(\frac{2}{x^2})=0$

2. (8%) Assume $f(x)=\begin{cases} x^4, & x\le 1 \\ ax+b, & x>1 \end{cases}$ is a differentiable function, what is the value of a and b?

Ans:

Since differentiable implies continuous, we have $f(x)=1=a+b$.

Since the function is differentiable, considering the alternative form of derivative:

$$\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^+}\frac{x^4-1}{x-1}=\lim_{x\to 1^+}\frac{(x^2+1)(x+1)(x-1)}{x-1}$$

$$=\lim_{x\to 1^+}(x^2+1)(x+1)=4$$

$$\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^-}\frac{ax+b-(a+b)}{x-1}=\lim_{x\to 1^-}\frac{a(x-1)}{x-1}=a$$

We get $a=4$ and $4+\mathrm{b}=1\rightarrow b=-3$.

3. (8%) For each of the following functions, first determine whether the Mean Value Theorem can be applied on the given closed interval $[a,b]$. If applicable, find all values of c in the open interval (a,b) that satisfy the conclusion of the Mean Value Theorem.
   (a) (4%) $f(x)=|4-x|$ on the interval [3,5]
   (b) (4%) $f(x)=x-cos(x)$ on the interval $[\frac{-\pi}{2},\frac{\pi}{2}]$

Ans:

(a) The Mean Value Theorem cannot be applied. f is not differentiable at $x=4$ in [3,5].

(b) The function f are continuous everywhere on R, including the closed interval $[\frac{-\pi}{2},\frac{\pi}{2}]$ and differentiable everywhere on R, including the open interval $[\frac{-\pi}{2},\frac{\pi}{2}]$.

   Therefore, the hypotheses of the Mean Value Theorem are satisfied.

$$f(\frac{-\pi}{2})=\frac{-\pi}{2},f(\frac{\pi}{2})=\frac{\pi}{2}$$

In addition, $f'(x)=1+\sin(x)$

By MVT, we have $f'(c)=\frac{f\left(\frac{\pi}{2}\right)-f(\frac{-\pi}{2})}{\frac{\pi}{2}-(\frac{-\pi}{2})}=1\rightarrow 1+\sin(c)=1\rightarrow c=0$

4. (15%) Remember that you can solve the derivative using the definition or the differentiation rule for the following question.

(a) (5%) Given $f(x) = \cos(x) - 3\tan(x)$ find $f'(x)$ and $f''(x)$

(b) (5%) Given $f(x) = \frac{x(x-1)(x-2)(x-3)}{(x+1)(x+2)(x+3)}$, what is the value of $f'(1)$?

(c) (5%) Let $(x+y)^3 = x^3 + y^3$, find $\frac{dy}{dx}$ by implicit differentiation. Then find the slope of the graph at $(-1,1)$.

Ans:

(a) $f'(x) = -\sin(x) - 3sec^2(x) \to f''(x) = -\cos(x) - 6\sec(x)\sec(x)\tan(x) = -\cos(x) - 6sec^2(x)\tan(x)$

(b) $f'(1) = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{\frac{x(x-1)(x-2)(x-3)}{(x+1)(x+2)(x+3)} - 0}{x-1} = \lim_{x\to 1} \frac{x(x-2)(x-3)}{(x+1)(x+2)(x+3)} = \frac{1}{12}$

(c) $(x+y)^3 = x^3 + y^3 \to 3(x+y)^2\left(1+\frac{dy}{dx}\right) = 3x^2 + 3y^2\frac{dy}{dx} \to (x^2 + 2xy + y^2)\left(1+\frac{dy}{dx}\right) = x^2 + y^2\frac{dy}{dx} \to (x^2 + 2xy + y^2 - y^2)\frac{dy}{dx} = x^2 - (x^2 + 2xy + y^2) \to \frac{dy}{dx} = \frac{-y(2x+y)}{x(x+2y)}$.

The slope at $(-1,1)$ is thus $\left.\frac{dy}{dx}\right]_{(-1,1)} = -1$.

5. (27%) Let $f(x) = \frac{x^3}{x^2-1}$

(a) (5%) Find the critical numbers and the possible points of inflection of $f(x)$

(b) (4%) Find the open intervals on which f is increasing or decreasing

(c) (4%) Find the open intervals of concavity

(d) (5%) Find all the asymptotes (Vertical/horizontal/Slant)

(e) (7%) Sketch the graph of $f(x)$ (Label any intercepts, relative extrema, points of inflection,and asymptotes)

(f) (2%) What is the domain and range of $f(x)$?

Ans:

(a) $\frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}$

Note that x is not define at $x = \pm 1$, we should not include it in the critical numbers or possible points of inflection

$f'(x) = 1 - \frac{x^2+1}{(x^2-1)^2} = \frac{x^2(x^2-3)}{(x^2-1)^2}$, $f''(x) = \frac{2x(x^2+3)}{(x^2-1)^3}$

The critical numbers are $x = \pm\sqrt{3}, 0$ **(**$f' = 0$**)**

Possible points of inflection: $x = 0$ $(f'' = 0)$

(b)

	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \sqrt{3})$	$(\sqrt{3}, \infty)$
測試值	-2	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	2
f'的正負號	+	-	-	-	-	+
f''的正負號	-	-	+	-	+	+
結論	遞增/向下凹	遞減/向下凹	遞減/向上凹	遞減/向下凹	遞減/向上凹	遞增/向上凹

Increasing on $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, \infty)$ since $f'(x) > 0$, Decreasing on $\left(-\sqrt{3}, -1\right), (-1, 1)$ and $\left(1, \sqrt{3}\right)$ since $f'(x) < 0$.

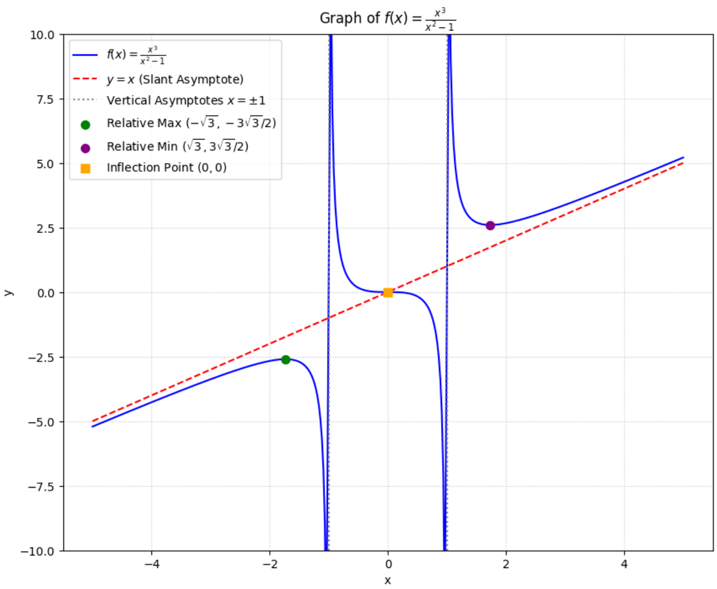
(c) f is concave downward on $(-\infty, -1)$ and $(0, 1)$ since $f''(x) < 0$, f is concave upward on $(-1, 0)$ and $(1, \infty)$ since $f''(x) > 0$.

(d) $x = \pm 1$ are the vertical asymptote. since $\lim_{x \to \pm 1} f(x) = \pm\infty$. A vertical asymptote occurs where the denominator is zero.

There is no horizontal asymptote since $\lim_{x \to \infty} f(x) = \infty$ and $\lim_{x \to -\infty} f(x) = -\infty$.

$\frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}$, therefore $y = x$ is the slant asymptote.

(e)



(f) Domain are all real number except $x = \pm 1$ and range is all real number.

6. (9%)
   (a) (3%) Find the equation of the tangent line to the curve $y = 3x - x^2$ at $x = a$.
   (b) (2%) Determine the x and y intercept of the tangent line found in (a)
   (c) (4%) Find the minimum area of the triangle formed by the tangent line in (a), along with the x-axis and y-axis, restricted to the first quadrant.

Ans:

(a) $y' = 3 - 2x$. The tangent line of $y = 3x - x^2$ at $x = a$ is $y - (3a - a^2) = (3 - 2a)(x - a)$

(b) The x intercept is obtain by setting y equals to 0, we have $0 - (3a - a^2) = (3 - 2a)(x - a) \to x = \frac{a^2}{2a-3}$. The y intercept is obtain by setting x equals to 0, we have $y - (3a - a^2) = (3 - 2a)(0 - a) \to x = a^2$

(c) The area of the triangle is given by $A = \frac{a^4}{2(2a-3)}$.

$$A' = \frac{3a^3(a-2)}{(2a-3)^2} = 0 \to a = 2 \text{ or } a = 0.$$

$a = 2$ is the feasible solution, using the first derivative test we can confirm it is a local minimum. Therefore, we have $A(2) = 8$ which is the smallest triangle.

7. (5%) Use Newton's Method with the initial approximation $x_1 = 1$ to find x_3, the third approximation to the solution of the equation $x^3 - 3x^2 + 3 = 0$

Ans:

Newton's Method iteratively improves an estimate x_n of a root of a function $f(x)$ using the formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f'(x) = 3x^2 - 6x$$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1	1	-3	$\frac{-1}{3}$	$\frac{4}{3}$
2	$\frac{4}{3}$	$\frac{1}{27}$	$\frac{-8}{3}$	$\frac{-1}{72}$	$\frac{97}{72}$
3	$\frac{97}{72}$				

8. (8%) Use differential to approximate $(3.02)^5$.

Ans: 3.02 is $3 + 0.02$. Let $f(x) = \mathrm{x}^5, f'(x) = 5x^4$

$$f(x + \Delta x) \approx f(x) + f'(x)dx = \mathrm{x}^5 + 5x^4 dx$$

Choosing $x = 3$ and $dx = 0.2$

$$(3.02)^5 = f(x + \Delta x) \approx 243 + 5 \times 81(0.02) = 243 + 5 \times 1.62 = 251.1$$