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Chapter **7**

APPLICATIONS OF INTEGRATION

7.1 Summary

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1. Area of a region between two curves If f and g are continuous on $[a, b]$ and $g(x) \leq f(x)$ for all x in $[a, b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$

and $x = b$ is

$$A = \int_a^b [f(x) - g(x)] dx.$$

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2. Integration as an accumulation process

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Representative element

\implies

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3. The Disk Method (圓盤法) To find the volume of a solid of revolution with the Disk Method, use one of the following, as shown in Figure 7.15.

Horizontal axis of revolution

$$\text{Volume} = V =$$

$$\pi \int_a^b [R(x)]^2 dx$$

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Vertical axis of revolution

$$\text{Volume} = V =$$

$$\pi \int_c^d [R(y)]^2 dy$$

4. **Washer Method** (**墊圈法**) Consider a region bounded by an outer radius $R(x)$ and an inner radius $r(x)$. If the region is revolved about its axis of revolution, the volume of the resulting solid is given by

$$V = \pi \int_a^b ([R(x)]^2 - [r(x)]^2) dx.$$

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5. **Volumes of solids with known cross sections**

(a) For cross sections of area $A(x)$ taken perpendicular to the x -axis,

$$\text{Volume} = \int_a^b A(x) \, dx. \quad \text{See Figure 7.24a}$$

(b) For cross sections of area $A(y)$ taken perpendicular to the y -axis,

$$\text{Volume} = \int_c^d A(y) \, dy. \quad \text{See Figure 7.24b}$$

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6. **The Shell Method** (柱殼法) To find the volume of a solid of revolution with the Shell Method, use one of the following, as show in

Figure 7.29.

Horizontal axis revolution

Vertical axis of revolution

$$\text{Volume} = V = 2\pi \int_c^d p(y)h(y) dy \quad \text{Volume} = V = 2\pi \int_a^b p(x)h(x) dx$$

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7. **Arc length** Let the function given by $y = f(x)$ represent a smooth curve on the interval $[a, b]$. The **arc length** (弧長) of f between a and b is

$$s = \int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

Similarly, for a smooth curve given by $x = g(y)$, the **arc length** (弧長)

of g between c and d is

$$s = \int_c^d \sqrt{1 + [g'(y)]^2} dy.$$

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8. **Surface of revolution** If the graph of a continuous function is revolved about a line, the resulting surface is a **surface of revolution** (**旋轉面**). 86

9. **Area of a surface of revolution** Let $y = f(x)$ have a continuous derivative on the interval $[a, b]$. The area S of the surface of revolution formed by revolving the graph of f about a horizontal or vertical axis is

$$S = 2\pi \int_a^b r(x) \sqrt{1 + [f'(x)]^2} dx \quad \mathbf{y \text{ is a function of } x}$$

where $r(x)$ is the distance between the graph of f and the axis of revolution. If $x = g(y)$ on the interval $[c, d]$, then the surface area is

$$S = 2\pi \int_c^d r(y) \sqrt{1 + [g'(y)]^2} dy \quad x \text{ is a function of } y$$

where $r(y)$ is the distance between the graph of g and the axis of revolution. 94

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10. **Work done by a constant force** If an object is moved a distance D in the direction of an applied constant force F , then the **work** (**功**) W done by the force is defined as $W = FD$ 99

11. **Work by a variable force** If an object is moved along a straight line by a continuously varying force $F(x)$, then the work W done by the

force as the object is moved from $x = a$ to $x = b$ is

$$W = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \Delta W_i = \int_a^b F(x) dx.$$

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Section 7.6 Moments, centers of mass, and centroids 118

12. Moments and center of mass: one-dimensional system

Let the point masses m_1, m_2, \dots, m_n be located at x_1, x_2, \dots, x_n .

(a) The **moment about the origin** (**對原點的力矩**) is $M_0 = m_1x_1 + m_2x_2 + \dots + m_nx_n$.

(b) The **center of mass** (**質心**) is $\bar{x} = \frac{M_0}{m}$, where $m = m_1 + m_2 + \dots + m_n$ is the **total mass** (**總質量**) of the system.

13. Moment and center of mass: two-dimensional system

Let the point masses m_1, m_2, \dots, m_n be located at $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$.

(a) The moment about the y -axis (對 y 軸的力矩) is $M_y = m_1x_1 + m_2x_2 + \dots + m_nx_n$.

(b) The moment about the x -axis (對 x 軸的力矩) is $M_x = m_1y_1 + m_2y_2 + \dots + m_ny_n$.

(c) The center of mass (重心) (\bar{x}, \bar{y}) (or center of gravity) is

$$\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m}$$

where $m = m_1 + m_2 + \dots + m_n$ is the total mass (總質量) of the system.

14. Moments and center of mass of a planar lamina

Let f and g be continuous functions such that $f(x) \geq g(x)$ on $[a, b]$, and consider the planar lamina of uniform density ρ bounded by the graphs of $y = f(x)$, $y = g(x)$, and $a \leq x \leq b$.

(a) The **moments about the x - and y -axes** are

$$M_x = \rho \int_a^b \left[\frac{f(x) + g(x)}{2} \right] [f(x) - g(x)] dx$$

$$M_y = \rho \int_a^b x (f(x) - g(x)) dx.$$

(b) The **center of mass** (質心) (\bar{x}, \bar{y}) is given by $\bar{x} = \frac{M_y}{m}$ and $\bar{y} = \frac{M_x}{m}$,

where $m = \rho \int_a^b [f(x) - g(x)] dx$ is the mass of the lamina.

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15. **The Theorem of Pappus** (帕卜定理) Let R be a region in a plane and let L be a line in the same plane such that L does not intersect the interior of R , as show in Figure 7.66. If r is the distance between the centroid of R and the line, then the volume V of the solid of revolution formed by revolving R about the line is
- $$V = 2\pi r A$$
- where A is the area of R . (Note that $2\pi r$ is the distance traveled by the centroid as the region is revolved about the line.) 149
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16. **Fluid pressure** The **pressure** (壓力) on an object at depth h

in a liquid is

$$\text{Pressure} = P = wh$$

where w is the weight-density of the liquid per unit of volume. 154

17. Force exerted by a fluid

The force F exerted by a fluid (流體施加力) of constant weight-density w (per unit of volume) against a submerged vertical plane region from $y = c$ to $y = d$ is

$$F = w \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n h(y_i) L(y_i) \Delta y = w \int_c^d h(y) L(y) dy$$

where $h(y)$ is the depth of the fluid at y and $L(y)$ is the horizontal length of the region at y 159

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