Chapter 7 Applications of Integration

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Table of Contents

Table of Contents

Area of a region between two curves

- With a few modifications, you can extend the application of definite integrals from the area under a curve to the area between two curves.
- Consider two functions f and g that are continuous on the interval $[a, b]$.

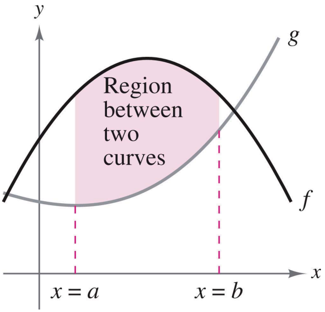


Figure 1: Area of a region between two curves.

- If, the graphs of both f and g lie above the x-axis, and the graph of g lies below the graph of f, you can geometrically interpret the area between the graphs as the area under the graph of g subtracted from the area under the graph of f, as shown in Figure 2.

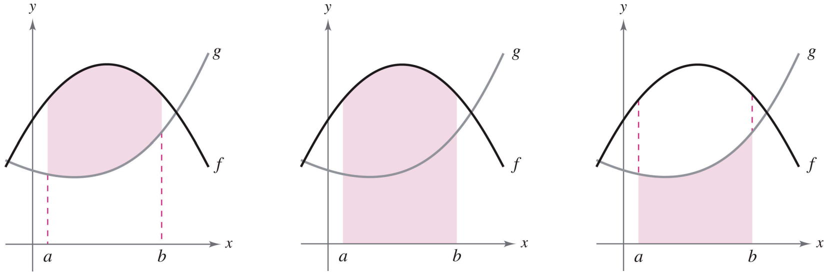


Figure 2: $\int_a^b [f(x) - g(x)]\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x - \int_a^b g(x)\,\mathrm{d}x$

- To verify the reasonableness of the result shown in Figure 2, you can partition the interval $[a, b]$ into n subintervals, each of width Δx.
- Then, as shown in Figure 3, sketch a representative rectangle of width Δx and height $f(x_i) - g(x_i)$, where x_i is in the ith subinterval.

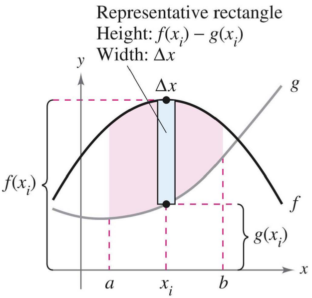


Figure 3: Representative rectangle. Height: $f(x_i) - g(x_i)$; Width: Δx.

- The area of this representative rectangle is

$$\Delta A_i = (\text{height})(\text{width}) = [f(x_i) - g(x_i)]\Delta x.$$

- By adding the areas of the n rectangles and taking the limit as $\|\Delta\| \to 0$ ($n \to \infty$), we obtain

$$\lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\Delta x.$$

- Because f and g are continuous on $[a, b]$, $f - g$ is also continuous on $[a, b]$ and the limit exists. So, the area of the given region is

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\Delta x = \int_a^b [f(x) - g(x)]\, \mathrm{d}x.$$

Area of a region between two curves

If f and g are continuous on $[a, b]$ and $g(x) \leq f(x)$ for all x in $[a, b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\,\mathrm{d}x.$$

- In Figure 1, the graphs of f and g are shown above the x-axis. This, however, is not necessary.
- The same integrand $[f(x) - g(x)]$ can be used as long as f and g are continuous and $g(x) \leq f(x)$ for all x in the interval $[a, b]$.
- Notice in Figure 4 that the height of a representative rectangle is $f(x) - g(x)$ regardless of the relative position of the x-axis.

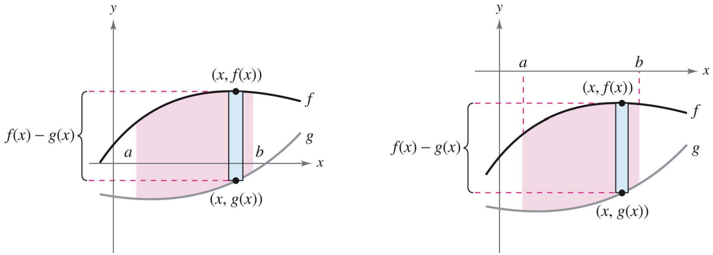


Figure 4: The height of a representative rectangle.

- Representative rectangles are used throughout this chapter in various applications of integration.
- A vertical rectangle (of width Δx) implies integration with respect to x, whereas a horizontal rectangle (of width Δy) implies integration with respect to y.

Example 1 (Finding the area of a region between two curves)

Find the area of the region bounded by the graphs of $f(x) = x^2 + 2$, $g(x) = -x$, $x = 0$, and $x = 1$.

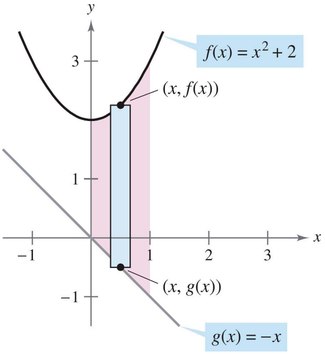


Figure 5: Region bounded by the graph of $f(x) = x^2 + 2$, $g(x) = -x$, $x = 0$, and $x = 1$.

- In Example 1, the graphs of $f(x) = x^2 + 2$ and $g(x) = -x$ do not intersect, and the values of a and b are given explicitly.
- A more common problem involves the area of a region bounded by two intersecting graphs, where the values of a and b must be calculated.

Example 2 (A region lying between two intersecting graphs)

Find the area of the region bounded by the graphs of $f(x) = 2 - x^2$ and $g(x) = x$.

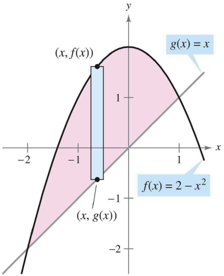


Figure 6: Region bounded by the graph of $f(x) = 2 - x^2$ and the graph of $g(x) = x$.

Example 4 (Curves that intersect at more than two points)

Find the area of the region between the graphs of $f(x) = 3x^3 - x^2 - 10x$ and $g(x) = -x^2 + 2x$.

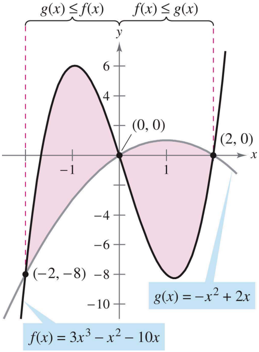
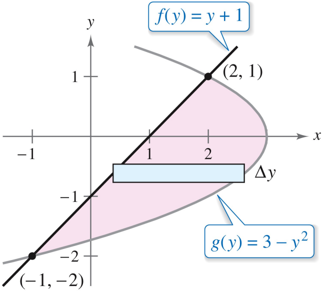


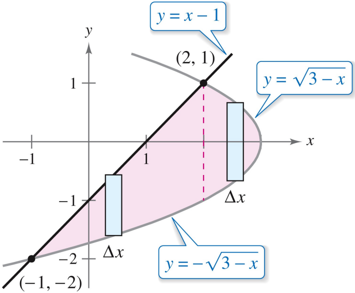
Figure 7: On $[-2, 0]$, $g(x) \le f(x)$, and on $[0, 2]$, $f(x) \le g(x)$.

Example 5 (Horizontal representative rectangles)

Find the area of the region bounded by the graphs of $x = 3 - y^2$ and $x = y + 1$.



(a) Horizontal rectangles
(integration with respect to y)



(b) Vertical rectangles
(integration with respect to x)

Figure 8: Horizontal rectangles v.s. vertical rectangles.

Table of Contents

The Disk Method

- If a region in the plane is revolved about a line, the resulting solid is a solid of revolution, and the line is called the axis of revolution.
- The simplest such solid is a right circular cylinder or disk, which is formed by revolving a rectangle about an axis adjacent to one side of the rectangle, as shown in Figure 9.

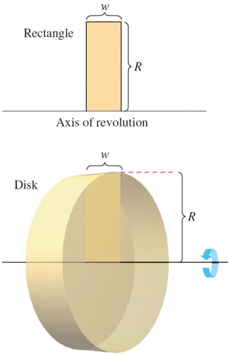


Figure 9: Volume of a disk: $\pi R^2 w$.

- The volume of such a disk is

$$\text{Volume of disk} = (\text{area of disk})(\text{width of disk}) = \pi R^2 w$$

where R is the radius of the disk and w is the width.

- To see how to use the volume of a disk to find the volume of a general solid of revolution, consider a solid of revolution formed by revolving the plane region in Figure 10 about the indicated axis.

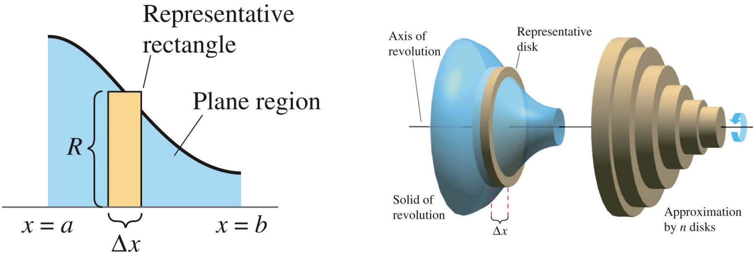


Figure 10: Disk Method.

- To determine the volume, consider a representative rectangle in the plane region. When this rectangle is revolved about the axis of revolution, it generates a representative disk whose volume is

$$\Delta V = \pi R^2 \Delta x.$$

- Approximating the volume of the solid by n such disks of width Δx and radius $R(x_i)$ produces

$$\text{Volume of solid} \approx \sum_{i=1}^{n} \pi [R(x_i)]^2 \Delta x = \pi \sum_{i=1}^{n} [R(x_i)]^2 \Delta x.$$

- This approximation appears to become better and better as $\|\Delta\| \to 0$ ($n \to \infty$). So, you can define the volume of the solid as

$$\text{Volume of solid} = \lim_{\|\Delta\| \to 0} \pi \sum_{i=1}^{n} [R(x_i)]^2 \Delta x = \pi \int_a^b [R(x)]^2 \, \mathrm{d}x.$$

- Schematically, the Disk Method looks like this.

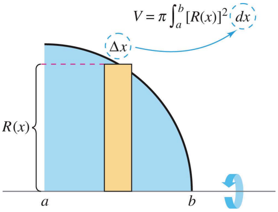
Known precalculus formula		Representative element		New integration formula
Volume of disk $V = \pi R^2 w$	$\Rightarrow$	$\Delta V = \pi [R(x_i)]^2 \Delta x$	$\Rightarrow$	Solid of revolution $V = \pi \int_a^b [R(x)]^2 \, \mathrm{d}x$

- A similar formula can be derived when the axis of revolution is vertical!

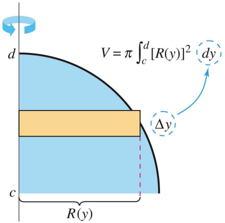
The Disk Method

To find the volume of a solid of revolution with the Disk Method, use one of the following, as shown below.

Horizontal axis of revolution	Vertical axis of revolution
Volume $= V = \pi \int_a^b [R(x)]^2 \, \mathrm{d}x$	Volume $= V = \pi \int_c^d [R(y)]^2 \, \mathrm{d}y$



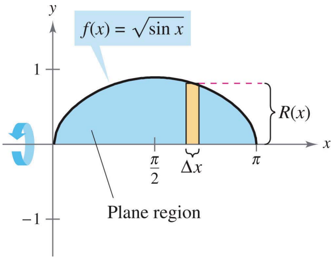
(a) Horizontal axis of revolution.



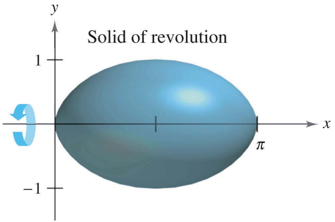
(b) Vertical axis of revolution.

Example 1 (Using the Disk Method)

Find the volume of the solid formed by revolving the region bounded by the graph of $f(x) = \sqrt{\sin x}$ and the x-axis ($0 \leq x \leq \pi$) about the x-axis.



(a) Plane region.

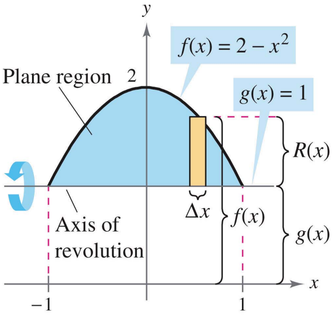


(b) Solid of revolution.

Figure 12: Disk Method: $f(x) = \sqrt{\sin x}$.

Example 2 (Revolving about a line that is not a coordinate axis)

Find the volume of the solid formed by revolving the region bounded by the graph of $f(x) = 2 - x^2$ and $g(x) = 1$ about the line $y = 1$, as shown in Figure 13.



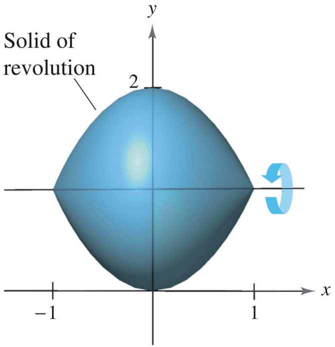


Figure 13: Revolving about a line that is not a coordinate axis.

The Washer Method

- The Disk Method can be extended to cover solids of revolution with holes by replacing the representative disk with a representative washer.
- The washer is formed by revolving a rectangle about an axis. If r and R are the inner and outer radii of the washer and w is the width of the washer, the volume is given by Volume of washer $= \pi(R^2 - r^2)w$.
- To see how this concept can be used to find the volume of a solid of revolution, consider a region bounded by an outer radius $R(x)$ and an inner radius $r(x)$, as shown in Figure 15.

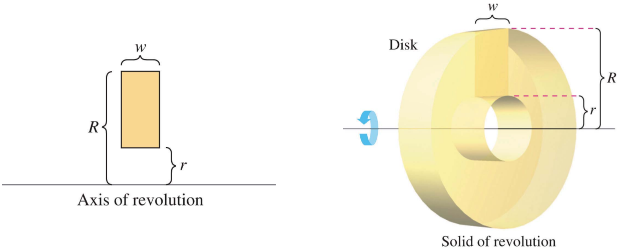


Figure 14: Washer Method.

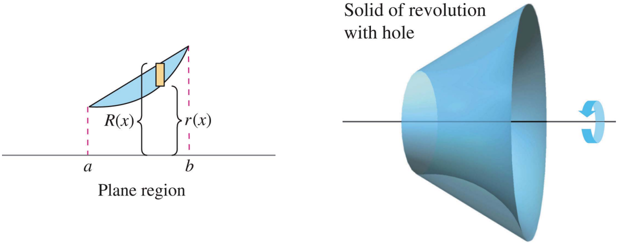


Figure 15: Solid of revolution with hole.

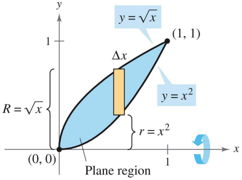
- If the region is revolved about its axis of revolution, the volume of the resulting solid is given by

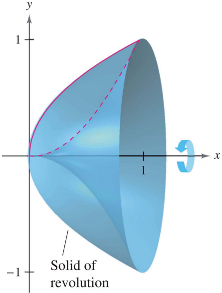
$$V = \pi \int_a^b \left([R(x)]^2 - [r(x)]^2\right) \mathrm{d}x. \qquad \text{Washer Method}$$

- Note that the integral involving the inner radius represents the volume of the hole and is subtracted from the integral involving the outer radius.

Example 3 (Using the Washer Method)

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sqrt{x}$ and $y = x^2$ about the x-axis, as shown below.





Example 4 (Integrating with respect to y, two-integral case)

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$ about y-axis, as shown in Figure 17.

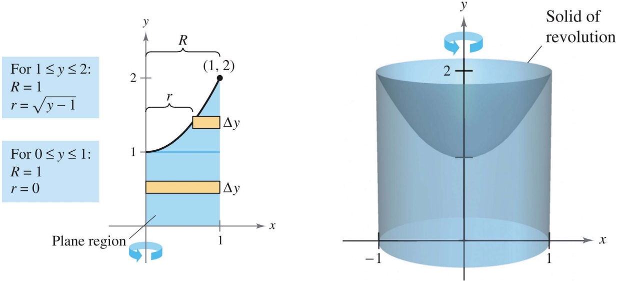
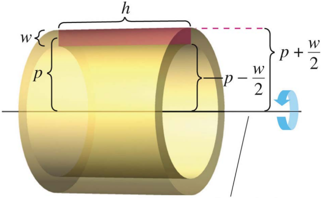


Figure 17: The volume of the solid formed by revolving the region bounded by the graphs of $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$ about y-axis.

Table of Contents

- An alternative method for finding the volume of a solid of revolution is called the Shell Method because it uses cylindrical shells.
- A comparison of the advantages of the disk and Shell Methods is given later in this section.
- Consider a representative rectangle as shown below, where w is the width of the rectangle, h is the height of the rectangle, and p is the distance between the axis of revolution and the center of the rectangle.



- When this rectangle is revolved about its axis of revolution, it forms a cylindrical shell (or tube) of thickness w.
- To find the volume of this shell, consider two cylinders. The radius of the larger cylinder corresponds to the outer radius of the shell, and the radius of the smaller cylinder corresponds to the inner radius of the shell. Because p is the average radius of the shell, you know the outer radius is $p + (w/2)$ and the inner radius is $p - (w/2)$.

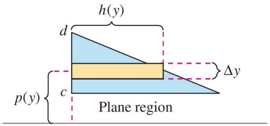
$$p + \frac{w}{2} \qquad \text{Outer radius} \qquad\qquad p - \frac{w}{2} \qquad \text{Inner radius}$$

- So, the volume of the shell is

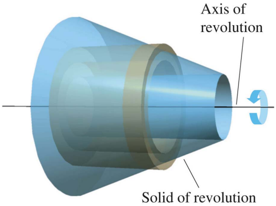
$$\begin{aligned}
\text{Volume of shell} &= (\text{volume of cylinder}) - (\text{volume of hole}) \\
&= \pi\left(p + \frac{w}{2}\right)^2 h - \pi\left(p - \frac{w}{2}\right)^2 h \\
&= 2\pi phw = 2\pi(\text{average radius})(\text{hight})(\text{thickness}).
\end{aligned}$$

- You can use this formula to find the volume of a solid of revolution. Assume that the plane region in Figure below is revolved about a line to form the indicated solid.
- If you consider a horizontal rectangle of width $\triangle y$, then, as the plane region is revolved about a line parallel to the x-axis, the rectangle generates a representative shell whose volume is

$$\triangle V = 2\pi[p(y)h(y)]\triangle y.$$



(a) Plane region.



(b) Solid of revolution.

- You can approximate the volume of the solid by n such shells of thickness $\triangle y$, height $h(y_i)$, and average radius $p(y_i)$.

$$\text{Volume of solid} \approx \sum_{i=1}^{n} 2\pi[p(y_i)h(y_i)]\triangle y = 2\pi \sum_{i=1}^{n}[p(y_i)h(y_i)]\triangle y$$

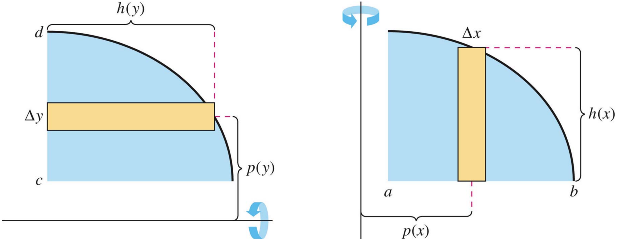
- This approximation appears to become better and better as $\|\triangle\| \to 0$ $(n \to \infty)$.
- So, the volume of the solid is

$$\text{Volume of solid} = \lim_{\|\triangle\|\to 0} 2\pi \sum_{i=1}^{n}[p(y_i)h(y_i)]\triangle y = 2\pi \int_c^d [p(y)h(y)]\,\mathrm{d}y.$$

The Shell Method

To find the volume of a solid of revolution with the Shell Method, use one of the following, as show in Figure 19.

Horizontal axis revolution	Vertical axis of revolution
$V = 2\pi \int_c^d p(y)h(y)\,\mathrm{d}y$	$V = 2\pi \int_a^b p(x)h(x)\,\mathrm{d}x$



(a) Horizontal axis of revolution.

(b) Vertical axis of revolution.

Figure 19: Shell Method: Horizontal versus vertical axis of revolution.

Example 1 (Using the Shell Method to find volume)

Find the volume of the solid formed by revolving the region bounded by $y = x - x^3$ and the x-axis $(0 \leq x \leq 1)$ about the y-axis.

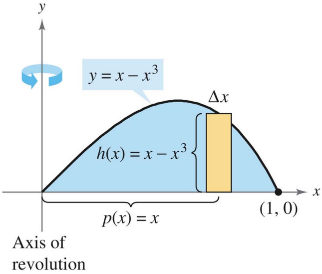


Figure 20: The volume of the solid of revolution formed by revolving the region bounded by $y = x - x^3$ and the x-axis ($0 \le x \le 1$) about the y-axis.

Example 2 (Using the Shell Method to find volume)

Find the volume of the solid formed by revolving the region bounded by the graph of $x = e^{-y^2}$ and the y-axis ($0 \leq y \leq 1$) about the x-axis.

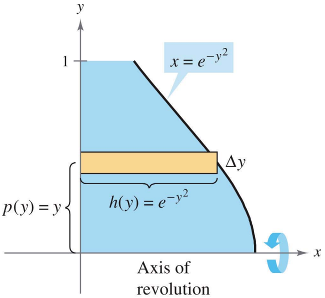
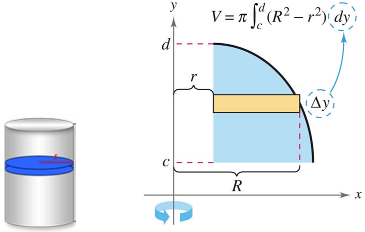


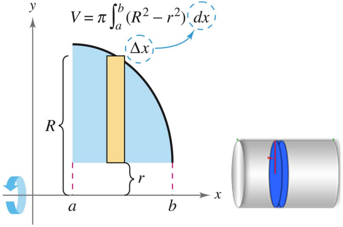
Figure 21: Using the Shell Method to find volume.

Comparison of Disk and Shell Methods

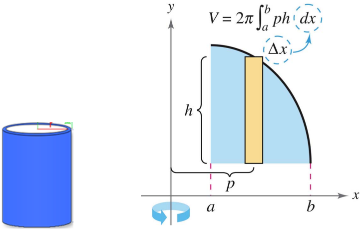
- The Disk and Shell Methods can be distinguished as follows.
- For the Disk Method, the representative rectangle is always perpendicular to the axis of revolution, whereas for the Shell Method, the representative rectangle is always parallel to the axis of revolution, as shown in Figure below.



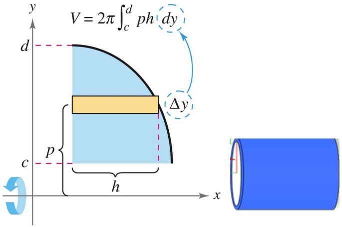
(a) Disk Method: Vertical axis of revolution.



(b) Disk Method: Horizontal axis of revolution.



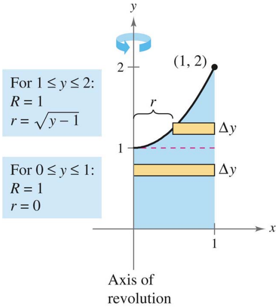
(c) Shell Method: Vertical axis of revolution.



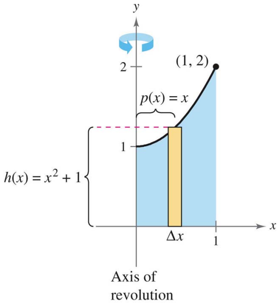
(d) Shell Method: Horizontal axis of revolution.

Example 3 (Shell Method preferable)

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$ about the y-axis.



(a) Disk Method.



(b) Shell Method.

Table of Contents

Arc length

- Definite integrals can also be use to find the arc length of curves and the areas of surfaces of revolution.
- In either case, an arc (a segment of a curve) is approximated by straight line segments whose lengths are given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

- A rectifiable curve is one that has a finite arc length. You will see that a sufficient condition for the graph of a function f to be rectifiable between $(a, f(a))$ and $(b, f(b))$ is that f' be continuous on $[a, b]$.
- Such a function is continuously differentiable on $[a, b]$, and its graph on the interval $[a, b]$ is a smooth curve.

- Consider a function $y = f(x)$ that is continuously differentiable on the interval $[a, b]$. You can approximate the graph of f by n line segments whose endpoints are determined by the partition $a = x_0 < x_1 < x_2 < \cdots < x_n = b$:

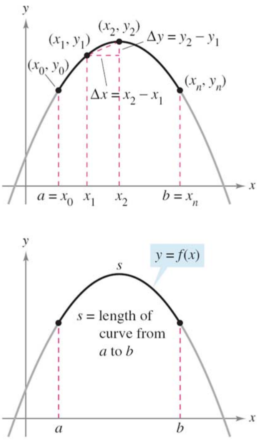


Figure 24: Arc length.

Definition 7.1 (Arc length)

Let the function given by $y = f(x)$ represent a smooth curve on the interval $[a, b]$. The arc length of f between a and b is

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, \mathrm{d}x.$$

Similarly, for a smooth curve given by $x = g(y)$, the arc length of g between c and d is

$$s = \int_c^d \sqrt{1 + [g'(y)]^2}\, \mathrm{d}y.$$

Example 1 (The length of a line segment)

Find the arc length from (x_1, y_1) to (x_2, y_2) on the graph of $f(x) = mx + b$

Example 2 (Finding arc length)

Find the arc length of the graph of $y = \frac{x^3}{6} + \frac{1}{2x}$ on the interval $\left[\frac{1}{2}, 2\right]$, as shown in Figure 25.

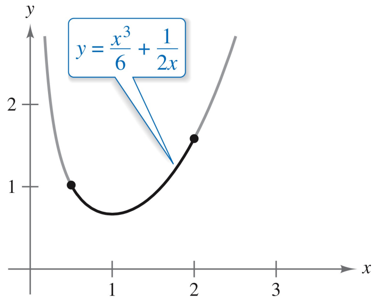


Figure 25: The arc length of the graph of $y = \frac{x^3}{6} + \frac{1}{2x}$ on $\left[\frac{1}{2}, 2\right]$.

Area of a surface of revolution

Definition 7.2 (Surface of revolution)

If the graph of a continuous function is revolved about a line, the resulting surface is a surface of revolution.

- The area of a surface of revolution is derived from the formula for the lateral surface area of the frustum of a right circular cone.
- Consider the line segment in Figure 26, where L is the length of the line segment, r_1 is the radius at the left end of the line segment, and r_2 is the radius at the right end of the line segment.

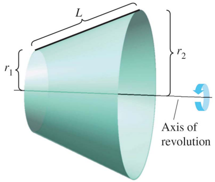


Figure 26: Surface of revolution.

- When the line segment is revolved about its axis of revolution, it forms a frustum of a right circular cone, with (Exercise 56)

$$S = 2\pi r L \qquad \text{Lateral surface area of frustum}$$

where

$$r = \frac{1}{2}(r_1 + r_2). \qquad \text{Average radius of frustum}$$

- Suppose the graph of a function f, having a continuous derivative on the interval $[a, b]$, is revolved about the x-axis to form a surface of revolution, as shown in Figure 27.

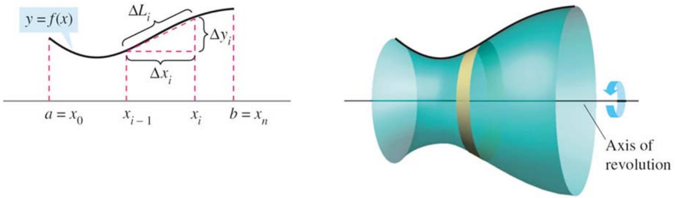


Figure 27: Surface of revolution.

- Let $\triangle$ be a partition of $[a, b]$, with width $\triangle x_i$. Then the line segment of length $\triangle L_i = \sqrt{\triangle x_i^2 + \triangle y_i^2}$ generates a frustum of a cone.
- Let r_i be the average radius of this frustum. By the Intermediate Value Theorem, a point d_i exists (in the ith subinterval) such that $r_i = f(d_i)$. The lateral surface area $\triangle S_i$ of the frustum is

$$\triangle S_i = 2\pi r_i \triangle L_i = 2\pi f(d_i)\sqrt{\triangle x_i^2 + \triangle y_i^2} = 2\pi f(d_i)\sqrt{1 + \left(\frac{\triangle y_i}{\triangle x_i}\right)^2}\triangle x_i.$$

- By Mean Value Theorem, a number c_i exists in (x_{i-1}, x_i) such that

$$f'(c_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = \frac{\triangle y_i}{\triangle x_i}.$$

- So, $\triangle S_i = 2\pi f(d_i)\sqrt{1 + [f'(c_i)]^2}\triangle x_i$, and the total surface area can be approximated by

$$S \approx 2\pi \sum_{i=1}^{n} f(d_i)\sqrt{1 + [f'(c_i)]^2}\triangle x_i.$$

- It can be shown that limit of the right side as $\|\triangle\| \to 0$ $(n \to \infty)$ is

$$S = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\,\mathrm{d}x.$$

- In a similar manner, if the graph of f is revolved about the y-axis, then S is

$$S = 2\pi \int_a^b x\sqrt{1 + [f'(x)]^2}\,\mathrm{d}x.$$

- In these two formulas for S, you can regard the products $2\pi f(x)$ and $2\pi x$ as the circumferences of the circles traced by a point (x, y) on the graph of f as it is revolved about the x-axis and the y-axis. The radius is $r = f(x)$, and $r = x$, respectively.

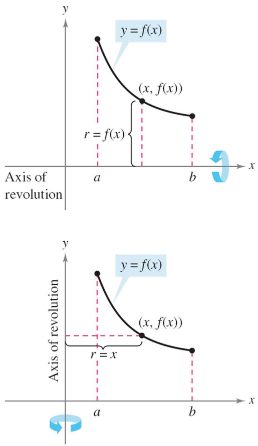


Figure 28: Revolve about x-axis and y-axis.

Definition 7.3 (Area of a surface of revolution)

Let $y = f(x)$ have a continuous derivative on the interval $[a, b]$. The area S of the surface of revolution formed by revolving the graph of f about a horizontal or vertical axis is

$$S = 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}\,\mathrm{d}x \qquad y \text{ is a function of } x$$

where $r(x)$ is the distance between the graph of f and the axis of revolution. If $x = g(y)$ on the interval $[c, d]$, then the surface area is

$$S = 2\pi \int_c^d r(y)\sqrt{1 + [g'(y)]^2}\,\mathrm{d}y \qquad x \text{ is a function of } y$$

where $r(y)$ is the distance between the graph of g and the axis of revolution.

Example 6 (The area of a surface of revolution)

Find the area of the surface formed by revolving the graph of $f(x) = x^3$ on the interval $[0, 1]$ about the x-axis, as shown in Figure 29.

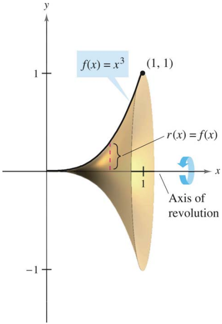


Figure 29: Area of a surface of revolution: $f(x) = x^3$ about x-axis.

Example 7 (The area of a surface of revolution)

Find the area of the surface formed by revolving the graph of $f(x) = x^2$ on the interval $[0, \sqrt{2}]$ about the y-axis, as shown in Figure 30.

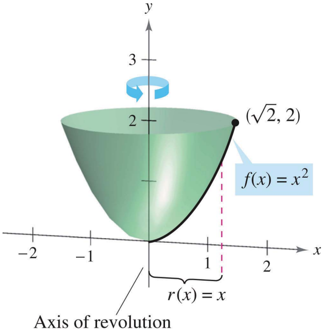


Figure 30: The area of a surface revolution: $f(x) = x^2$ about y-axis.