Problem 1

Verify that f has an inverse function. Then use the function f and the given function value $f(6) = 3$ to find $(f^{-1})'(3)$.

$$f(x) = \frac{x+6}{x-2}, \quad x > 2$$

Solution

Step 1: Verify that f has an inverse function. To show that the inverse exists, we check if f is strictly monotonic by finding its derivative using the quotient rule:

$$\begin{aligned} f'(x) &= \frac{(1)(x-2) - (x+6)(1)}{(x-2)^2} \\ &= \frac{x - 2 - x - 6}{(x-2)^2} \\ &= \frac{-8}{(x-2)^2} \end{aligned}$$

Since $(x-2)^2 > 0$ for all $x > 2$, we have $f'(x) < 0$. Thus, f is strictly decreasing and one-to-one, which implies that f^{-1} exists.

Step 2: Find $(f^{-1})'(3)$. According to the Inverse Function Theorem:

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

Given $a = 3$ and $f(6) = 3$, we know that $f^{-1}(3) = 6$. Substituting these values into the theorem:

$$(f^{-1})'(3) = \frac{1}{f'(6)}$$

We evaluate $f'(x)$ at $x = 6$:

$$f'(6) = \frac{-8}{(6-2)^2} = \frac{-8}{16} = -\frac{1}{2}$$

Finally, compute the derivative of the inverse:

$$(f^{-1})'(3) = \frac{1}{-\frac{1}{2}} = -2$$

Problem 2

Find the derivative of the function:

$$y = \ln\left(\frac{1+e^x}{1-e^x}\right)$$

Solution

Step 1: Simplify using Logarithmic Properties Before differentiating, we use the property $\ln(A/B) = \ln A - \ln B$ to simplify the function:

$$y = \ln(1+e^x) - \ln(1-e^x)$$

Step 2: Differentiate term by term Now we differentiate with respect to x using the chain rule $\frac{d}{dx}\ln(u) = \frac{u'}{u}$:

$$\begin{aligned} y' &= \frac{d}{dx}\ln(1+e^x) - \frac{d}{dx}\ln(1-e^x) \\ &= \frac{e^x}{1+e^x} - \frac{-e^x}{1-e^x} \\ &= \frac{e^x}{1+e^x} + \frac{e^x}{1-e^x} \end{aligned}$$

Step 3: Simplify the result To combine the fractions, we find a common denominator $(1+e^x)(1-e^x) = 1-e^{2x}$:

$$\begin{aligned} y' &= \frac{e^x(1-e^x) + e^x(1+e^x)}{(1+e^x)(1-e^x)} \\ &= \frac{e^x - e^{2x} + e^x + e^{2x}}{1-e^{2x}} \\ &= \frac{2e^x}{1-e^{2x}} \end{aligned}$$

Final Answer:

$$y' = \frac{2e^x}{1-e^{2x}}$$

Problem 3

Find the derivative of the function:

$$g(x) = \log_5 \left(\frac{4}{x^2\sqrt{1-x}} \right)$$

Solution

Step 1: Simplify using Logarithmic Laws Instead of differentiating directly, we expand the logarithmic expression to simplify the calculation:

$$\begin{aligned} g(x) &= \log_5(4) - \log_5(x^2\sqrt{1-x}) \\ &= \log_5(4) - \left[\log_5(x^2) + \log_5((1-x)^{1/2})\right] \\ &= \log_5(4) - 2\log_5(x) - \frac{1}{2}\log_5(1-x) \end{aligned}$$

Step 2: Differentiate term by term Recall that $\frac{d}{dx}(\log_a u) = \frac{u'}{u \ln a}$. Here $a = 5$.

- The derivative of the constant $\log_5(4)$ is 0.
- The derivative of $-2\log_5(x)$ is $-2 \cdot \frac{1}{x \ln 5}$.
- The derivative of $-\frac{1}{2}\log_5(1-x)$ involves the chain rule:

$$-\frac{1}{2} \cdot \frac{1}{(1-x)\ln 5} \cdot \frac{d}{dx}(1-x) = -\frac{1}{2(1-x)\ln 5} \cdot (-1)$$

Combining these terms:

$$\begin{aligned} g'(x) &= 0 - \frac{2}{x \ln 5} + \frac{1}{2(1-x)\ln 5} \\ &= \frac{1}{\ln 5}\left(\frac{1}{2(1-x)} - \frac{2}{x} \right) \end{aligned}$$

Step 3: Simplify the final answer (Optional) Finding a common denominator $2x(1-x)$:

$$\begin{aligned} g'(x) &= \frac{1}{\ln 5}\left(\frac{x - 4(1-x)}{2x(1-x)} \right) \\ &= \frac{1}{\ln 5}\left(\frac{x - 4 + 4x}{2x(1-x)} \right) \\ &= \frac{5x-4}{2x(1-x)\ln 5} \end{aligned}$$

Problem 4

Evaluate the limit, using L'Hôpital's Rule if necessary:

$$\lim_{x \to 1} \frac{\ln x^3}{x^2 - 1}$$

Solution

Step 1: Check the limit form First, substitute $x = 1$ to check if the limit is indeterminate:

- Numerator: $\ln(1^3) = \ln(1) = 0$
- Denominator: $1^2 - 1 = 0$

Since we have the indeterminate form $\frac{0}{0}$, we can apply L'Hôpital's Rule.

Step 2: Simplify and Apply L'Hôpital's Rule Before differentiating, we use the logarithmic property $\ln(a^b) = b \ln a$ to simplify the numerator. This avoids using the chain rule unnecessarily.

$$\lim_{x \to 1} \frac{\ln x^3}{x^2 - 1} = \lim_{x \to 1} \frac{3 \ln x}{x^2 - 1}$$

Now, differentiate the numerator and the denominator with respect to x:

$$\begin{aligned} \lim_{x \to 1} \frac{\frac{d}{dx}(3 \ln x)}{\frac{d}{dx}(x^2 - 1)} &= \lim_{x \to 1} \frac{3 \cdot \frac{1}{x}}{2x} \\ &= \lim_{x \to 1} \frac{3}{2x^2} \end{aligned}$$

Step 3: Evaluate the limit Finally, substitute $x = 1$ into the simplified expression:

$$= \frac{3}{2(1)^2} = \frac{3}{2}$$