Assignment 4 Solutions

1. Implicit Differentiation

Given

$$x = \sec\left(\frac{1}{y}\right),$$

differentiate both sides with respect to x:

$$1 = \sec\left(\frac{1}{y}\right)\tan\left(\frac{1}{y}\right)\cdot\frac{d}{dx}\left(\frac{1}{y}\right) = \sec\left(\frac{1}{y}\right)\tan\left(\frac{1}{y}\right)\left(-\frac{1}{y^2}\right)\frac{dy}{dx}.$$

Thus,

$$\boxed{\frac{dy}{dx} = -\frac{y^2}{\sec(1/y)\,\tan(1/y)}}.$$

Alternatively, since $x = \sec(1/y)$, we may write

$$\frac{dy}{dx} = -\frac{y^2}{x\,\tan(1/y)}.$$

2. Absolute Extrema on $[0,2]$

Consider

$$y = \tan\left(\frac{\pi}{8}x\right), \quad x \in [0,2].$$

We compute

$$y' = \frac{\pi}{8}\sec^2\left(\frac{\pi}{8}x\right) > 0,$$

so the function is strictly increasing and has no interior extrema.
Evaluate endpoints:

$$y(0) = \tan(0) = 0, \qquad y(2) = \tan\left(\frac{\pi}{4}\right) = 1.$$

Hence

$$\boxed{\text{Absolute minimum } = 0 \text{ at } x = 0, \quad \text{Absolute maximum } = 1 \text{ at } x = 2.}$$

3. Rolle's Theorem

Let

$$f(x) = 3 - |x - 3|, \quad x \in [0, 6].$$

Check conditions:

- f is continuous on $[0, 6]$ because the absolute value function is continuous.
- f is not differentiable at $x = 3$ (a cusp).
- $f(0) = 3 - 3 = 0, \quad f(6) = 3 - 3 = 0.$

Although the endpoints are equal, f fails differentiability on $(0, 6)$, so Rolle's Theorem does not apply.
Indeed,

$$f'(x) = \begin{cases} 1, & x < 3, \\ -1, & x > 3, \end{cases}$$

and no point satisfies $f'(c) = 0$.

$$\boxed{\text{Rolle's Theorem does not apply; no } c \text{ with } f'(c) = 0.}$$

4. Mean Value Theorem

Given

$$f(x) = x^3 - 3x^2 + 9x + 5, \quad x \in [0, 1].$$

Since f is a polynomial, it is continuous and differentiable everywhere. Therefore, the Mean Value Theorem applies.
Compute the average rate of change:

$$\frac{f(1) - f(0)}{1 - 0} = \frac{(1 - 3 + 9 + 5) - 5}{1} = 7.$$

Find c such that $f'(c) = 7$.
The derivative is

$$f'(x) = 3x^2 - 6x + 9.$$

Setting $f'(c) = 7$:

$$3c^2 - 6c + 9 = 7 \;\Rightarrow\; 3c^2 - 6c + 2 = 0.$$

Solve:

$$c = \frac{6 \pm \sqrt{36 - 24}}{6} = 1 \pm \frac{\sqrt{3}}{3}.$$

Only the smaller root lies in $(0, 1)$:

$$\boxed{c = 1 - \frac{\sqrt{3}}{3} \approx 0.4229.}$$

Hence,

$$\boxed{\text{The MVT applies with } c = 1 - \frac{\sqrt{3}}{3}.}$$