Problem 1

Find the derivative of the function:

$$f(x) = \arcsin x + \arccos x$$

Solution (Method 1: Direct Differentiation)

We know the derivatives of the inverse trigonometric functions:

$$\begin{aligned}\frac{d}{dx}(\arcsin x) &= \frac{1}{\sqrt{1-x^2}} \\ \frac{d}{dx}(\arccos x) &= -\frac{1}{\sqrt{1-x^2}}\end{aligned}$$

By differentiating $f(x)$ term by term:

$$\begin{aligned}f'(x) &= \frac{d}{dx}(\arcsin x) + \frac{d}{dx}(\arccos x) \\ &= \frac{1}{\sqrt{1-x^2}} + \left(-\frac{1}{\sqrt{1-x^2}}\right) \\ &= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} \\ &= 0\end{aligned}$$

Problem 2

Evaluate the definite integral:

$$\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x}\, dx$$

Solution (Substitution Method)

Let $u = \sin x$. Then, the differential is:

$$du = \cos x\, dx$$

We also need to change the limits of integration:

- When $x = 0$, $u = \sin(0) = 0$.
- When $x = \pi/2$, $u = \sin(\pi/2) = 1$.

Now, substitute u and the new limits into the integral:

$$\begin{aligned}\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x}\,dx &= \int_0^1 \frac{1}{1+u^2}\,du \\ &= \Big[\arctan u\Big]_0^1 \\ &= \arctan(1) - \arctan(0) \\ &= \frac{\pi}{4} - 0 \\ &= \frac{\pi}{4}\end{aligned}$$

Problem 3

Find the area of the region bounded by:

$$y = -x^3 + 2, \quad y = x - 3, \quad x = -1, \quad x = 1$$

Solution

First, we determine which curve is the upper curve in the interval $[-1, 1]$. Let's test a point inside the interval, say $x = 0$:

$$\begin{aligned} y_1 &= -(0)^3 + 2 = 2 \\ y_2 &= 0 - 3 = -3 \end{aligned}$$

Since $2 > -3$, $y = -x^3 + 2$ is the upper curve and $y = x - 3$ is the lower curve.
The area is given by the integral of the upper curve minus the lower curve:

$$\begin{aligned}\text{Area} &= \int_{-1}^{1} \big[(-x^3 + 2) - (x - 3)\big]\,dx \\ &= \int_{-1}^{1} (-x^3 - x + 5)\,dx\end{aligned}$$

Method: Using Symmetry (Odd and Even Functions) Since the integration interval $[-1, 1]$ is symmetric about the origin:

- $-x^3$ and $-x$ are odd functions, so their integrals over $[-1, 1]$ are 0.
- 5 is a constant (even function).

Therefore, the calculation simplifies to:

$$\begin{aligned}\text{Area} &= \int_{-1}^{1} 5\,dx \\ &= 5 \cdot [x]_{-1}^{1} \\ &= 5 \cdot (1 - (-1)) \\ &= 5 \cdot 2 \\ &= 10\end{aligned}$$

Problem 4

Evaluate the limit, using L'Hôpital's Rule if necessary:

$$\lim_{x \to 1} \frac{\ln x}{\sin(\pi x)}$$

Solution

First, we check the limit form by substituting $x = 1$:

- Numerator: $\ln(1) = 0$
- Denominator: $\sin(\pi \cdot 1) = 0$

Since this results in the indeterminate form $\frac{0}{0}$, we can apply L'Hôpital's Rule.

Differentiating the numerator and the denominator with respect to x:

$$\lim_{x \to 1} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(\sin(\pi x))} = \lim_{x \to 1} \frac{\frac{1}{x}}{\pi \cos(\pi x)}$$

Note: Be careful with the chain rule in the denominator: $\frac{d}{dx} \sin(\pi x) = \cos(\pi x) \cdot \pi$.

Now, we evaluate the limit again by substituting $x = 1$:

$$\begin{aligned}&= \frac{\frac{1}{1}}{\pi \cos(\pi)} \\ &= \frac{1}{\pi \cdot (-1)} \\ &= -\frac{1}{\pi}\end{aligned}$$