Problem 1

Find the derivative of the function:

$$f(x) = \ln\left(\frac{2x}{x+3}\right)$$

Solution

Step 1: Simplify the function using logarithmic properties.

Using the property $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$ and $\ln(ab) = \ln(a) + \ln(b)$, we can rewrite $f(x)$ as:

$$f(x) = \ln(2x) - \ln(x+3)$$
$$f(x) = \ln(2) + \ln(x) - \ln(x+3)$$

Step 2: Differentiate with respect to x.

Now, we find the derivative of each term:

- The derivative of the constant $\ln(2)$ is 0.
- The derivative of $\ln(x)$ is $\frac{1}{x}$.
- The derivative of $\ln(x+3)$ is $\frac{1}{x+3} \cdot (x+3)' = \frac{1}{x+3}$.

So, the derivative is:

$$f'(x) = 0 + \frac{1}{x} - \frac{1}{x+3}$$

Step 3: Simplify the result (Optional).

Combine the fractions by finding a common denominator:

$$f'(x) = \frac{1(x+3) - 1(x)}{x(x+3)}$$
$$f'(x) = \frac{x+3-x}{x(x+3)}$$
$$f'(x) = \frac{3}{x(x+3)}$$

Final Answer:

$$f'(x) = \frac{3}{x(x+3)}$$

Problem 2

Find the indefinite integral:

$$\int \frac{x^2 + 4x}{x^3 + 6x^2 + 5}\, dx$$

Solution

Step 1: Use Integration by Substitution (u-substitution).

Let u be the denominator:

$$u = x^3 + 6x^2 + 5$$

Step 2: Find the differential du.

Differentiate u with respect to x:

$$du = (3x^2 + 12x)\, dx$$
$$du = 3(x^2 + 4x)\, dx$$

Divide by 3 to isolate the term in the numerator:

$$\frac{1}{3} du = (x^2 + 4x)\, dx$$

Step 3: Substitute u and du into the integral.

Rewrite the integral in terms of u:

$$\int \frac{1}{u} \cdot \frac{1}{3}\, du = \frac{1}{3} \int \frac{1}{u}\, du$$

Step 4: Integrate.

Using the formula $\int \frac{1}{u}\, du = \ln|u| + C$:

$$= \frac{1}{3} \ln|u| + C$$

Step 5: Substitute back x.

Replace u with $x^3 + 6x^2 + 5$:

$$= \frac{1}{3} \ln|x^3 + 6x^2 + 5| + C$$

Final Answer:

$$\int \frac{x^2 + 4x}{x^3 + 6x^2 + 5}\, dx = \frac{1}{3} \ln|x^3 + 6x^2 + 5| + C$$

Problem 3

Find the indefinite integral by making a change of variables.
Hint: Let u be the denominator of the integrand.

$$\int \frac{4}{1+\sqrt{5x}}\,dx$$

Solution

Step 1: Set the variable u.
Let u be the denominator:

$$u = 1+\sqrt{5x}$$

Step 2: Express x and dx in terms of u.
Rearrange the equation to solve for x:

$$\begin{aligned} u-1 &= \sqrt{5x} \\ (u-1)^2 &= 5x \\ x &= \frac{1}{5}(u-1)^2 \end{aligned}$$

Now, differentiate with respect to u to find dx:

$$\begin{aligned} dx &= \frac{1}{5}\cdot 2(u-1)\,du \\ dx &= \frac{2}{5}(u-1)\,du \end{aligned}$$

Step 3: Substitute into the integral.
Replace $1+\sqrt{5x}$ with u, and replace dx with the expression found above:

$$\int \frac{4}{u}\cdot\left[\frac{2}{5}(u-1)\,du\right]$$

Step 4: Simplify and integrate.
Factor out the constants and simplify the integrand:

$$\begin{aligned} &= \int \frac{8}{5}\cdot\frac{u-1}{u}\,du \\ &= \frac{8}{5}\int\left(\frac{u}{u}-\frac{1}{u}\right)\,du \\ &= \frac{8}{5}\int\left(1-\frac{1}{u}\right)\,du \end{aligned}$$

Perform the integration:

$$= \frac{8}{5}\Big(u-\ln|u|\Big)+C$$

Step 5: Substitute back $u = 1 + \sqrt{5x}$.

$$= \frac{8}{5}\left((1 + \sqrt{5x}) - \ln|1 + \sqrt{5x}|\right) + C$$

Since $1+\sqrt{5x}$ is always positive, we can remove the absolute value bars. You can also distribute the constant or leave it factored.

Final Answer:

$$\int \frac{4}{1 + \sqrt{5x}}\, dx = \frac{8}{5}\left(1 + \sqrt{5x} - \ln(1 + \sqrt{5x})\right) + C$$