Problem 1 Solution

Problem: Evaluate the definite integral $\int_0^4 |x^2 - 4x + 3| dx$.

Step 1: Analyze the Absolute Value Function

First, we determine the roots of the quadratic equation inside the absolute value to identify the intervals where the function is positive or negative.

$$x^2 - 4x + 3 = (x - 1)(x - 3) = 0$$

The roots are $x = 1$ and $x = 3$. These points split the integration interval $[0, 4]$ into three sub-intervals. We check the sign of $f(x) = x^2 - 4x + 3$ in each interval:

- **Interval** $[0, 1]$**:** Let $x = 0.5$.

$$(0.5)^2 - 4(0.5) + 3 = 0.25 - 2 + 3 = 1.25 > 0$$

  So, $|x^2 - 4x + 3| = x^2 - 4x + 3$.

- **Interval** $[1, 3]$**:** Let $x = 2$.

$$2^2 - 4(2) + 3 = 4 - 8 + 3 = -1 < 0$$

  So, $|x^2 - 4x + 3| = -(x^2 - 4x + 3) = -x^2 + 4x - 3$.

- **Interval** $[3, 4]$**:** Let $x = 3.5$.

$$(3.5)^2 - 4(3.5) + 3 = 12.25 - 14 + 3 = 1.25 > 0$$

  So, $|x^2 - 4x + 3| = x^2 - 4x + 3$.

Step 2: Split the Integral

Based on the intervals above, we split the definite integral into three parts:

$$I = \int_0^1 (x^2 - 4x + 3)dx + \int_1^3 -(x^2 - 4x + 3)dx + \int_3^4 (x^2 - 4x + 3)dx$$

Step 3: Evaluate the Integrals

The antiderivative for $x^2 - 4x + 3$ is:

$$F(x) = \int (x^2 - 4x + 3)dx = \frac{x^3}{3} - 2x^2 + 3x$$

Now, we evaluate each part:

1. **From 0 to 1:**

$$\left[\frac{x^3}{3} - 2x^2 + 3x\right]_0^1 = \left(\frac{1}{3} - 2 + 3\right) - 0 = \frac{4}{3}$$

2. **From 1 to 3 (with negative sign):**

$$\int_1^3 -(x^2 - 4x + 3)dx = -[F(3) - F(1)]$$

$$F(3) = \frac{3^3}{3} - 2(3^2) + 3(3) = 9 - 18 + 9 = 0$$

$$F(1) = \frac{4}{3}$$

$$-\left(0 - \frac{4}{3}\right) = \frac{4}{3}$$

3. **From 3 to 4:**

$$\left[\frac{x^3}{3} - 2x^2 + 3x\right]_3^4 = F(4) - F(3)$$

$$F(4) = \frac{4^3}{3} - 2(4^2) + 3(4) = \frac{64}{3} - 32 + 12 = \frac{64}{3} - \frac{60}{3} = \frac{4}{3}$$

$$\text{Result} = \frac{4}{3} - 0 = \frac{4}{3}$$

Final Answer

Summing the results from the three intervals:

$$\text{Total Integral} = \frac{4}{3} + \frac{4}{3} + \frac{4}{3} = \frac{12}{3} = 4$$

Problem 2 Solution

Problem: Find the indefinite integral $\int \frac{6x^2}{(4x^3-9)^3} dx$ and check the result by differentiation.

Step 1: Integration by Substitution

Let $u = 4x^3 - 9$. Then, compute the differential du:

$$du = 12x^2 dx$$

Notice that the numerator in the integral is $6x^2 dx$. We can adjust our differential equation:

$$\frac{1}{2} du = 6x^2 dx$$

Step 2: Substitute and Integrate

Substitute u and du into the integral:

$$\int \frac{1}{u^3} \cdot \frac{1}{2} du = \frac{1}{2} \int u^{-3} du$$

Apply the power rule for integration $\int u^n du = \frac{u^{n+1}}{n+1}$:

$$\frac{1}{2} \cdot \frac{u^{-2}}{-2} + C = -\frac{1}{4} u^{-2} + C$$

Substitute $u = 4x^3 - 9$ back into the expression:

$$\text{Answer} = -\frac{1}{4(4x^3 - 9)^2} + C$$

Step 3: Check by Differentiation

Differentiate the result to verify:

$$\frac{d}{dx} \left(-\frac{1}{4} (4x^3 - 9)^{-2} \right)$$

Using the Chain Rule:

$$= -\frac{1}{4} \cdot (-2)(4x^3 - 9)^{-3} \cdot \frac{d}{dx}(4x^3 - 9)$$

$$= \frac{1}{2} (4x^3 - 9)^{-3} \cdot (12x^2)$$

$$= 6x^2 (4x^3 - 9)^{-3} = \frac{6x^2}{(4x^3 - 9)^3}$$

The derivative matches the original integrand.

Problem 3 Solution

Problem: Find the indefinite integral by making a change of variables: $\int \sec^5(7x)\tan(7x)dx$.

Step 1: Rewrite the Integral

Separate one factor of $\sec(7x)$ to group it with $\tan(7x)$:

$$\int \sec^4(7x) \cdot [\sec(7x)\tan(7x)]dx$$

Step 2: Substitution

Let $u = \sec(7x)$. Differentiate with respect to x using the chain rule:

$$du = \sec(7x)\tan(7x) \cdot 7dx$$

$$du = 7\sec(7x)\tan(7x)dx$$

Solve for the term in the integral:

$$\frac{1}{7}du = \sec(7x)\tan(7x)dx$$

Step 3: Integrate

Substitute u and du back into the integral:

$$\int u^4 \cdot \frac{1}{7}du = \frac{1}{7}\int u^4 du$$

Apply the power rule:

$$\frac{1}{7} \cdot \frac{u^5}{5} + C = \frac{1}{35}u^5 + C$$

Step 4: Final Substitution

Replace u with $\sec(7x)$:

$$\text{Answer} = \frac{1}{35}\sec^5(7x) + C$$