Problem 1

Question: Find the indefinite integral and check the result by differentiation.

$$\int (\sec y)(\tan y - \sec y)\, dy$$

Solution

Step 1: Expand the integrand Multiply the terms inside the integral:

$$\begin{aligned} I &= \int (\sec y \tan y - \sec^2 y)\, dy \\ &= \int \sec y \tan y\, dy - \int \sec^2 y\, dy \end{aligned}$$

Step 2: Apply integration formulas Using the standard trigonometric integrals:

- $\int \sec y \tan y\, dy = \sec y$
- $\int \sec^2 y\, dy = \tan y$

Thus, the result is:

$$\boxed{\sec y - \tan y + C}$$

Check by Differentiation

We differentiate the result with respect to y:

$$\begin{aligned} \frac{d}{dy}(\sec y - \tan y + C) &= \frac{d}{dy}(\sec y) - \frac{d}{dy}(\tan y) \\ &= \sec y \tan y - \sec^2 y \\ &= \sec y(\tan y - \sec y) \end{aligned}$$

The result matches the original integrand. **Verified.**

Problem 2

Question: Use the summation formulas to rewrite the expression without the summation notation. Use the result to find the sums for $n = 10, 100, 1000, 10000$.

$$\sum_{i=1}^{n} \frac{2i^3 - 3i}{n^4}$$

Solution

Part 1: Rewrite the expression

Factor out the constant term $\frac{1}{n^4}$ and split the summation:

$$\begin{aligned} S_n &= \frac{1}{n^4} \sum_{i=1}^{n} (2i^3 - 3i) \\ &= \frac{1}{n^4} \left(2 \sum_{i=1}^{n} i^3 - 3 \sum_{i=1}^{n} i \right) \end{aligned}$$

Substitute the standard summation formulas $\sum i^3 = \frac{n^2(n+1)^2}{4}$ and $\sum i = \frac{n(n+1)}{2}$:

$$\begin{aligned} S_n &= \frac{1}{n^4} \left(2 \cdot \frac{n^2(n+1)^2}{4} - 3 \cdot \frac{n(n+1)}{2} \right) \\ &= \frac{1}{n^4} \left(\frac{n^2(n+1)^2}{2} - \frac{3n(n+1)}{2} \right) \end{aligned}$$

Simplify the expression:

$$\begin{aligned} S_n &= \frac{1}{2n^4} \left[n^2(n^2 + 2n + 1) - (3n^2 + 3n) \right] \\ &= \frac{1}{2n^4} (n^4 + 2n^3 + n^2 - 3n^2 - 3n) \\ &= \frac{n^4 + 2n^3 - 2n^2 - 3n}{2n^4} \end{aligned}$$

Dividing each term by $2n^4$ gives the simplified form for calculation:

$$\boxed{S_n = \frac{1}{2} + \frac{1}{n} - \frac{1}{n^2} - \frac{3}{2n^3}}$$

Part 2: Numerical Values

錯 1~2: 扣 1 分; 錯 3~4: 扣 2 分

Using the derived formula:

- For $n = 10$:

$$0.5 + 0.1 - 0.01 - 0.0015 = \mathbf{0.5885}$$

- For $n = 100$:

$$0.5 + 0.01 - 0.0001 - 0.0000015 = \mathbf{0.5098985}$$

- For $n = 1000$:

$$0.5 + 0.001 - 0.000001 - 0.0000000015 = \mathbf{0.5009989985}$$

- For $n = 10000$:

$$0.5 + 0.0001 - 0.00000001 - 0.0000000000015 = \mathbf{0.5000999899985}$$

Problem 3

Question: Find the upper and lower sums for the region bounded by the graph of the function and the x-axis on the given interval. Leave your answer in terms of n.

$$f(x) = 9 - x^2, \quad x \in [0, 2]$$

Solution

Setup:

- Interval width: $\Delta x = \frac{2-0}{n} = \frac{2}{n}$.
- Since $f(x)$ is decreasing on $[0, 2]$, the maximum value occurs at the left endpoint and the minimum at the right endpoint of each subinterval.

1. Lower Sum (s_n)

We use the right endpoints $x_i = \frac{2i}{n}$ for $i = 1, \dots, n$.

$$\begin{aligned} s_n &= \sum_{i=1}^{n} f(x_i)\Delta x \\ &= \sum_{i=1}^{n} \left(9 - \left(\frac{2i}{n}\right)^2\right) \frac{2}{n} \\ &= \frac{2}{n} \left(\sum_{i=1}^{n} 9 - \frac{4}{n^2} \sum_{i=1}^{n} i^2\right) \end{aligned}$$

Substitute $\sum i^2 = \frac{n(n+1)(2n+1)}{6}$:

$$
\begin{aligned}
s_n &= \frac{2}{n}\left(9n - \frac{4}{n^2} \cdot \frac{n(n+1)(2n+1)}{6}\right) \\
&= 18 - \frac{4}{3n^2}(2n^2 + 3n + 1) \\
&= 18 - \left(\frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}\right) \\
&= \frac{54}{3} - \frac{8}{3} - \frac{4}{n} - \frac{4}{3n^2}
\end{aligned}
$$

$$
\boxed{s_n = \frac{46}{3} - \frac{4}{n} - \frac{4}{3n^2}}
$$

2. Upper Sum (S_n)

We use the left endpoints $x_{i-1} = \frac{2(i-1)}{n}$ with the summation index i running from 1 to n.

$$
\begin{aligned}
S_n &= \sum_{i=1}^{n} f(x_{i-1})\Delta x \\
&= \sum_{i=1}^{n} \left[9 - \left(\frac{2(i-1)}{n}\right)^2\right] \frac{2}{n} \\
&= \frac{2}{n} \sum_{i=1}^{n} \left(9 - \frac{4(i-1)^2}{n^2}\right) \\
&= \frac{2}{n} \left(\sum_{i=1}^{n} 9 - \frac{4}{n^2} \sum_{i=1}^{n} (i-1)^2\right)
\end{aligned}
$$

Note that $\sum_{i=1}^{n}(i-1)^2 = 0^2 + 1^2 + \cdots + (n-1)^2$, which is the sum of the first $n-1$ squares:

$$
\sum_{i=1}^{n}(i-1)^2 = \frac{(n-1)n(2n-1)}{6}
$$

Substitute this back into the expression:

$$
\begin{aligned}
S_n &= \frac{2}{n}\left(9n - \frac{4}{n^2} \cdot \frac{n(n-1)(2n-1)}{6}\right) \\
&= 18 - \frac{8}{n^3} \cdot \frac{n(2n^2 - 3n + 1)}{6} \\
&= 18 - \frac{4}{3n^2}(2n^2 - 3n + 1) \\
&= 18 - \left(\frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}\right) \\
&= \frac{54}{3} - \frac{8}{3} + \frac{4}{n} - \frac{4}{3n^2}
\end{aligned}
$$

$$
\boxed{S_n = \frac{46}{3} + \frac{4}{n} - \frac{4}{3n^2}}
$$