1. Find the following limit. (If the limit does not exist or has an infinite limit, you should point it out. In addition, also remember the definition of definite integral). (20%)

(a) $\lim_{x\to 0^+} \frac{\int_0^{x^2}(\sin(\sqrt{t})-\sqrt{t})dt}{\int_0^{x^2}(tan(\sqrt{t})-\sqrt{t})dt}$

(b) $\lim_{x\to 0^+} tan(x)\ln(sin^2(x))$

(c) $\lim_{x\to 1^+}(x-1)^{\ln x}$

(d) $\lim_{x\to\infty}(e^x - x^2)$

Ans:

(a) $\lim_{x\to 0^+} \frac{\int_0^{x^2}(\sin(\sqrt{t})-\sqrt{t})dt}{\int_0^{x^2}(tan(\sqrt{t})-\sqrt{t})dt} = \lim_{x\to 0^+}\frac{(\sin(\sqrt{x^2})-\sqrt{x^2})2x}{(\tan(\sqrt{x^2})-\sqrt{x^2})2x}$ (Fundamental theorem of calculus and chain rule)

$\lim_{x\to 0^+}\frac{(\sin(\sqrt{x^2})-\sqrt{x^2})2x}{(\tan(\sqrt{x^2})-\sqrt{x^2})2x} = \lim_{x\to 0^+}\frac{(\sin(x)-x)2x}{(\tan(x)-x)2x} = \lim_{x\to 0^+}\frac{\cos(x)-1}{sec^2(x)-1}$ (L' Hôpital' s rule)=

$\lim_{x\to 0^+}\frac{-\sin(x)}{2sec^2(x)tan(x)}$ (L' Hôpital' s rule)= $\lim_{x\to 0^+}\frac{-1}{2sec^2(x)}\lim_{x\to 0^+}\frac{\sin(x)}{\tan(x)} = \frac{-1}{2}$

(b) $\lim_{x\to 0^+}\tan(x)\ln(sin^2(x)) = \lim_{x\to 0^+}\frac{\ln(sin^2(x))}{cotx} = \lim_{x\to 0^+}\frac{\frac{2\sin(x)\cos(x)}{sin^2x}}{-csc^2x}$ (L' Hôpital' s rule) $= \lim_{x\to 0^+} -2\sin(x)\cos(x) = 0$

(c) $y = \lim_{x\to 1^+}(x-1)^{\ln x}$

$\ln y = \ln\lim_{x\to 1^+}(x-1)^{\ln x} = \lim_{x\to 1^+}\ln(x-1)^{\ln x} = \lim_{x\to 1^+}\ln(x)\ln(x-1) =$

$\lim_{x\to 1^+}\frac{\ln(x-1)}{\frac{1}{\ln(x)}} = \lim_{x\to 1^+}\frac{\frac{1}{x-1}}{\frac{1}{x}\left[\frac{-1}{(\ln x)^2}\right]}$ (L' Hôpital' s rule)= $\lim_{x\to 1^+}\frac{-(\ln x)^2}{\frac{x-1}{x}} =$

$\lim_{x\to 1^+}\frac{-2\frac{1}{x}\ln x}{\frac{1}{x^2}}$ (L' Hôpital' s rule) $= \lim_{x\to 1^+} -2x\ln x = 0$

Since $\ln y = 0$ Therefore, $y = 1$

(d) $\lim_{x\to\infty}(e^x - x^2) = \lim_{x\to\infty} x^2(\frac{e^x}{x^2} - 1)$

$= \lim_{x\to\infty} x^2 \lim_{x\to\infty}(\frac{e^x}{x^2} - 1)$

$= \infty \times \left[\lim_{x\to\infty}(\frac{e^x}{x^2}) - \lim_{x\to\infty}(1)\right] = \infty \times \left[\lim_{x\to\infty}(\frac{e^x}{2x}) - 1\right]$ (L' Hôpital' s rule)

$= \infty \times \left[\lim_{x\to\infty} (\frac{e^x}{2}) - 1\right]$ (L' Hôpital' s rule)

$= \infty$

2. Solve the following problems (10%):

(a) Show that $f(x) = \int_1^x \sqrt{1+t^2}\, dt$ has an inverse function

(b) Find $(f^{-1})'(0)$

Ans:

(a) Note that $f'(x) = \sqrt{1+x^2} > 0$ for all $x \to f$ is strictly increasing therefore is one to one and has an inverse function.

(b) Let $y = f^{-1}(0) \to f(y) = 0 \to y = 1$

$$f(1) = 0 \to (f^{-1})'(0) = \frac{1}{f'(1)} = \frac{1}{\sqrt{2}}$$

3. Evaluate the following integrals. (Hint: Try to use change of variables for all the problems) (15%)

(a) $\int_3^4 (4-x)6^{(4-x)^2} dx$

(b) $\int \sqrt{e^t - 3}\ \ dt$

(c) $\int_0^1 \frac{dx}{2\sqrt{3-x}\sqrt{x+1}}$

Ans:

(a) $\int_3^4 (4-x)6^{(4-x)^2} dx = \frac{-1}{2}\int_1^0 6^u du$ (Let $u = (4-x)^2, du = -2(4-x)dx$)

$$= \frac{-6^u}{2\ln 6}\Big]_1^0 = \frac{-1}{2\ln 6}(1-6) = \frac{5}{2\ln 6}$$

(b) $\int \sqrt{e^t - 3}\ \ dt = \int \frac{2u^2}{u^2+3} du$ (Let $u = \sqrt{e^t - 3} \to u^2 + 3 = e^t, 2udu = e^t dt,$

$\frac{2udu}{u^2+3} = dt)$

$$= \int 2\, du - \int 6\frac{1}{u^2+3} du = 2u - 2\sqrt{3}\tan^{-1}\frac{u}{\sqrt{3}} + C$$

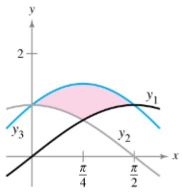
$$= 2\sqrt{e^t - 3} - 2\sqrt{3}\tan^{-1}\sqrt{\frac{e^t - 3}{3}} + C$$

(c) $\int_0^1 \frac{dx}{2\sqrt{3-x}\sqrt{x+1}} = \int_1^{\sqrt{2}} \frac{2udu}{2\sqrt{4-u^2}u}$ (Let $u = \sqrt{x+1} \to u^2 = x+1, 2udu = dx$, $\sqrt{3-x} = \sqrt{4-u^2}$)

$$= \int_1^{\sqrt{2}} \frac{du}{\sqrt{4-u^2}} = \sin^{-1}\frac{u}{2}\Big]_1^{\sqrt{2}} = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$$

4. Find the area of the given region bounded by the graph y_1, y_2 and y_3 (7%)

$$y_1 = sinx, y_2 = cosx, y_3 = cosx + sinx$$

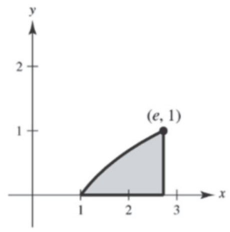


Ans:

$$A = \int_0^{\frac{\pi}{4}} (y_3 - y_2)dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (y_3 - y_1)dx = \int_0^{\frac{\pi}{4}} sinx\ \ dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} cosx\ \ dx$$

$$= -cosx\Big|_0^{\frac{\pi}{4}} + \sin x\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = 2 - \sqrt{2}$$

5. Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \ln x$, $y = 0$ and $x = \mathrm{e}$ about the x-axis. (8%)



Ans:

$$V = \pi \int_1^e (\ln x)^2 dx$$

$$dv = dx, u = (ln\, x)^2 \to v = x, du = 2\frac{1}{x} ln\, x\, dx$$

$$\int (ln\, x)^2 dx = x(ln\, x)^2 - \int x\, 2\frac{1}{x} ln\, x\, dx = x(ln\, x)^2 - 2\int ln\, x\, dx$$

$$(dv = dx, u = ln\, x \to v = x, du = \frac{1}{x}dx$$

$$\int \ln x\, dx = x \ln x - \int x \frac{1}{x} dx = x \ln x - x)$$

$$\int (ln\, x)^2 dx = x(ln\, x)^2 - 2\int ln\, x\, dx = x(ln\, x)^2 - 2x \ln x + 2x$$

$$\mathrm{V} = \pi \int_1^e (\ln x)^2 dx = \pi(x(ln\, x)^2 - 2x \ln x + 2x)]_1^e = \pi(e-2)$$

6. Use the sehll method to find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the y-axis. (6%)

$$y = \frac{1}{x^2}, y = 0, x = 2, x = 5$$

Ans:

$$\mathrm{V} = 2\pi \int_2^5 x\left(\frac{1}{x^2}\right) dx = 2\pi \int_2^5 \frac{1}{x} dx = 2\pi \ln|x|]_2^5 = 2\pi \ln\frac{5}{2}$$

7. Find the arc length of the graph of the function $y = \ln(1-x^2)$ on the interval $0 \le x \le \frac{1}{3}$. (9%)

Ans:

$$\mathrm{y} = \ln(1-x^2), y' = \frac{-2x}{(1-x^2)}$$

Arc length $= \int_0^{\frac{1}{3}} \sqrt{1+(y')^2}\, dx = \int_0^{\frac{1}{3}} \frac{1+x^2}{1-x^2} dx = \int_0^{\frac{1}{3}} (-1 + \frac{1}{x+1} + \frac{1}{1-x})\, dx =$

$-x + \ln(1+x) - \ln(1-x)]_0^{\frac{1}{3}} = \ln 2 - \frac{1}{3}$

8. Evaluate the following integrals. (25%)

(a) $\int \frac{\ln x}{x^3} dx$

(b) $\int sin^2 x cos^3 x dx$

(c) $\int \frac{1}{\sqrt{1+\sqrt{x}}} dx$

(d) $\int \frac{1}{1+\tan(\theta)} d\theta$

(e) $\int_1^4 \frac{1}{(x-2)^2} dx$

Ans:

(a) Let $u = \ln x, dv = x^{-3}dx \rightarrow du = \frac{1}{x}dx, v = \frac{-1}{2}x^{-2}$

$$\int \frac{\ln x}{x^3} dx = \frac{-1}{2}x^{-2}\ln x - \int \left(\frac{-1}{2}x^{-2}\right)\frac{1}{x}dx = \frac{-1}{2}x^{-2}\ln x + \frac{1}{2}\int x^{-3}dx$$

$$= \frac{-1}{2}x^{-2}\ln x + \frac{1}{2}\frac{x^{-2}}{-2} + C = \frac{-1}{2}\frac{\ln x}{x^2} - \frac{1}{4x^2} + C$$

(b) $\int sin^2(x)cos^3(x)dx = \int sin^2(x)cos^2(x)cos(x)dx = \int sin^2(x)[1 - sin^2(x)]cos(x)dx = \int u^2(1-u^2)du$ (Let $u = sin(x), du = cos(x)dx$)

$$= \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{sin^3(x)}{3} - \frac{sin^5(x)}{5} + C$$

(c) Let $x = tan^4(\theta) \rightarrow dx = 4tan^3(\theta)sec^2(\theta)d\theta$

$$\int \frac{1}{\sqrt{1+\sqrt{x}}} dx = \int \frac{4tan^3(\theta)sec^2(\theta)d\theta}{\sec(\theta)} = 4\int tan^3(\theta)\sec(\theta)d\theta$$

Let $u = \sec(\theta) \rightarrow \quad du = \sec(\theta)\tan(\theta)d\theta$

$4\int tan^3(\theta)\sec(\theta)d\theta = 4\int(u^2-1)du = \frac{4}{3}u^3 - 4u + C = \frac{4}{3}sec^3\theta -$

$4\sec(\theta) + C = \frac{4}{3}\sqrt{1+\sqrt{x}}(1+\sqrt{x}-3) + \mathrm{C} = \frac{4}{3}(\sqrt{x}-2)(\sqrt{1+\sqrt{x}}) + C$

(d) Let $\mathrm{u} = \tan(\theta), \mathrm{du} = sec^2(\theta), du = (1+u^2)d\theta$

$$\int \frac{1}{1+\tan(\theta)} d\theta = \int \frac{1}{(1+u)(1+u^2)} du = \frac{1}{2}\int \frac{1}{1+u} + \frac{1-u}{1+u^2} du$$

$$= \frac{1}{2}\left[\ln|1+u| - \frac{1}{2}\ln|1+u^2| + \tan^{-1}u\right] + C$$

$$= \frac{1}{2}\left[\ln|1+\tan(\theta)| - \frac{1}{2}\ln|1+tan^2(\theta)| + \tan^{-1}\tan(\theta)\right] + C$$

$$= \frac{1}{2}[\ln|1+\tan(\theta)| + \ln|\cos(\theta)| + \theta] + C$$

$$= \frac{1}{2}[\ln|\cos(\theta)+\sin(\theta)| + \theta] + C$$

(e) $\int_1^4 \frac{1}{(x-2)^2} dx = \int_1^2 \frac{1}{(x-2)^2} dx + \int_2^4 \frac{1}{(x-2)^2} dx = \lim_{c\to 2^-} \int_1^c \frac{1}{(x-2)^2} dx + \lim_{c\to 2^+} \int_c^4 \frac{1}{(x-2)^2} dx$

$$\int \frac{1}{(x-2)^2}\,dx = \frac{-1}{(x-2)} + C$$

$$\lim_{c\to 2^-}\int_1^c \frac{1}{(x-2)^2}\,dx + \lim_{c\to 2^+}\int_c^4 \frac{1}{(x-2)^2}\,dx = \lim_{c\to 2^-} \left.\frac{-1}{(x-2)}\right]_1^c + \lim_{c\to 2^+} \left.\frac{-1}{(x-2)}\right]_c^4 \text{ is diverge}$$