1. Limits

(a)

$$\lim_{x\to 4}\frac{x^2-16}{|x-4|}=\lim_{x\to 4}\frac{(x-4)(x+4)}{|x-4|}=\begin{cases}-(x+4), & x\to 4^-\\ +(x+4), & x\to 4^+\end{cases}=\begin{cases}-8, & x\to 4^-\\ 8, & x\to 4^+\end{cases}$$

Since the left and right limits differ, the limit does not exist.

(b)

$$\lim_{x\to-\infty}\left(\sqrt{4x^2-2x}+2x\right)$$

Multiply by the conjugate:

$$=\lim_{x\to-\infty}\frac{\left(\sqrt{4x^2-2x}+2x\right)\left(\sqrt{4x^2-2x}-2x\right)}{\sqrt{4x^2-2x}-2x}$$

$$=\lim_{x\to-\infty}\frac{(4x^2-2x)-4x^2}{\sqrt{4x^2-2x}-2x}=\lim_{x\to-\infty}\frac{-2x}{\sqrt{4x^2-2x}-2x}$$

Factor out $|x|=-x$ since $x\to-\infty$:

$$=\lim_{x\to-\infty}\frac{-2x}{|x|\sqrt{4-\frac{2}{x}}-2x}=\lim_{x\to-\infty}\frac{-2x}{(-x)\sqrt{4-\frac{2}{x}}-2x}$$

$$=\lim_{x\to-\infty}\frac{-2x}{-x\left(\sqrt{4-\frac{2}{x}}+2\right)}=\lim_{x\to-\infty}\frac{-2}{-\left(\sqrt{4-\frac{2}{x}}+2\right)}$$

$$=\lim_{x\to-\infty}\frac{2}{\sqrt{4-\frac{2}{x}}+2}=\frac{2}{\sqrt{4}+2}=\frac{2}{2+2}=\frac{1}{2}$$

(c)

$$\lim_{x\to 0}x(\cos 2x+\cos(1/x))$$

Since

$$-1\le\cos 2x\le 1,\quad -1\le\cos(1/x)\le 1$$

we have

$$-2|x|\le x(\cos 2x+\cos(1/x))\le 2|x|\Rightarrow\lim_{x\to 0}x(\cos 2x+\cos(1/x))=0$$

(d)

$$\lim_{x\to 1}\frac{x^2+5x-6}{x^2-1}=\lim_{x\to 1}\frac{(x-1)(x+6)}{(x-1)(x+1)}=\lim_{x\to 1}\frac{x+6}{x+1}=\frac{7}{2}$$

2. Differentiability of a piecewise function

Let

$$f(x) = \begin{cases} x^2 - a, & x \geq 2, \\ bx + 6, & x < 2. \end{cases}$$

To make $f(x)$ differentiable at $x = 2$, we require

Continuity at $x = 2$:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) \Rightarrow 2b + 6 = 4 - a \Rightarrow a + 2b = -2$$

Derivative continuity:

$$f'(x) = \begin{cases} 2x, & x > 2, \\ b, & x < 2 \end{cases} \Rightarrow f'(2) = 2 \cdot 2 = 4 = b$$

$$b = 4 \Rightarrow a + 2(4) = -2 \Rightarrow a = -10$$

$$\boxed{a = -10, \quad b = 4}$$

3. Verification of IVT, MVT and uniqueness of solution

Consider the function

$$f(x) = x^5 + x + \frac{3}{2}.$$

Verification of IVT

The Intermediate Value Theorem (IVT) states that if a function is continuous on a closed interval $[a, b]$, then for any value L between $f(a)$ and $f(b)$, there exists a $c \in (a, b)$ such that $f(c) = L$.

Since $f(x)$ is a polynomial, and all polynomials are continuous on $\mathbb{R}$, we conclude that f is continuous on any interval $[a, b]$. Therefore, f satisfies the hypotheses of the IVT.

Verification of MVT

The Mean Value Theorem (MVT) requires:

- f is continuous on $[a, b]$
- f is differentiable on (a, b)

As noted above, f is continuous everywhere. Since f is a polynomial, it is also differentiable on $\mathbb{R}$, hence differentiable on (a, b).

Thus, f satisfies the hypotheses of the Mean Value Theorem on any interval $[a, b]$.

$$\Rightarrow \text{IVT and MVT both apply to } f(x).$$

Existence of a real root

Evaluate $f(x)$ at two points:

$$f(-1) = (-1)^5 + (-1) + \frac{3}{2} = -1 - 1 + \frac{3}{2} = -\frac{1}{2} < 0, \quad f(0) = 0 + 0 + \frac{3}{2} = \frac{3}{2} > 0.$$

Since $f(-1) < 0 < f(0)$ and f is continuous, by the IVT, there exists $c \in (-1, 0)$ such that $f(c) = 0$. Hence, $f(x)$ has at least one real root.

Uniqueness of the real root

To show uniqueness, examine the derivative:

$$f'(x) = 5x^4 + 1.$$

Since $5x^4 \geq 0$ for all real x and $1 > 0$, we have

$$f'(x) > 0 \quad \forall x \in \mathbb{R}.$$

Thus, f is strictly increasing on $\mathbb{R}$. A strictly increasing function can cross the x-axis at most once. Since we already established the existence of a root in $(-1, 0)$, this root must be unique.

$$\boxed{f(x) \text{ has exactly one real root}}.$$

4. Derivatives and Applications

(a)

$$\lim_{x \to 1} \frac{\frac{x}{\sqrt{x^2+1}} - \frac{1}{\sqrt{2}}}{x - 1}$$

Let

$$g(x) = \frac{x}{\sqrt{x^2 + 1}}.$$

Then the limit equals $g'(1)$.

$$g'(x) = \frac{\sqrt{x^2+1} - x \cdot \frac{x}{\sqrt{x^2+1}}}{x^2+1} = \frac{(x^2+1) - x^2}{(x^2+1)^{3/2}} = \frac{1}{(x^2+1)^{3/2}}$$

Thus,

$$g'(1) = \frac{1}{(1+1)^{3/2}} = \frac{1}{2\sqrt{2}}.$$

$$\boxed{\frac{1}{2\sqrt{2}}}$$

(b)

$$f(x) = x^3 \sec\left(\frac{1}{x^2}\right)$$

Let $u = x^{-2}$, then

$$f'(x) = 3x^2 \sec\left(\frac{1}{x^2}\right) + x^3 \sec(u)\tan(u) \cdot u'$$

$$u' = \frac{d}{dx}(x^{-2}) = -2x^{-3}$$

$$f'(x) = 3x^2 \sec\left(\frac{1}{x^2}\right) - 2\sec\left(\frac{1}{x^2}\right)\tan\left(\frac{1}{x^2}\right)$$

$$\boxed{f'(x) = 3x^2 \sec\left(\frac{1}{x^2}\right) - 2\sec\left(\frac{1}{x^2}\right)\tan\left(\frac{1}{x^2}\right)}$$

(c) We have the implicit equation

$$3xy + \sin x = 2.$$

Differentiating both sides implicitly with respect to x:

$$\frac{d}{dx}(3xy) + \frac{d}{dx}(\sin x) = 0.$$

Using the product rule on $3xy$:

$$3\left(x\frac{dy}{dx} + y\right) + \cos x = 0$$

$$3x\frac{dy}{dx} + 3y + \cos x = 0$$

Solving for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{3y + \cos x}{3x}.$$

$$\boxed{\frac{dy}{dx} = -\frac{(3y + \cos x)}{3x}}$$

Now differentiate again to obtain the second derivative. Differentiating both sides:

$$\frac{d}{dx}\left(3x\frac{dy}{dx} + 3y + \cos x\right) = 0.$$

Apply product rule to $3x\frac{dy}{dx}$ and chain rule to $\cos x$:

$$3\left(x\frac{d^2y}{dx^2} + \frac{dy}{dx}\right) + 3\frac{dy}{dx} - \sin x = 0.$$

Combine like terms:

$$3x\frac{d^2y}{dx^2} + 6\frac{dy}{dx} - \sin x = 0.$$

Solve for $\frac{d^2y}{dx^2}$:

$$3x\frac{d^2y}{dx^2} = \sin x - 6\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{\sin x - 6\frac{dy}{dx}}{3x}$$

Substitute $\frac{dy}{dx} = -\frac{(3y + \cos x)}{3x}$:

$$\frac{d^2y}{dx^2} = \frac{\sin x - 6\left(-\frac{3y+\cos x}{3x}\right)}{3x}$$

$$= \frac{\sin x + \frac{6(3y+\cos x)}{3x}}{3x} = \frac{\sin x + \frac{2(3y+\cos x)}{x}}{3x}$$

Write with common denominator:

$$= \frac{x\sin x + 6y + 2\cos x}{3x^2}$$

Thus,

$$\boxed{\frac{d^2y}{dx^2} = \frac{x\sin x + 6y + 2\cos x}{3x^2}}$$

This expression contains only x and y (no $\frac{dy}{dx}$ term), as desired.

(d)

$$f(x) = x^3 - \sqrt{x}, \quad (1,0)$$

$$f'(x) = 3x^2 - \frac{1}{2\sqrt{x}} \Rightarrow f'(1) = 3 - \frac{1}{2} = \frac{5}{2}$$

Equation of tangent line:

$$y - 0 = \frac{5}{2}(x-1) \quad \Rightarrow \quad \boxed{y = \frac{5}{2}(x-1)}$$

5. Curve Analysis of $f(x) = \dfrac{(x+1)^2}{x^2+1}$

Given

$$f(x) = \frac{(x+1)^2}{x^2+1}.$$

(a) Critical numbers and possible inflection points

Compute the first derivative:

$$f'(x) = \frac{2(x+1)(1-x)}{(x^2+1)^2} = \frac{2(1-x^2)}{(x^2+1)^2}.$$

Set $f'(x) = 0 \Rightarrow 1 - x^2 = 0 \Rightarrow x = \pm 1$.
Evaluate the function at these points:

$$f(-1) = \frac{(0)^2}{(-1)^2+1} = 0, \qquad f(1) = \frac{(2)^2}{1^2+1} = 2.$$

$$\boxed{\text{Critical points: } (-1,0),\ (1,2)}$$

Second derivative:

$$f''(x) = \frac{4x(x^2-3)}{(x^2+1)^3}.$$

Set numerator $= 0$:

$$4x(x^2-3)=0 \Rightarrow x=0,\ x=\pm\sqrt{3}.$$

Evaluate $f(x)$ at these points:

$$f(0)=\frac{1}{1}=1,$$

$$f(\pm\sqrt{3})=\frac{(\sqrt{3}+1)^2}{(\sqrt{3})^2+1}=\frac{4+2\sqrt{3}}{4}=1+\frac{\sqrt{3}}{2}\quad \text{(for } x=\sqrt{3}\text{)}$$

$$f(-\sqrt{3})=\frac{(-\sqrt{3}+1)^2}{4}=\frac{4-2\sqrt{3}}{4}=1-\frac{\sqrt{3}}{2}$$

Thus the possible inflection points are

$$\boxed{(-\sqrt{3},\ 1-\frac{\sqrt{3}}{2}),\quad (0,1),\quad (\sqrt{3},\ 1+\frac{\sqrt{3}}{2})}$$

(b) Increasing / decreasing intervals

Since

$$f'(x)=\frac{2(1-x^2)}{(x^2+1)^2},$$

the sign depends on $1-x^2$:

$$\begin{cases} (-\infty,-1): & 1-x^2<0 \Rightarrow f'<0 \Rightarrow \text{decreasing,} \\ (-1,1): & 1-x^2>0 \Rightarrow f'>0 \Rightarrow \text{increasing,} \\ (1,\infty): & 1-x^2<0 \Rightarrow f'<0 \Rightarrow \text{decreasing.} \end{cases}$$

$$\boxed{\text{Inc: } (-1,1),\quad \text{Dec: } (-\infty,-1),(1,\infty)}$$

(c) Concavity

$$f''(x)=\frac{4x(x^2-3)}{(x^2+1)^3}$$

Sign chart based on x and x^2-3:

$$\begin{cases} (-\infty, -\sqrt{3}): & f'' < 0 \Rightarrow \text{concave down}, \\ (-\sqrt{3}, 0): & f'' > 0 \Rightarrow \text{concave up}, \\ (0, \sqrt{3}): & f'' < 0 \Rightarrow \text{concave down}, \\ (\sqrt{3}, \infty): & f'' > 0 \Rightarrow \text{concave up}. \end{cases}$$

(d) Asymptotes.

Since

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{(x+1)^2}{x^2+1} = 1,$$

the horizontal asymptote is

$$y = 1.$$

A vertical asymptote occurs where the denominator equals zero. However,

$$x^2 + 1 \neq 0 \quad \text{for all } x \in \mathbb{R},$$

so there are no vertical asymptotes.

A slant asymptote exists only when the degree of the numerator is exactly one greater than the degree of the denominator. Here both degrees are 2, so no slant asymptote exists.

(e) Key points and graph sketch

Inflection points occur at

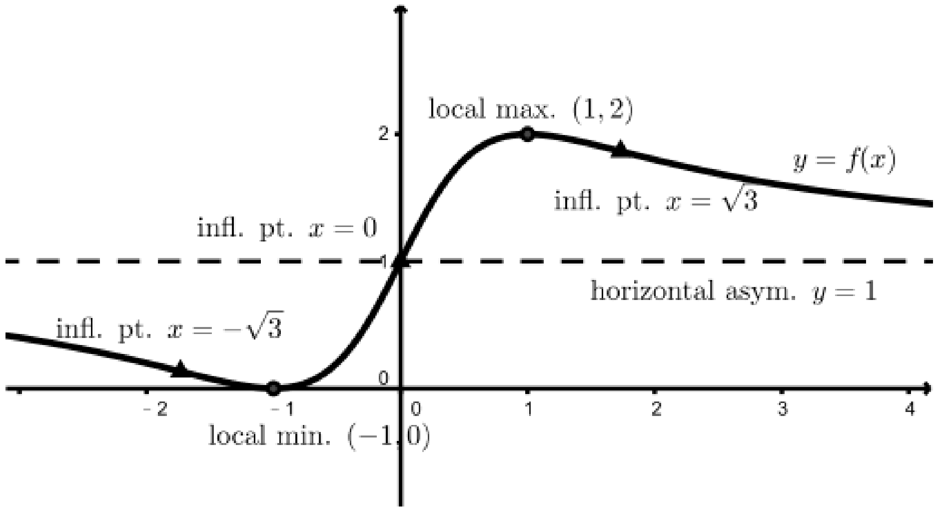
$$(-\sqrt{3},\, 1 - \frac{\sqrt{3}}{2}), \qquad (0, 1), \qquad (\sqrt{3},\, 1 + \frac{\sqrt{3}}{2}).$$

The function is increasing on $(-1, 1)$ and decreasing on $(-\infty, -1)$ and $(1, \infty)$.

The graph is concave down on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$, and concave up on $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$.

Y-intercept: $(0, 1)$,X-intercept: $(-1, 0)$

A horizontal asymptote occurs at $y = 1$.



圖表 1: Graph of $f(x) = \frac{(x+1)^2}{x^2+1}$

(f) Domain and Range

To determine the domain, note that the denominator is

$$x^2 + 1,$$

which is always strictly positive for all real numbers. Therefore, the function is defined for every real x.

$$\boxed{\text{Domain} = \mathbb{R}}$$

To find the range, observe the behavior of $f(x)$ and its critical points.

The critical points are

$$(-1, 0) \quad \text{and} \quad (1, 2),$$

corresponding to a local (and global) minimum and a local (and global) maximum, respectively.

Thus,

$$0 \leq f(x) \leq 2.$$

Furthermore, since

$$\lim_{x \to \pm\infty} f(x) = 1,$$

the function approaches but never exceeds these bounds outside the interval containing the critical points.

Therefore, the range is

$$\boxed{\text{Range} = [0, 2]}.$$

6. Maximum Area Rectangle in an Ellipse

We wish to find the rectangle with maximum area that can be inscribed in the ellipse

$$\frac{x^2}{144} + \frac{y^2}{16} = 1,$$

where the rectangle's sides are parallel to the coordinate axes.

By symmetry, it suffices to consider the point (x, y) in the first quadrant; the rectangle will have vertices $(\pm x, \pm y)$, so its area is

$$A = 4xy.$$

From the ellipse equation, solve for y:

$$\frac{x^2}{144} + \frac{y^2}{16} = 1 \quad \Rightarrow \quad y^2 = 16\left(1 - \frac{x^2}{144}\right) \quad \Rightarrow \quad y = 4\sqrt{1 - \frac{x^2}{144}}.$$

Thus the area becomes a function of x:

$$A(x) = 4x \cdot 4\sqrt{1 - \frac{x^2}{144}} = 16x\sqrt{1 - \frac{x^2}{144}}.$$

To maximize $A(x)$, differentiate. Let

$$u = 1 - \frac{x^2}{144}, \quad A(x) = 16x\sqrt{u}.$$

Then

$$A'(x) = 16\sqrt{u} + 16x \cdot \frac{1}{2\sqrt{u}} \cdot \left(-\frac{2x}{144}\right) = 16\sqrt{u} - \frac{16x^2}{144\sqrt{u}}.$$

Set $A'(x) = 0$:

$$16\sqrt{u} = \frac{16x^2}{144\sqrt{u}} \quad \Rightarrow \quad 144u = x^2 \quad \Rightarrow \quad 144\left(1 - \frac{x^2}{144}\right) = x^2$$

$$144 - x^2 = x^2 \quad \Rightarrow \quad 2x^2 = 144 \quad \Rightarrow \quad x^2 = 72 \quad \Rightarrow \quad x = 6\sqrt{2}.$$

Then

$$y = 4\sqrt{1 - \frac{72}{144}} = 4\sqrt{\frac{1}{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}.$$

Thus the rectangle dimensions are

$$\text{Width} = 2x = 12\sqrt{2}, \qquad \text{Height} = 2y = 4\sqrt{2}.$$

$$\boxed{\text{Width} = 12\sqrt{2}}, \qquad \boxed{\text{Height} = 4\sqrt{2}}$$

7. Newton's Method

We seek the x-value where the graphs of

$$f(x) = 1 - x \quad \text{and} \quad g(x) = x^5 + 2$$

intersect. This is equivalent to solving

$$h(x) = f(x) - g(x) \quad \Longleftrightarrow \quad h(x) = 1 - x - (x^5 + 2) \quad \Longleftrightarrow \quad h(x) = -x^5 - x - 1$$

Let

$$h(x) = -x^5 - x - 1, \qquad h'(x) = -5x^4 - 1.$$

Newton's iteration formula:

$$x_{n+1} = x_n - \frac{h(x_n)}{h'(x_n)}.$$

Start with the initial guess $x_1 = -1$:

$$\begin{aligned}
x_1 &= -1, \\
x_2 &= x_1 - \frac{h(x_1)}{h'(x_1)} = -1 - \frac{-1-1-1}{5(1)+1} = -1 - \frac{-3}{6} = -0.8333, \\
x_3 &= -0.8333 - \frac{h(-0.8333)}{h'(-0.8333)} \approx -0.7644, \\
x_4 &= -0.7644 - \frac{h(-0.7644)}{h'(-0.7644)} \approx -0.7550,
\end{aligned}$$

Since

$$|x_4 - x_3| < 0.01,$$

the iteration stops.

Thus, the intersection point occurs approximately at

$$\boxed{x \approx -0.7550}.$$

8. Differential Approximation

We use differentials to approximate

$$\sqrt{63.9}.$$

Let

$$f(x) = \sqrt{x}, \qquad f'(x) = \frac{1}{2\sqrt{x}}.$$

Choose a nearby value $a = 64$ for which $f(a)$ is known:

$$f(64) = 8.$$

Then

$$dx = 63.9 - 64 = -0.1.$$

The differential approximation gives

$$f(63.9) \approx f(64) + f'(64)\, dx.$$

Compute $f'(64)$:

$$f'(64) = \frac{1}{2\sqrt{64}} = \frac{1}{2 \cdot 8} = \frac{1}{16}.$$

Thus,

$$f(63.9) \approx 8 + \frac{1}{16}(-0.1) = 8 - \frac{0.1}{16} = 8 - 0.00625 = 7.99375.$$

$$\boxed{\sqrt{63.9} \approx 7.99375}$$