If the limit does not exist or has an infinite limit, you should point it out. In addition, do not use the L'Hôpital's rule to solve the limit problem.

1. (16%) Find the following limit

   (a) $\lim_{x\to 2}\frac{2x^3-3x^2-3x+2}{x^2+x-6}$

   (b) $\lim_{x\to 0}\frac{(\sqrt{16+x}-4)}{x}$

   (c) $\lim_{x\to\infty}\sqrt{3x^2+1}\tan\frac{1}{x}$

   (d) $\lim_{x\to 0} x\sqrt{1+\frac{4}{x^2}}$

Ans:

(a) $\lim_{x\to 2}\frac{2x^3-3x^2-3x+2}{x^2+x-6}=\lim_{x\to 2}\frac{(x+1)(x-2)(2x-1)}{(x-2)(x+3)}=\lim_{x\to 2}\frac{(x+1)(2x-1)}{(x+3)}=\frac{(2+1)(2\times 2-1)}{(2+3)}=\frac{9}{5}$

(b) $\lim_{x\to 0}\frac{(\sqrt{16+x}-4)}{x}=\lim_{x\to 0}\frac{\sqrt{16+x}-4}{x}\frac{\sqrt{16+x}+4}{\sqrt{16+x}+4}=\lim_{x\to 0}\frac{16+x-16}{x(\sqrt{16+x}+4)}$

$$=\lim_{x\to 0}\frac{1}{\sqrt{16+x}+4}=\frac{1}{8}$$

(c) $\lim_{x\to\infty}\sqrt{3x^2+1}\tan\frac{1}{x}=\lim_{x\to\infty}\frac{\sqrt{3x^2+1}}{x}x\tan\frac{1}{x}=$ (Let $t=\frac{1}{x}$)

$$\lim_{x\to\infty}\frac{\sqrt{3x^2+1}}{x}\lim_{t\to 0^+}\frac{\tan(t)}{t}=\lim_{x\to\infty}\sqrt{3+\frac{1}{x^2}}=\sqrt{3}$$

(d) $\lim_{x\to 0^+} x\sqrt{1+\frac{4}{x^2}}=\lim_{x\to 0^+}\sqrt{x^2(1+\frac{4}{x^2})}=\lim_{x\to 0^+}\sqrt{x^2+4}=2$ and $\lim_{x\to 0^-} x\sqrt{1+\frac{4}{x^2}}=$

$\lim_{x\to 0^-}-\sqrt{x^2(1+\frac{4}{x^2})}=\lim_{x\to 0^-}-\sqrt{x^2+4}=-2$. Therefore, the limit does not exist!

2. (10%) Assume the following function is a differentiable function

$$f(x)=\begin{cases} x^2 sin(\frac{1}{x}), & x>0 \\ ax+b, & x\le 0\end{cases}$$

What is the value of a and b?

Ans:

(a) Since $-1 \le \sin(\frac{1}{x}) \le 1$ for all $x \ne 0$, $-x^2 \le x^2\sin(\frac{1}{x}) \le x^2$ for all $x \ne 0$

Furthermore $\lim_{x\to 0^+} x^2 = \lim_{x\to 0^+} -x^2 = 0$. According to the squeeze therorem

$\lim_{x\to 0^+} x^2\sin(\frac{1}{x}) = 0$.

Sinc the function is differentiable thus it is continuous, we have $\lim_{x\to 0^+} f(x) =$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} ax + b = 0 \quad \to b = 0$

(b) Considering the alternative form of derivative:

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{x^2 sin(\frac{1}{x}) - 0}{x} \lim_{x\to 0^+} x \sin\left(\frac{1}{x}\right)$$

Since $-1 \le \sin(\frac{1}{x}) \le 1$ for all $x \ne 0$, $-|x| \le xsin(\frac{1}{x}) \le |x|$ for all $x \ne 0$

Furthermore $\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} -|x| = 0$. According to the squeeze therorem

$\lim_{x\to 0^+} x\sin(\frac{1}{x}) = 0$

Sinc the function is differentiable, we have $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} =$

$\lim_{x\to 0^-} \frac{ax+b}{x} = 0 \quad \to a = 0$

3. (9%) Assume $f(1) = 8$ and $\forall x \in (1,4)$ we have $f'(x) \ge 2$. What is the minimum possible value for $f(4)$ (Hint: use the mean value theorem)

Ans:

According to the Mean value theorem, $\exists\ c \in (1,4)$ such that $f'(c) = \frac{f(4)-f(1)}{4-1} =$

$\frac{f(4)-8}{3}$. From the problem we know that $f'(c) = \frac{f(4)-8}{3} \ge 2 \to f(4) \ge 14$

Therefore, the minimum possible value for $f(4)$ is 14.

4. (12%) Remember that you can solve the derivative using the definition or the differentiation rule for the following question.

(a) Let $f(x) = \frac{x(x-2)(x-3)(x-4)}{(x+2)(x+3)(x+4)}$, find $f'(2)$.

(b) Find the derivative of $f(x) = 2csc^2(\pi x)$

(c) Let $x^3 + y^3 = 2$, find the value of $\frac{d^2y}{dx^2}$ when $x = 1$

Ans:

(a) $f'(2) = \lim_{x\to 2}\frac{f(x)-f(2)}{x-2} = \lim_{x\to 2}\frac{\frac{x(x-2)(x-3)(x-4)}{(x+2)(x+3)(x+4)}-0}{x-2} = \lim_{x\to 2}\frac{x(x-3)(x-4)}{(x+2)(x+3)(x+4)} = \frac{4}{120} = \frac{1}{30}$

(b) $f'(x) = 2 \times 2\,csc(\pi x) \times -csc(\pi x)\,cot(\pi x) \times \pi = -4\pi csc^2(\pi x)cot(\pi x)$

(c) Differentiate $x^3 + y^3 = 2$ with respect to x. We have $3x^2 + 3y^2\frac{dy}{dx} = 0 \rightarrow$

$\frac{dy}{dx} = \frac{-x^2}{y^2}$. When $x = 1 \rightarrow\ y = 1$, $\frac{dy}{dx} = \frac{-x^2}{y^2} = -1$.

$$\frac{d^2y}{dx^2} = \frac{-2xy^2 + x^2 2y\frac{dy}{dx}}{y^4}$$

When $x = 1, y = 1$ we have $\frac{d^2y}{dx^2} = \frac{-2+2(-1)}{1} = -4$.

5. (20%) Let $f(x) = x^2 + \frac{1}{x}$

(a) Find the critical numbers and the possible points of inflection of $f(x)$
(b) Find the open intervals on which f is increasing or decreasing
(c) Find the open intervals of concavity
(d) Find all the asymptotes (Vertical/horizontal/Slant)
(e) Sketch the graph of $f(x)$ (Label any intercepts, relative extrema, points of inflection,and asymptotes)

Ans: Note that the original function is undefined at $x = 0$, therefore we should include it in the following table.

$f(x) = x^2 + \frac{1}{x} = \frac{x^3+1}{x}, f'(x) = 2x - \frac{1}{x^2} = \frac{2x^3-1}{x^2}$

$$f''(x) = 2 + \frac{2}{x^3} = \frac{2(x^3+1)}{x^3}$$

	$(-\infty,-1)$	$(-1,0)$	$(0,\frac{1}{\sqrt[3]{2}})$	$(\frac{1}{\sqrt[3]{2}},\infty)$
測試值	-2	$\frac{-1}{2}$	$\frac{1}{2}$	1
f'的正負號	-	-	-	+
f''的正負號	+	-	+	+
結論	遞減/向上凹	遞減/向下凹	遞減/向上凹	遞增/向上凹

(a) Note that x is not define at $x = 0$, we should not include it in the critical numbers or possible points of inflection

The critical numbers are $x = \frac{1}{\sqrt[3]{2}}$ $(f' = 0)$

Possible points of inflection: $x = -1$ $(f'' = 0)$

(b) Increasing $(\frac{1}{\sqrt[3]{2}},\infty)$. Decreasing $(-\infty,0),(0,\frac{1}{\sqrt[3]{2}})$.

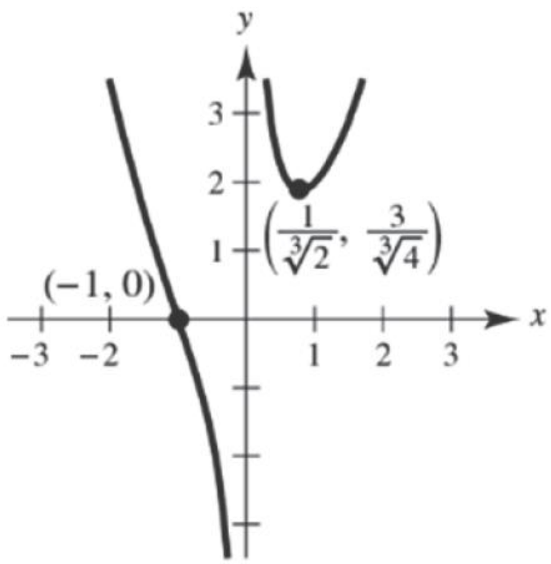
(c) Upward: $(-\infty,-1),(0,\infty)$. Downward $(-1,0)$

(d) Since $\lim_{x\to\pm\infty} f(x) = \infty$ $\to$ No horizontal asymptote

Since $\lim_{x\to-0^+} f(x) = \infty$ and $\lim_{x\to-0^-} f(x) = -\infty$ vertical asymptote at $x = 0$

There is no slant asymptote

(e) Graph



There is a local minimum at $x = \frac{1}{\sqrt[3]{2}}$ and an inflection point at (-1,0)

6. (9%) Find a point on the graph $x = \sqrt{10y}$ that is closetst to point (0,4). (Becarefull about the domain)

Ans: The distance between (0,4) and a point (x, y) on the graph of $\sqrt{10y}$ is

$$d = \sqrt{(x-0)^2 + (y-4)^2} = \sqrt{10y + (y-4)^2}$$

Minimize $d^2 = f(y) = 10y + (y-4)^2$, $y \geq 0$. Note that $f'(y) = 10 + 2(y-4) = 2y + 2$. The only critical point is $y = -1$. However, since it is ouside the domain, so the minimum should occur in the end point where $y = 0$. The point is thus $(0,0)$.

7. (9%) Use differential to approximat $\tan(46°)$

Ans: $46°$ is $\frac{\pi}{4} + \frac{\pi}{180}$. Let $f(x) = tan(x), f'(x) = sec^2(x)$

$$f(x + \Delta x) \approx f(x) + f'(x)dx = tan(x) + sec^2(x)dx$$

Choosing $x = \frac{\pi}{4}$ and $dx = \frac{\pi}{180}$.

$$f(x + \Delta x) = \tan(46°) \approx tan\left(\frac{\pi}{4}\right) + sec^2\left(\frac{\pi}{4}\right)\frac{\pi}{180} = 1 + \frac{\pi}{90}$$

8. (15%) Remember the meaning and the definition of definite integral when solving the following question

(a) $\int 3 + cot^2(t)dt$

(b) $\int_0^5 5 - |x-5|dx$

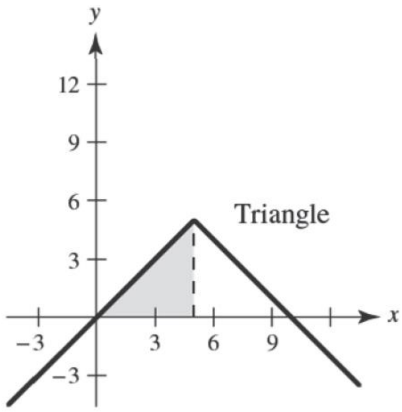
(c) $\lim_{n\to\infty} 2(\frac{1+2+\cdots+n}{n^2})$

Ans:

(a) $\int 3 + cot^2(t)dt = \int 2 + 1 + cot^2(t)dt = \int 2 + csc^2(t)dt = 2t - cot(t) + C$

(b) $\int_0^5 5 - |x-5|dx = \frac{1}{2} \times 5 \times 5 = \frac{25}{2}$

Note the area can be considered as the area in the following graph:



(c) $\lim_{n\to\infty} 2\left(\frac{1+2+\cdots+n}{n^2}\right) = \lim_{n\to\infty} \frac{2}{n}\left(\frac{1+2+\cdots+n}{n}\right) = \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n} \frac{i}{n} = 2\int_0^1 x\,dx = 1$