1. (12%) Determine the following limit.

(a) (3%)
$$\lim_{x \to 1} \frac{x^2 + 2x - 3}{x^2 - 1}$$

(b) (3%) $\lim_{x \to 2} \frac{x^2 - 4}{|x - 2|}$
(c) (3%) $\lim_{x \to 0} \frac{3}{1 + \frac{2}{x}}$
(d) (3%) $\lim_{x \to \infty} \frac{\cos(\pi x)}{x + 1}$

Ans:

(a)
$$\lim_{x \to 1} \frac{x^2 + 2x - 3}{x^2 - 1} = \lim_{x \to 1} \frac{(x + 3)(x - 1)}{(x + 1)(x - 1)} = \lim_{x \to 1} \frac{(x + 3)}{(x + 1)} = 2$$

(b)
$$\lim_{x \to 2^+} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2^+} x + 2 = 4 \text{ and } \lim_{x \to 2^+} \frac{x^2 - 4}{-(x - 2)} = \lim_{x \to 2^-} -(x + 2) = -4$$

Therefore, the limit does not exist.
(c)
$$\lim_{x \to 0} \frac{1}{1 + x^2} = \lim_{x \to 0} \frac{3x}{x + 2} = \lim_{x \to 0} \frac{3x}{x + 2} = 0$$

(d) Since $\frac{-1}{x + 1} \le \frac{\cos(\pi x)}{x + 1} \le \frac{1}{x + 1}$ and $\lim_{x \to 1^-} \frac{1}{x + 1} = 0 = \lim_{x \to \infty} \frac{-1}{x + 1}$. By the squeeze theorem, $\lim_{x \to 0^+} \frac{\cos(\pi x)}{x + 1} = 0$.
2. (6%)
If $f(x)$ and $g(x)$ are both continuous function with $\lim_{x \to 2^-} [3f(x) + g(x)] = 4$ and $\lim_{x \to 2^+} [f(x) - 2g(x)] = 6$. Find
(a) (2%) $\lim_{x \to 2^+} f(x)$ (b) (2%) $g(2)$ (c) (2%) $\lim_{x \to 2^-} f(x)g(x)$
Ans:
Since $f(x)$ and $g(x)$ are continuous at $x = 2$, we have:
 $\lim_{x \to 2^+} f(x) = f(2)$ and $\lim_{x \to 2^+} g(x) = g(2)$
From the given limits, let $L = f(2)$ and $M = g(2)$
 $3L + M = 4, L - 2M = 6 \rightarrow L = 2, M = -2$
(a) $\lim_{x \to 2^+} f(x) = L = 2$
(b) $g(2) = M = -2$
(c) $\lim_{x \to 2^-} f(x)g(x) = -4$
3. (10%)
Let $f(x) = {\sin(3x) \text{ for } x \le 0 \atop mx \text{ for } x > 0}$
(a) (5%) Find all values of m that make f continuous at 0
(b) (5%) Find all the values of m that make f differentiable at 0
Ans:
(a)
 $\lim_{x \to 0^+} f(x) = 0 = \lim_{x \to 0^+} f(x) \to m$ can be any real number.

(b) Considering the alternative form of derivative:

$$\lim_{x \to 0^{-}} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^{-}} \frac{\sin(3x)}{x} = \lim_{x \to 0^{-}} \frac{3\sin(3x)}{3x} = 3\lim_{t \to 0^{-}} \frac{\sin(t)}{t} (\text{Let } t = 3x)$$

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{mx}{x} = m$$

Since it is differentiable, we have m = 3

4. (5%) If $f(x) = x^2 + 2x - 3$, use the definition of the derivative of a function to compute f'(x)

Ans:

$$f'(x) = \lim_{\Delta x \to 0} \frac{(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 + 2(x + \Delta x) - 3 - (x^2 + 2x - 3)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2 + 2\Delta x}{\Delta x} = \lim_{\Delta x \to 0} 2x + \Delta x + 2 = 2x + 2$$

5. (5%) Verify that $f(x) = x^3 + 2x + 4$ satisfies the hypotheses of the Mean Value Theorem on [-1,1]. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

Ans:

The function $f(x) = x^3 + 2x + 4$ is a polynomial function. Polynomial functions are continuous everywhere on R, including the closed interval [-1,1]. Again, since f(x)is a polynomial function, it is differentiable everywhere on R, including the open interval (-1,1). Therefore, the hypotheses of the Mean Value Theorem are satisfied. 1

$$f(1) = 7, f(-1) =$$

In addition, $f'(x) = 3x^2 + 2$ By MVT, we have $f'(c) = \frac{f(1) - f(-1)}{1 - (-1)} = 3 \rightarrow 3c^2 + 2 = 3 \rightarrow 3c^2 = 1 \rightarrow c = \pm \frac{\sqrt{3}}{3}$ Both of them lies in (-1,1). Therefore, $c = \pm \frac{\sqrt{3}}{3}$.

- 6. (12%)
 - (a) (5%) Find the equation of the tangent line to the graph of $f(x) = \frac{x+8}{\sqrt{3x+1}}$ at the point (0.8)
- (b) (3%) Use chain rule to find the derivative of $g(x) = sin(2x^2 + 3cos(x))$
- (c) (4%) Use implicit differentiation to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ of the expression $2xy xy = \frac{d^2y}{dx^2}$ $1 = 3x + y^2$

Ans:

(a)
$$f'(x) = \frac{(3x+1)^{\frac{1}{2}}(1) - (x+8)^{\frac{1}{2}}(3x+1)^{\frac{-1}{2}}(3)}{3x+1}$$

$$f'(0) = \frac{1-4(3)}{1} = -11.$$
 The tangent line is $y - 8 = -11(x - 0) \rightarrow y = -11x + 8$
(b) $g'(x) = \cos(2x^2 + 3\cos(x)) \times (4x - 3\sin(x))$

(c) Differentiate both side with respect to x, we have

$$\frac{2y + 2x\frac{dy}{dx} = 3 + 2y\frac{dy}{dx} \rightarrow \frac{dy}{dx} = \frac{3 - 2y}{2x - 2y}}{\frac{d^2y}{dx^2}} = \frac{-2\frac{dy}{dx}(2x - 2y) - 2(3 - 2y)(1 - \frac{dy}{dx})}{4(x - y)^2} = \frac{(-4x + 6)\frac{dy}{dx} + (4y - 6)}{4(x - y)^2}}{\frac{4(x - y)^2}{4(x - y)^2}} = \frac{-12x + 8xy + 9 - 4y^2}{4(x - y)^3}$$

- 7. (20%) Let $f(x) = \frac{x^4 + x^2 + 4x}{x}$
 - (a) (4%) Find the critical points and possible points of inflection for f(x)
 - (b) (3%) Find the open intervals on which f(x) is increasing or decreasing
 - (c) (3%) Find the open intervals of concavity for f(x)
 - (d) (4%) Find all asymptotes of f(x)
 - (e) (6%) Sketch the graph of f(x), labeling intercepts, relative extrema, points of inflection, and asymptotes.

Ans: Note that the original function is undefined at x = 0, therefore we should include it in the following table.

$$f(x) = x^{3} + x + 4, \ x \neq 0, \ f'(x) = 3x^{2} + 1 > 0$$

$$f''(x) = 6x$$

	(−∞, 0)	(0 ,∞)
測試值	-1	1
f ′的正負號	+	+
f "的正負號	-	+
結論	遞增/向下凹	遞減/上下凹

(a) Note that x is not define at x = 0, we should not include it in the critical numbers or possible points of inflection

There is no critical numbers (f' = 0)

There is no possible points of inflection (f'' = 0)

- (b) Increasing $(-\infty, 0), (0, \infty)$.
- (c) Upward: $(0, \infty)$. Downward $(-\infty, 0)$
- (d) Since $\lim_{x \to \pm \infty} f(x) = \pm \infty$ \to No horizontal asymptote. No vertical asymptote since f(x) is undefined at x = 0
- (e) Graph



There is no relative extrema or point of inflection. No y intercept Using Newton's method or bisection method, we have x intercept roughly equals -1.38. (Other approximation methods are also acceptable, like trial and error)

- 8. (8%)
- (a) (4%) Find the point on the graph $y = \sqrt{x 8}$ that is closest to the point (12,0).

(b) (4%) Use Newton's method with the initial approximation $x_1 = -1$ to find x_3 ,

- the third approximation to the solution of the equation $2x^3 3x^2 + 2 = 0$ Ans:
- (a) The distance between (12,0) and a point (*x*, *y*) on the graph of $\sqrt{x-8}$ is

 $d = \sqrt{(x-12)^2 + y^2} = \sqrt{(x-12)^2 + x - 8}$ Minimize $d^2 = f(x) = (x-12)^2 + x - 8$. Note that f'(x) = 2(x-12) + 1 = 2x - 23. The only critical point is $x = \frac{23}{2}$. When $x = \frac{23}{2}$, $y = \sqrt{\frac{7}{2}}$, $d = \frac{\sqrt{15}}{2}$. Therefore, $x = \frac{23}{2}$ is relative minimum (Testing the critical number using First Derivative Test). Therefore, the closest point is $(\frac{23}{2}, \sqrt{\frac{7}{2}})$.

(b) Newton's Method iteratively improves an estimate x_n of a root of a function f(x) using the formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$f'(x) = 6x^2 - 6x$$

n	x _n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1	-3	12	-0.25	-0.75
2	-0.75	-0.53125	7.875	0.06746031746	-0.68254
3	-0.68254				

9. (8%) Use differentials to approximate $\sqrt{1 + sin(0.01)}$

Ans: Let
$$f(x) = \sqrt{1 + \sin(x)}, f'(x) = \frac{\cos(x)}{2\sqrt{1 + \sin(x)}}$$

 $f(x + \Delta x) \approx f(x) + f'(x)dx = \sqrt{1 + \sin(x)} + \frac{\cos(x)}{2\sqrt{1 + \sin(x)}}dx$
Choosing $x = 0$ and $dx = 0.01$.
 $f(x + \Delta x) = \sqrt{1 + \sin(0.01)} \approx \sqrt{1 + \sin(0)} + \frac{\cos(0)}{2\sqrt{1 + \sin(0)}} 0.01 = 1 + \frac{0.01}{2}$
 $= 1.005$

10. (14%) Solve the following problems

(a) (7%) Evaluate
$$\int \frac{3}{\sqrt[3]{x}} + x^2 + 2dx$$

(b) (7%) Find $\lim_{n \to \infty} \frac{1}{n} \left[\sqrt{\frac{n^2 - 1^2}{n^2}} + \sqrt{\frac{n^2 - 2^2}{n^2}} + \sqrt{\frac{n^2 - 3^2}{n^2}} + \dots + \sqrt{\frac{n^2 - n^2}{n^2}} \right]$

Ans:

$$(a) \int \frac{3}{\sqrt[3]{x}} + x^2 + 2dx = \int 3x^{\frac{-1}{3}} + x^2 + 2dx = \frac{9}{2}x^{\frac{2}{3}} + \frac{1}{3}x^3 + 2x + C$$

$$(b) \lim_{n \to \infty} \frac{1}{n} \left[\sqrt{\frac{n^2 - 1^2}{n^2}} + \sqrt{\frac{n^2 - 2^2}{n^2}} + \sqrt{\frac{n^2 - 3^2}{n^2}} + \dots + \sqrt{\frac{n^2 - n^2}{n^2}} \right] = \lim_{n \to \infty} \sum_{k=1}^n \frac{\sqrt{n^2 - k^2}}{n^2} = \lim_{n \to \infty} \sum_{k=1}^n \frac{\sqrt{n^2 - k^2}}{n^2} = \lim_{n \to \infty} \sum_{k=1}^n \frac{1}{n} \sqrt{1 - \left(\frac{k}{n}\right)^2} = \int_0^1 \sqrt{1 - x^2} dx = \frac{\pi}{4}$$