Calculus (I), Final Exam

Note that e is euler constant in all the following questions

1. (15%) Find the following limit. (If the limit does not exist or has an infinite limit, you should point it out)

   (a) (5%) $\lim_{x\to 0^+} \sin(x)^{\frac{1}{\ln(x)}}$

   (b) (5%) $\lim_{x\to 2^+} \frac{1}{x^2-4} - \frac{\sqrt{x-1}}{x^2-4}$

   (c) (5%) $\lim_{x\to\infty} 2x tan(\frac{1}{x})$

Ans:

(a) $y = \lim_{x\to 0^+} \sin(x)^{\frac{1}{\ln(x)}}$

$\ln y = \ln \lim_{x\to 0^+} \sin(x)^{\frac{1}{\ln(x)}} = \lim_{x\to 0^+} \ln \sin(x)^{\frac{1}{\ln(x)}} = \lim_{x\to 0^+} \frac{\ln sin(x)}{\ln(x)} = \lim_{x\to 0^+} \frac{\frac{\cos(x)}{\sin(x)}}{\frac{1}{x}}$(L' Hôpital' s rule)$= \lim_{x\to 0^+} \frac{-csc^2(x)}{\frac{-1}{x^2}}$ (L' Hôpital' s rule) $= \lim_{x\to 0^+} \frac{x^2}{sin^2(x)} = 1$

Since $\ln y = 1$ Therefore, $y = e$

(b) $\lim_{x\to 2^+} \frac{1}{x^2-4} - \frac{\sqrt{x-1}}{x^2-4} = \lim_{x\to 2^+} \frac{1-\sqrt{x-1}}{x^2-4} = \lim_{x\to 2^+} \frac{\frac{-1}{2\sqrt{x-1}}}{2x}$ (L' Hôpital' s rule) $=$

$\lim_{x\to 2^+} \frac{-1}{4x\sqrt{x-1}} = \frac{-1}{8}$

(c) $\lim_{x\to\infty} 2x tan(\frac{1}{x}) = \lim_{x\to\infty} \frac{2tan(\frac{1}{x})}{\frac{1}{x}}$ (L′ Hôpital′ s rule) $= \lim_{x\to\infty} \frac{\frac{-2}{x^2}sec^2(\frac{1}{x})}{\frac{-1}{x^2}} = 2$

2. (4%) Suppose f is a function such that $f(1) = 1, f'(1) = 3, f''(1) = 5$, $f(2) = -2, f'(2) = -4, f''(2) = -6$, f' and f'' are both continuous everywhere. Evaluate $\int_1^2 f'(x)dx$ and $\int_1^2 f''(x)dx$.

   Ans:

By the Fundamental Theorem of Calculus,

$$\int_1^2 f'(x)dx = f(2) - f(1) = -3.$$

$$\int_1^2 f''(x)dx = f'(2) - f'(1) = -7$$

3. (9%) Let $f(x) = x^3 + 3x + 1$

(a) (3%) Show that $f(x)$ has an inverse function

(b) (3%) What is the value of $f^{-1}(x)$ when $x = 5$

(c) (3%) What is the value of $(f^{-1})'(x)$ when $x = 5$

Ans:

(a) Note that f is strictly increasing and therefore has an inverse function ($f' = 3x^2 + 3 > 0$)

(b) Let $\mathrm{y} = f^{-1}(5) \to f(\mathrm{y}) = 5 \to y^3 + 3y + 1 = 5 \to (y^2 + \mathrm{y} + 4)(\mathrm{y} - 1) = 0 \to \mathrm{y} = 1$

Because $f(1) = 5$, we know that $f^{-1}(5) = 1$

(c) $f'(x) = 3x^2 + 3$

Because f is differentiable and has an inverse function, we have

$$(f^{-1})'(5) = \frac{1}{f'(1)} = \frac{1}{6}$$

4. (20%) Evaluate the following integral.

(a) (4%) $\int \cos(x) \times \sin(\sin(x))\, \mathrm{dx}$

(b) (4%) $\int_0^1 (5^x - 3^x)dx$

(c) (4%) $\int_1^{e^2} \ln(x)\, dx$

(d) (4%) $\int tan^3(\theta) sec^4(\theta)\, d\theta$

(e) (4%) $\int_1^2 \frac{3x^2+6x+2}{x^2+3x+2} dx$

Ans:

(a) Let $\mathrm{u} = \sin(x), du = \cos(x)dx$

$$\int \cos(x) \times \sin(\sin(x))\, \mathrm{dx} = \int \sin(u)\, \mathrm{du} = -\cos(u) + C = -\cos(\sin(x)) + C$$

(b)

$$\int_0^1 (5^x - 3^x)dx = \frac{1}{\ln 5}5^x - \frac{1}{\ln 3}3^x\Big|_0^1 = \frac{4}{\ln 5} - \frac{2}{\ln 3}$$

(c)

Let $\mathrm{u} = \ln(x), dv = dx \to du = \frac{1}{x}dx, v = x$

$$\int \ln(x)\, dx = x\ln(x) - \int x\frac{1}{x}dx = x\ln(x) - x + C$$

$$\int_1^{e^2} \ln(x)\, dx = x\ln(x) - x\Big|_1^{e^2} = e^2 + 1$$

(d)

$\int tan^3(\theta)sec^4(\theta)\,d\theta = \int tan^2(\theta)sec^3(\theta)\sec(\theta)\,tan(\theta)d\theta =$
$\int(sec^2(\theta){-}1)sec^3(\theta)\sec(\theta)\,tan(\theta)d\theta$ (Let $\mathrm{u} = \sec(\theta), \mathrm{du} =$

$\sec(\theta)\,tan(\theta)d\theta) =$ $\int(u^2-1)u^3\,du = \frac{1}{6}u^6 - \frac{1}{4}u^4 + \mathrm{C} = \frac{1}{6}sec^6(\theta) -$

$\frac{1}{4}sec^4\big((\theta)\big) + C$

(e)

$$\frac{3x^2+6x+2}{x^2+3x+2} = 3 + \frac{-3x-4}{x^2+3x+2}$$

$$\frac{-3x-4}{x^2+3x+2} = \frac{A}{x+1} + \frac{B}{x+2} \rightarrow -3\mathrm{x} - 4 = \mathrm{A(x+2) + B(x+1)}$$

$$= \mathrm{(A+B)x + (2A+B)} \rightarrow \mathrm{A} = -1, \mathrm{B} = -2$$

$$\int_1^2 \frac{3x^2+6x+2}{x^2+3x+2}dx = \int_1^2 3 - \frac{1}{x+1} - \frac{2}{x+2}\mathrm{dx}$$

$$= 3\mathrm{x} - \ln|x+1| - 2\ln|x+2|\,\Big|_1^2 = 3 + \ln\frac{3}{8}$$

5. (4%) Sketch the region enclosed by the given curves and find its area: $y = \frac{1}{x}, y = x,$ and $y = 4x$ for $\mathrm{x} \geq 0$

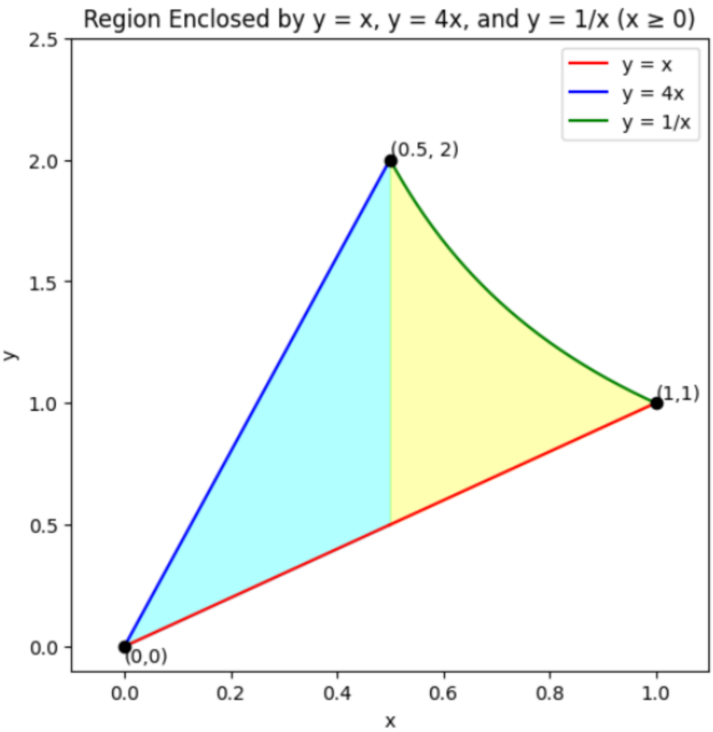
Ans:

To find the region, calculate the points of intersection of these curves:

$y = \frac{1}{x}, y = x \rightarrow \frac{1}{x} = x \rightarrow \mathrm{x} = 1,$ the intersection point is $(1,1)$

$y = \frac{1}{x}, y = 4x \rightarrow \frac{1}{x} = 4x \rightarrow \mathrm{x} = \frac{1}{2},$ the intersection point is $(\frac{1}{2}, 2)$

$y = \mathrm{x}, y = 4x \rightarrow x = 4x \rightarrow \mathrm{x} = 0,$ the intersection point is $(0,0)$



$$A = \int_0^{\frac{1}{2}} 4x - x\,dx + \int_{\frac{1}{2}}^{1} \frac{1}{x} - x\,dx = \int_0^{\frac{1}{2}} 3x\,dx + \int_{\frac{1}{2}}^{1} \frac{1}{x}dx - \int_{\frac{1}{2}}^{1} x\,dx = \frac{3}{2}x^2\Big|_0^{\frac{1}{2}} + \ln x - \frac{1}{2}x^2\Big|_{\frac{1}{2}}^{1} = \ln 2$$

6. (12%) Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2, y = 1$ and the y-axis ($x = 0$) about the following lines:
   (a) (6%) About $x-$axis (by the disk/washer method)
   (b) (6%) About y −axis (by the shell method)

Ans:

(a)

$$V = \pi \int_0^1 (1)^2 - (x^2)^2 dx = \pi \int_0^1 1 - x^4 dx = \pi \left[x - \frac{x^5}{5}\right]_0^1 = \frac{4\pi}{5}$$

(b)

$$V = 2\pi \int_0^1 x(1 - x^2) dx = 2\pi \int_0^1 x - x^3 dx = 2\pi \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{\pi}{2}$$

7. (12%)

(a) (3%) Write down the formula for finding the arc length of a curve defined by $y = \frac{x^2}{4} - 2x$ in terms of the variable x.

(b) (3%) Use trigonometric substitution to express the integral for the arc length obtained in part (a) in terms of the variable θ.

(c) (6%) Solve the integral you obtained in part (b) and calculate the total arc length of the curve over the interval [4,8]

Ans:

(a) $y' = \frac{x}{2} - 2$

$$s = \int \sqrt{1 + y'^2}dx = \int \sqrt{1 + (\frac{x}{2} - 2)^2}dx = \int \sqrt{\frac{1}{4}(x^2 - 8x + 20)dx}$$

(b)

$$s = \int \sqrt{\frac{1}{4}(x^2 - 8x + 20)dx} = \frac{1}{2}\int \sqrt{x^2 - 8x + 16 + 4dx} = \frac{1}{2}\int \sqrt{(x-4)^2 + 4dx}$$

Let $u = x - 4 = 2\tan(\theta)$, $dx = 2sec^2(\theta)d\theta$

$$\frac{1}{2}\int \sqrt{(x-4)^2 + 4dx} = \frac{1}{2}\int 2sec(\theta) \times 2sec^2(\theta)d\theta = 2\int sec^3(\theta)d\theta$$

(c)

Let $u = \sec(\theta), dv = sec^2(\theta)d\theta$

$$du = \sec(\theta)\tan(\theta), v = \tan(\theta)$$

$$\int sec^3\theta d\theta = \sec(\theta)\tan(\theta) - \int \sec(\theta)\, tan^2\theta d\theta$$

$$= \sec(\theta)\tan(\theta) - \int \sec(\theta)\,(sec^2\theta - 1)d\theta$$

$$= \sec(\theta)\tan(\theta) - \int sec^3\theta d\theta + \int \sec(\theta)$$

$$\int sec^3\theta d\theta = \frac{1}{2}[\sec(\theta)\tan(\theta) + \ln|\sec(\theta) + \tan(\theta)|]$$

$$\frac{1}{2}\int \sqrt{(x-4)^2 + 4dx} = 2\int sec^3(\theta)d\theta = [\sec(\theta)\tan(\theta) + \ln|\sec(\theta) + \tan(\theta)|]$$

$$= \left[\frac{\sqrt{(x-4)^2 + 4}}{2}\frac{(x-4)}{2} + \ln\left|\frac{\sqrt{(x-4)^2 + 4}}{2} + \frac{(x-4)}{2}\right|\right]$$

$$s = \left[\frac{\sqrt{(x-4)^2+4}}{2}\frac{(x-4)}{2} + \ln\left|\frac{\sqrt{(x-4)^2+4}}{2} + \frac{(x-4)}{2}\right| \right]\Big|_4^8$$

$$= \sqrt{20} + \ln\left(4+\sqrt{20}\right) - \ln 2$$

8. (4%) If we have an arc which is part of the circules $x^2 + y^2 = 4$ between the points $(-\sqrt{3}, 1)$ and $(\sqrt{3}, 1)$. Find the area of the surface generated by revolving the arc about the x-axis.

Ans:

$$\mathrm{y} = \sqrt{4-x^2}$$

$$\sqrt{1+y'^2} = \frac{2}{\sqrt{4-x^2}}$$

$$\mathrm{S} = 2\pi \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{4-x^2}\frac{2}{\sqrt{4-x^2}}dx = 8\pi\sqrt{3}$$

9. (20%) Evaluate the following integral. (If the integral is diverge, you should point it out)
   (a) (6%) $\int x \cdot arcsin(x^2)\, dx$
   (b) (6%) $\int \frac{x^2+2x}{x^3-x^2+x-1}dx$
   (c) (8%) $\int_0^\infty \frac{1}{e^x+e^{-x}}dx$

Ans:

(a) Let $\mathrm{u} = arcsin(x^2), dv = xdx \rightarrow v = \frac{x^2}{2}, du = \frac{2x}{\sqrt{1-x^4}}$

$$\int x \cdot arcsin(x^2)\,\mathrm{dx} = \frac{x^2}{2}arcsin(x^2) - \int \frac{x^3}{\sqrt{1-x^4}}\mathrm{dx}$$

Let $\mathrm{u} = 1 - x^4, \mathrm{du} = -4x^3dx$

$$\int \frac{x^3}{\sqrt{1-x^4}}\mathrm{dx} = \frac{-1}{4}\int \frac{1}{\sqrt{u}}\mathrm{du} = \frac{-1}{2}\sqrt{u} + \mathrm{C} = \frac{-1}{2}\sqrt{1-x^4} + C$$

$$\int x \cdot arcsin(x^2)\,\mathrm{dx} = \frac{x^2}{2}arcsin(x^2) - \int \frac{x^3}{\sqrt{1-x^4}}\mathrm{dx}$$

$$= \frac{x^2}{2}arcsin(x^2) + \frac{1}{2}\sqrt{1-x^4} + C$$

(b) $\frac{x^2+2x}{x^3-x^2+x-1} = \frac{x^2+2x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$

$$x^2 + 2x = A(x^2+1) + (Bx+C)(x-1)$$

When $x = 1, 3 = 2A \rightarrow A = \frac{3}{2}$

When $x = 0, 0 = A - C \rightarrow C = \frac{3}{2}$

When $x = 2, 8 = 5A + 2B + C \rightarrow B = \frac{-1}{2}$

$$\int \frac{x^2 + 2x}{x^3 - x^2 + x - 1} dx = \frac{3}{2} \int \frac{1}{x - 1} dx - \frac{1}{2} \int \frac{x - 3}{x^2 + 1} dx$$

$$= \frac{3}{2} \int \frac{1}{x - 1} dx - \frac{1}{2} \int \frac{x}{x^2 + 1} dx + \frac{3}{2} \int \frac{1}{x^2 + 1} dx$$

$$= \frac{3}{2} \ln|x - 1| - \frac{1}{4} \ln(x^2 + 1) + \frac{3}{2} \arctan(x) + C$$

(c) $\int_0^\infty \frac{1}{e^x + e^{-x}} dx = \int_0^\infty \frac{e^x}{e^{2x} + 1} dx$

Let $u = e^x, du = e^x dx$

$$\int_0^\infty \frac{e^x}{e^{2x} + 1} dx = \int_1^\infty \frac{du}{1 + u^2} = \lim_{b \to \infty} \int_1^b \frac{du}{1 + u^2} = \lim_{b \to \infty} arctan(u)]_1^b = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$