1. (20%) Find the following limit. (If the limit does not exist or has an infinite limit, you should point it out)

(a) (5%) $\lim_{x\to 0^+}(e^x - 1)\cot x$

(b) (5%) $\lim_{x\to 0^+}(e^x + x)^{\frac{1}{x}}$

(c) (5%) $\lim_{x\to 3^+}\frac{1-\sqrt{x-2}}{x^2-9}$

(d) (5%) $\lim_{x\to\infty}\frac{\int_0^x \sqrt{1+t^4}dt}{x^3}$

**Ans**:

(a) $\lim_{x\to 0^+}(e^x - 1)\cot x = \lim_{x\to 0^+}\frac{e^x-1}{\tan(x)}$(L' Hôpital' s rule)$= \lim_{x\to 0^+}\frac{e^x}{sec^2(x)} = 1$

(b) $y = \lim_{x\to 0^+}(e^x + x)^{\frac{1}{x}}$

$$\ln y = \ln \lim_{x\to 0^+}(e^x + x)^{\frac{1}{x}} = \lim_{x\to 0^+}\ln(e^x + x)^{\frac{1}{x}}$$

$$= \lim_{x\to 0^+}\frac{1}{x}\ln(e^x + x)\text{ (L' Hôpital' s rule)} = \lim_{x\to 0^+}\frac{\frac{e^x+1}{e^x+x}}{1} =$$

$$= 2$$

Since $\ln y = 2$ Therefore, $y = e^2$

(c) $\lim_{x\to 3^+}\frac{1-\sqrt{x-2}}{x^2-9}$(L' Hôpital' s rule) $= \lim_{x\to 3^+}\frac{\frac{-1}{2\sqrt{x-2}}}{2x} = \frac{-1}{12}$

(d) $\lim_{x\to\infty}\frac{\int_0^x \sqrt{1+t^4}dt}{x^3}$(L'Hôpital's rule) $= \lim_{x\to\infty}\frac{\sqrt{1+x^4}}{3x^2} = \lim_{x\to\infty}\frac{\sqrt{\frac{1}{x^4}+1}}{3} = \frac{1}{3}$

2. (15%) Solve the following problems

($a$) Evaluate $\int_{-3}^{3} \frac{sin^3(x)}{2+\cos(x)} dx$

(b) Find $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left[\frac{1}{1+\frac{i}{n}}\right]$

(c) $\int_{-3}^{0} \sqrt{9-x^2}\, dx$

**Ans:**

(a) Note that let $f(x) = \frac{sin^3(x)}{2+\cos(x)}$, we have $f(-x) = -f(-x)$. It is an odd function, therefore, $\int_{-3}^{3} \frac{sin^3(x)}{2+\cos(x)} dx = 0$

(b) $\lim_{n\to\infty}\left[\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \frac{1}{1+\frac{3}{n}} + \cdots + \frac{1}{1+\frac{n}{n}}\right]\frac{1}{n} = \int_0^1 \frac{1}{1+x} dx = ln(1+x)\Big|_0^1 = \ln(2)$

(c) It is the region below a quarter of a circle center at zero with radius 3 , thus $\int_{-3}^{0} \sqrt{9-x^2}\, dx = \frac{1}{4}\pi 3^2 = \frac{9}{4}\pi$

3. (9%) Let $f(x) = \int_3^x \frac{1}{\sqrt{1+t^2}} dt$

(a) (3%) Show that $f(x)$ has an inverse function

(b) (3%) What is the value of $f^{-1}(x)$ when $x = 0$

(c) (3%) What is the value of $(f^{-1})'(x)$ when $x = 0$

**Ans:**

(a) Note that $f'(x) = \frac{1}{\sqrt{1+x^2}} > 0$ for all $x \to f$ is strictly increasing therefore is one to one and has an inverse function.

(b) Because $f(3) = 0$, we know that $f^{-1}(0) = 3$

(c) $f'(x) = \frac{1}{\sqrt{1+x^2}}$

Because $f$ is differentiable and has an inverse function, we have

$$(f^{-1})'(0) = \frac{1}{f'(3)} = \sqrt{10}$$

4. (10%) Compute the derivative.

(a) (5%) $f(x) = 5^{3x} + sin^{-1}(e^{-x}) + \log_3 x$

(b) (5%) $f(x) = \ln(\ln x^2) + \log_2 \frac{x\sqrt{x-1}}{2}$

**Ans:**

(a) $f'(x) = 3(\ln 5)5^{3x} - \frac{e^{-x}}{\sqrt{1-e^{-2x}}} + \frac{1}{\ln(3)x}$

(b) $f(x) = \ln(\ln x^2) + \log_2 x + \frac{1}{2}\log_2(x-1) - 1$

$$f'(x) = \frac{2x}{x^2 \ln x^2} + \frac{1}{\ln 2}(\frac{1}{x} + \frac{1}{2(x-1)})$$

5. (30%) Evaluate the following integral.

(a) (5%) $\int_0^1 7^x + 2^x dx$

(b) (5%) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec(x)(\tan(x) - \sec(x))dx$

(c) (5%) $\int \frac{e^{4x}-e^{-4x}}{e^{4x}+e^{-4x}} dx$

(d) (5%) $\int \frac{\sin(3+\ln\theta)}{\theta} d\theta$

(e) (5%) $\int \frac{\sin(\theta)\cos(\theta)}{1+sin^4\theta} \mathrm{d}\theta$

(f) (5%) $\int \frac{1}{\sqrt{5-4x-x^2}} dx$

**Ans**:

(a) $\int_0^1 7^x + 2^x dx = \left[\frac{7^x}{\ln 7} + \frac{2^x}{\ln 2}\right]\Big|_0^1 = \frac{6}{\ln 7} + \frac{1}{\ln 2}$

(b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec(x)(\tan(x) - \sec(x))dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec(x)\tan(x) - seci^2(x)dx = \sec(x) -$

$\tan(x)|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = 2 - \sqrt{3} - \frac{1}{\sqrt{3}}$

(c) $\mathrm{u} = e^{4x} + e^{-4x},\ du = 4(e^{4x} - e^{-4x})dx$

$$\int \frac{e^{4x} - e^{-4x}}{e^{4x} + e^{-4x}} dx = \int \frac{du}{4u} = \frac{1}{4}\ln|u| + C = \frac{1}{4}\ln(e^{4x} - e^{-4x}) + \mathrm{C}$$

(d) Let $u = 3 + \ln\theta$, $du = \frac{1}{\theta}d\theta$

$$\int \frac{\sin(3 + \ln\theta)}{\theta} d\theta = \int \sin(u)du = -\cos(u) + C = -\cos(3 + \ln\theta) + \mathrm{C}$$

(e) Let $u = \sin(\theta), du = \cos(\theta)d\theta$

$$\int \frac{\sin(\theta)\cos(\theta)}{1 + sin^4\theta} \mathrm{d}\theta = \int \frac{\mathrm{u}}{1 + u^4} \mathrm{du}$$

Let $v = u^2, dv = 2udu$

$$\int \frac{\mathrm{u}}{1 + u^4}\mathrm{du} = \frac{1}{2}\int \frac{1}{1 + v^2}\mathrm{dv} = \frac{1}{2}tan^{-1}(v) + C = \frac{1}{2}tan^{-1}(u^2) + C$$

$$= \frac{1}{2}tan^{-1}(sin^2(\theta)) + C$$

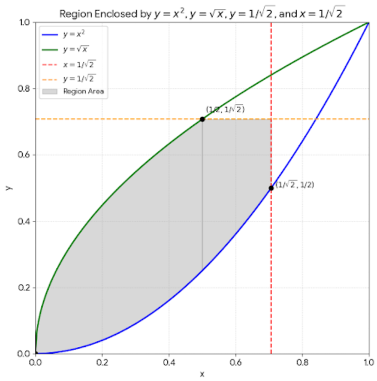
(f) $\int \frac{1}{\sqrt{5-4x-x^2}}dx = \int \frac{1}{\sqrt{9-(x+2)^2}}dx$. Let $\mathrm{u} = \mathrm{x} + 2, \mathrm{a} = 3$

$$\int \frac{1}{\sqrt{a^2 - u^2}}dx = \sin^{-1}\frac{x + 2}{3} + \mathrm{C}$$

6. (6%) Firstly, sketch the region bounded by the graphs of $y = x^2$, $y = \sqrt{x}, y = \frac{1}{\sqrt{2}}, \mathrm{x} = \frac{1}{\sqrt{2}}$, then find its area.

**Ans:**

$$\mathrm{A} = \int_0^{\frac{1}{2}}[\sqrt{x} - x^2]\,dx + \int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}}\left[\frac{1}{\sqrt{2}} - x^2\right]dx = \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\Big|_0^{\frac{1}{2}} + \frac{x}{\sqrt{2}} - \frac{x^3}{3}\Bigg]_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} = \frac{3-\sqrt{2}}{6}$$



7. (12%) Find the volume of the solid formed by revolving the region bounded by the graphs of $y=\sqrt{x}+1$, $\mathrm{x}=1$ and the $y=1$ about the following lines:
(a) About $y-$ axis (by the disk/washer method)
(b) About $x-$axis (by the shell method)

**Ans:**
The intersection between $y=\sqrt{x}+1$ and $y=1$ is at (0,1) and the intersection between $y=\sqrt{x}+1$ and $\mathrm{x}=1$ is at (1,2).
We also rewrite $y=\sqrt{x}+1$ as $\mathrm{x}=(y-1)^2$

(a) $\mathrm{V}=\pi\int_1^2[1^2-((y-1)^2)^2]dy=\pi\left[\int_1^2 1-(y-1)^4dy\right]=\pi\left[y-\frac{(y-1)^5}{5}\right]|_1^2=\frac{4\pi}{5}$

(b) $\mathrm{V}=2\pi\int_1^2 y[1-(y-1)^2]dy=2\pi\left[\int_1^2 2y^2-y^3dy\right]=2\pi\left[\frac{2}{3}y^3-\frac{1}{4}y^4\right]|_1^2=\frac{11\pi}{6}$

8. (6%) Find the arc length of the graph $\mathrm{y}=\frac{x^2}{4}-\ln\sqrt{x}$ on the interval $1\le\mathrm{x}\le\mathrm{e}$.

**Ans:**

$$y'=\frac{x}{2}-\frac{1}{2x},1+(y')^2=(\frac{x}{2}+\frac{1}{2x})^2$$

$$\mathrm{s}=\int_1^e\sqrt{(\frac{x}{2}+\frac{1}{2x})^2}\,dx=\int_1^e\frac{x}{2}+\frac{1}{2x}dx=\frac{x^2}{4}+\frac{1}{2}\ln|x|\,|_1^e=\frac{e^2+1}{4}$$

9. (7%) Find the area of the surface generated by revolving the curve $\mathrm{y}=\frac{x^2}{2}+4$ on the interval $0\le\mathrm{x}\le 2$ about the $y$-axis.

**Ans:**
$y'=x,1+(y')^2=1+x^2$

$$\mathrm{S}=2\pi\int_0^2\mathrm{x}\sqrt{1+x^2}dx$$

Let $\mathrm{u}=1+x^2,du=2xdx$

$$\mathrm{S}=2\pi\int_0^2\mathrm{x}\sqrt{1+x^2}dx=\pi\int_1^5\sqrt{u}du=\frac{2\pi}{3}u^{\frac{3}{2}}|_1^5=\frac{2\pi}{3}(5\sqrt{5}-1)$$