Chapter 5 Logarithmic, Exponential, and Other Transcendental Functions

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The natural logarithmic function

- The General Power Rule

$$\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

  has an important disclaimer—it doesn't apply when $n = -1$. Consequently, we have not yet found an antiderivative for the function $f(x) = 1/x$.
- In fact, it is neither algebraic nor trigonometric, but falls into a new class of functions called logarithmic functions.
- This particular function is the natural logarithmic function.

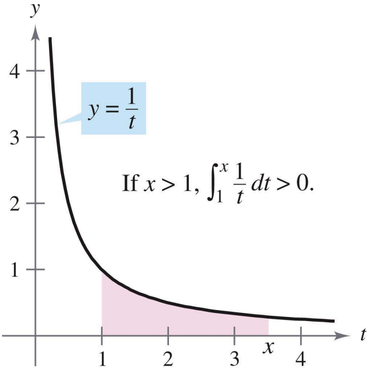
Definition 5.1 (The natural logarithmic function)

The natural logarithmic function is defined by

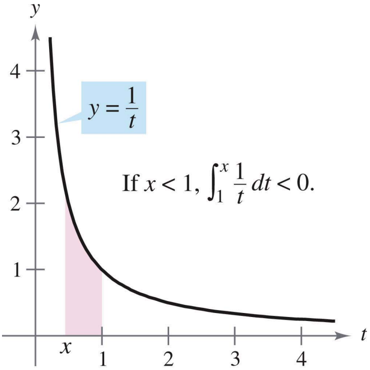
$$\ln x = \int_1^x \frac{1}{t}\,\mathrm{d}t, \quad x > 0.$$

The domain of the natural logarithmic function is the set of all positive real numbers.

- From this definition, you can see that $\ln x$ is positive for $x > 1$ and negative for $0 < x < 1$.
- Moreover, $\ln(1) = 0$, because the upper and lower limits of integration are equal when $x = 1$.



(a) If $x > 1$, then $\ln x > 0$



(b) If $0 < x < 1$, then $\ln x < 0$.

Figure 1: The natural logarithmic function $\ln x$.

- To sketch the graph of $y = \ln x$, you can think of the natural logarithmic function as an antiderivative given by the differential equation

$$\frac{dy}{dx} = \frac{1}{x}$$

- Figure 2 is a computer-generated graph, called a slope (or direction) field, showing small line segments of slope $1/x$.
- The graph of $y = \ln x$ is the one that passes through the point $(1, 0)$.

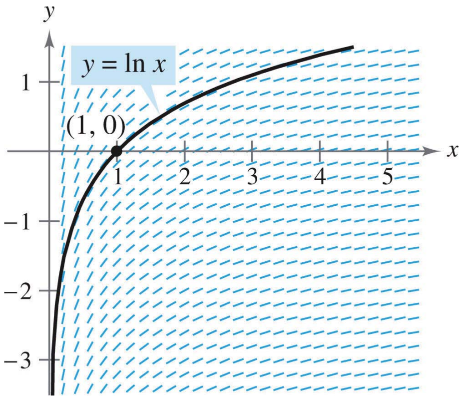


Figure 2: Each small line segment has a slope of $\frac{1}{x}$.

Theorem 5.1 (Properties of the natural logarithmic function)

The natural logarithmic function has the following properties.

1. *The domain is* $(0, \infty)$ *and the range is* $(-\infty, \infty)$.
2. *The function is continuous, increasing, and one-to-one.*
3. *The graph is concave downward.*

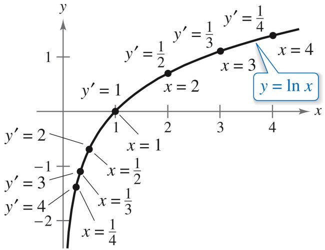


Figure 3: The natural logarithmic function is increasing, and its graph is concave downward.

Theorem 5.2 (Logarithmic properties)

If a and b are positive numbers and n is rational, then the following properties are true.

1. $\ln(1) = 0$
2. $\ln(ab) = \ln a + \ln b$
3. $\ln(a^n) = n \ln a$
4. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

- When rewriting the logarithmic functions, you must check to see whether the domain of the rewritten function is the same as the domain of the original.
- For instance, the domain of $f(x) = \ln x^2$ is all real numbers except $x = 0$, and the domain of $g(x) = 2\ln x$ is all positive real numbers.

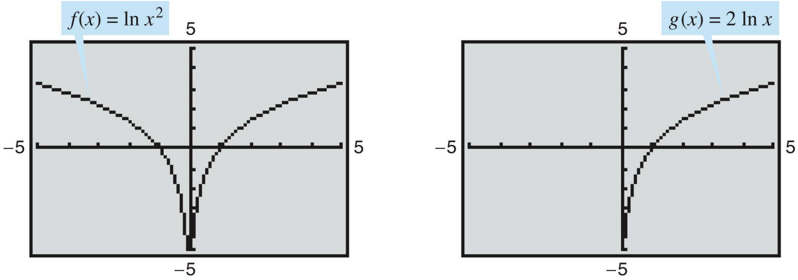


Figure 4: Domain of $f(x) = \ln x^2$ and $g(x) = 2\ln x$.

The number e

- It is likely that you have studied logarithms in an algebra course. There, without the benefit of calculus, logarithms would have been defined in terms of a base number.
- For example, common logarithms have a base of 10 since $\log_{10} 10 = 1$.
- The base for the natural logarithm is defined using the fact that the natural logarithmic function is continuous, is one-to-one, and has a range of $(-\infty, \infty)$.
- So, there must be a unique real number x such that $\ln x = 1$.

- This number is denoted by the letter e. It can be shown that e is irrational and has the following decimal approximation.

$$e \approx 2.71828182846$$

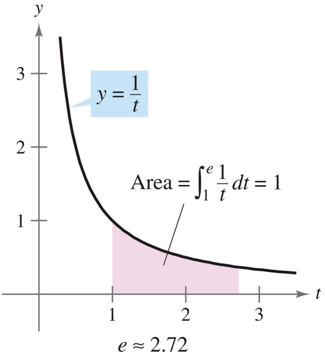


Figure 5: e is the base for the natural logarithm because $\ln e = 1$.

Definition 5.2 (e)

The letter e denotes the positive real number such that

$$\ln e = \int_1^e \frac{1}{t}\,\mathrm{d}t = 1.$$

- $\ln(e^n) = n \ln e = n(1) = n$, we can evaluate the natural logarithms:

x	$\frac{1}{e^3} \approx 0.050$	$\frac{1}{e^2} \approx 0.135$	$\frac{1}{e} \approx 0.368$	$e^0 = 1$	$e \approx 2.718$	$e^2 \approx 7.389$
$\ln x$	-3	-2	-1	0	1	2

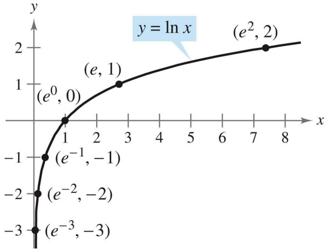


Figure 6: If $x = e^n$, then $\ln x = n$.

- Some useful or interesting values related to e and $\ln x$ are listed below.

Example 1 (Evaluating natural logarithmic expressions)

a. $\ln 2 \approx 0.693$ **b.** $\ln 32 \approx 3.466$ **c.** $\ln 0.1 \approx -2.303$ ■

Euler's Formula

$$e^{ix} = \cos x + i \sin x$$

Euler's Identity: One of the most beautiful theorem in mathematics.

$$e^{i\pi} + 1 = 0$$

The derivative of the natural logarithmic function

- The derivative of the natural logarithmic function is given in Theorem 5.3.
- The first part of the theorem follows from the definition of the natural logarithmic function as an antiderivative.
- The second part of the theorem is simply the Chain Rule version of the first part.

Theorem 5.3 (Derivative of the natural logarithmic function)

Let u be a differentiable function of x.

1. $\frac{\mathrm{d}}{\mathrm{d}x}[\ln x] = \frac{1}{x}$, $x > 0$ **2.** $\frac{\mathrm{d}}{\mathrm{d}x}[\ln u] = \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{u'}{u}$, $u > 0$

Example 2 (Differentiation of logarithmic functions)

a. $\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln(2x)\right]$

b. $\frac{\mathrm{d}}{\mathrm{d}x}[\ln(x^2+1)]$

c. $\frac{\mathrm{d}}{\mathrm{d}x}\left[x\ln x\right]$

d. $\frac{\mathrm{d}}{\mathrm{d}x}\left[(\ln x)^3\right]$

Example 3 (Logarithmic properties as aids to differentiation)

Differentiate $f(x) = \ln \sqrt{x+1}$.

Example 4 (Logarithmic properties as aids to differentiation)

Differentiate $f(x) = \ln \frac{x(x^2+1)^2}{\sqrt{2x^3-1}}$.

- Using logarithms as aids in differentiating nonlogarithmic functions is called logarithmic differentiation.

Example 5 (Logarithmic differentiation)

Find the derivative of

$$y = \frac{(x-2)^2}{\sqrt{x^2+1}}, \quad x \neq 2.$$

Theorem 5.4 (Derivative involving absolute value)

If u is a differentiable function of x such that $u \neq 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln |u| = \frac{u'}{u}.$$

Example 6 (Derivative involving absolute value)

Find the derivative of

$$f(x) = \ln|\cos x|.$$

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The differentiation rules

$$\frac{\mathrm{d}}{\mathrm{d}x}[\ln|x|] = \frac{1}{x} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}[\ln|u|] = \frac{u'}{u}$$

produce the following integration rule.

Theorem 5.5 (Log Rule for integration)

Let u be a differentiable function of x.
1. $\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C$ **2.** $\int \frac{1}{u}\,\mathrm{d}u = \ln|u| + C$

Because $\mathrm{d}u = u'\,\mathrm{d}x$, the second formula can also be written as

$$\int \frac{u'}{u}\,\mathrm{d}x = \ln|u| + C. \qquad \text{Alternative form of Log Rule}$$

Example 1 (Using the Log Rule for integration)

Find $\int \frac{2}{x}\,\mathrm{d}x$

Example 2 (Using the log rule with a change of variables)

Find $\int \frac{1}{4x-1}\,\mathrm{d}x$.

Example 3 (Finding area with the log rule)

Find the area of the region bounded by the graph of $y = \frac{x}{x^2+1}$ the x-axis, and the lines $x = 0$ and $x = 3$.

Example 4 (Recognizing quotient forms of the Log Rule)

a. $\int \frac{3x^2+1}{x^3+x}\,\mathrm{d}x$

b. $\int \frac{\sec^2 x}{\tan x}\,\mathrm{d}x$

c. $\int \frac{x+1}{x^2+2x}\,\mathrm{d}x$

d. $\int \frac{1}{3x+2}\,\mathrm{d}x$

- If a rational function has a numerator of degree greater than or equal to that of the denominator, division may reveal a form to which you can apply the Log Rule!

Example 5 (Using long division before integrating)

Find $\int \frac{x^2+x+1}{x^2+1}\,\mathrm{d}x$.

Example 6 (Change of variables with the Log Rule)

Find $\int \frac{2x}{(x+1)^2}\,\mathrm{d}x$.

Guidelines for integration

1. Learn a basic list of integration formulas.
2. Find an integration formula that resembles all or part of the integrand, and, by trial and error, find a choice of u that will make the integrand conform to the formula.
3. If you cannot find a u-substitution that works, try altering the integrand. You might try a trigonometric identity, multiplication and division by the same quantity, addition and subtraction of the same quantity, or long division. Be creative!
4. (Not for exam) If you have access to computer software that will find antiderivatives symbolically, use it.
5. Check your result by differentiating to obtain the original integrand.

Example 7 (u-Substitution and the Log Rule)

Solve the differential equation $\frac{dy}{dx} = \frac{1}{x \ln x}$.

Integrals of trigonometric functions

Example 8 (Using a trigonometric identity)

Find $\int \tan x \, \mathrm{d}x$.

Example 9 (Derivation of the Secant Formula)

Find $\int \sec x \, \mathrm{d}x$.

Table 1: Integrals of the six basic trigonometric functions

$\int \sin u \, du = -\cos u + C$	$\int \cos u \, du = \sin u + C$
$\int \tan u \, du = -\ln \lvert \cos u \rvert + C$	$\int \cot u \, du = \ln \lvert \sin u \rvert + C$
$\int \sec u \, du = \ln \lvert \sec u + \tan u \rvert + C$	$\int \csc u \, du = -\ln \lvert \csc u + \cot u \rvert + C$

Example 10 (Integrating trigonometric functions)

Evaluate $\int_0^{\pi/4} \sqrt{1+\tan^2 x}\,\mathrm{d}x$.

Example 11 (Finding an average value)

Find the average value of $f(x) = \tan x$ on the interval $\left[0, \frac{\pi}{4}\right]$.

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Inverse functions

- The function $f(x) = x + 3$ from $A = \{1, 2, 3, 4\}$ to $B = \{4, 5, 6, 7\}$ can be written as

$$f\colon \{(1,4),\ (2,5),\ (3,6),\ (4,7)\}.$$

- By interchanging the first and second coordinates of each ordered pair, you can form the inverse function of f. This function is denoted by f^{-1}. It is a function from B to A, and can be written as

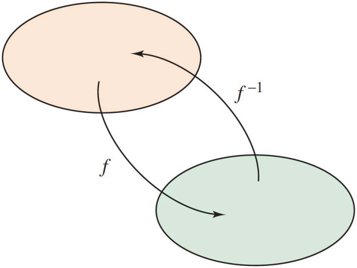
$$f^{-1}\colon \{(4,1),\ (5,2),\ (6,3),\ (7,4)\}.$$

- The domain of f is equal to the range of f^{-1}, and vice versa. When you form the composition of f with f^{-1} or the composition of f^{-1} with f, you obtain the identity function.

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x$$

Definition 5.3 (Inverse function)

A function g is the inverse function of the function f if $f(g(x)) = x$ for each x in the domain of g and $g(f(x)) = x$ for each x in the domain of f. The function g is denoted by f^{-1} (read "f inverse").



Here are some important observations about inverse functions.

1. If g is the inverse function of f, then f is the inverse function of g.
2. The domain of f^{-1} is equal to the range of f, and the range of f^{-1} is equal to the domain of f.
3. A function need not have an inverse function, but if it does, the inverse function is unique!

- You can think of f^{-1} as undoing what has been done by f.
- $f(x) = x + c$ and $f^{-1}(x) = x - c$ are inverse functions of each other.
- $f(x) = cx$ and $f^{-1}(x) = \frac{x}{c}$, $c \neq 0$, are inverse functions of each other.

Example 1 (Verifying inverse functions)

Show that the functions are inverse functions of each other.

$$f(x) = 2x^3 - 1 \quad \text{and} \quad g(x) = \sqrt[3]{\frac{x+1}{2}}$$

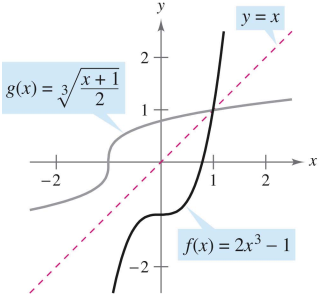


Figure 7: $f(x) = 2x^3 - 1$ and $g(x) = \sqrt[3]{\frac{x+1}{2}}$ are inverse functions of each other.

- In Figure 7, the graphs of f and $g = f^{-1}$ appear to be mirror images of each other with respect to the line $y = x$.
- The graph of f^{-1} is a reflection of the graph of f in the line $y = x$!
- The idea of a reflection of the graph of f in the line $y = x$ is generalized in the following theorem.

Theorem 5.6 (Reflective property of inverse functions)

The graph of f contains the point (a, b) if and only if the graph of f^{-1} contains the point (b, a).

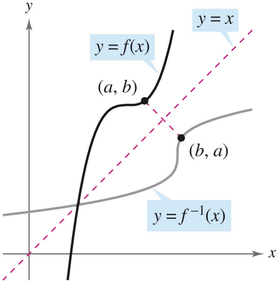


Figure 8: The graph of f^{-1} is a reflection of the graph of f in the line $y = x$.

Existence of an inverse function

- Not every function has an inverse function, and Theorem 5.6 suggests a graphical test for those that do—the Horizontal Line Test for an inverse function.
- This test states that a function f has an inverse function if and only if every horizontal line intersects the graph of f at most once.

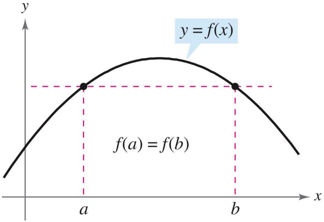


Figure 9: If a horizontal line intersects the graph of f twice, then f is not one-to-one.

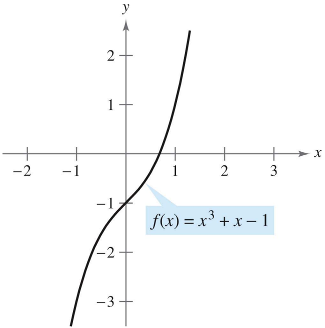
Theorem 5.7 (The existence of an inverse function)

1. *A function has an inverse function if and only if it is one-to-one.*
2. *If f is strictly monotonic on its entire domain, then it is one-to-one and therefore has an inverse function.*

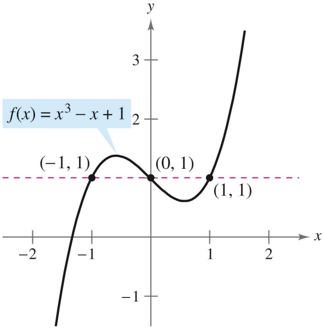
Example 2 (The existence of an inverse function)

Which of the functions has an inverse function?

a. $f(x) = x^3 + x - 1$ **b.** $f(x) = x^3 - x + 1$



(a) Because $f(x) = x^3 + x - 1$ is increasing over its entire domain, it has an inverse function.



(b) Because $f(x) = x^3 - x + 1$ is not one-to-one, it does not have an inverse function.

Figure 10: The existence of an inverse function.

- The following guidelines suggest a procedure for finding an inverse function.

Guidelines for finding an inverse function

1. Use Theorem 5.7 to determine whether the function given by $y = f(x)$ has an inverse function.
2. Solve for x as a function of $y : x = g(y) = f^{-1}(y)$.
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.
4. Define the domain of f^{-1} as the range of f.
5. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Example 3 (Finding an inverse function)

Find the inverse function of $f(x) = \sqrt{2x - 3}$.

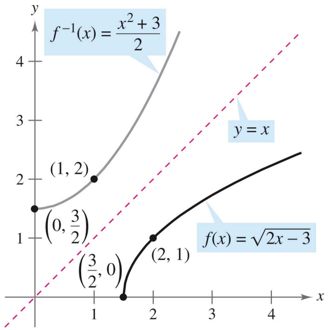


Figure 11: The domain of $f^{-1}(x) = \frac{x^2+3}{2}$, $[0, \infty)$ is the range of $f(x) = \sqrt{2x-3}$.

- Suppose you are given a function that is not one-to-one on its domain.
- By restricting the domain to an interval on which the function is strictly monotonic, you can conclude that the new function is one-to-one on the restricted domain.

Example 4 (Testing whether a function is one-to-one)

Show that the sine function

$$f(x) = \sin x$$

is not one-to-one on the entire real line. Then show that $[-\pi/2, \pi/2]$ is the largest interval, centered at the origin, on which f is strictly monotonic.

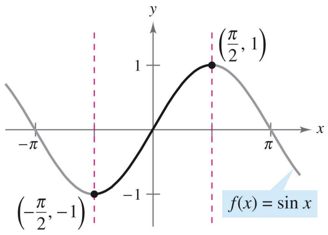


Figure 12: $f(x) = \sin x$ is one-to-one on the interval $[-\pi/2, \pi/2]$.

Derivative of an inverse function

The next two theorems discuss the derivative of an inverse function.

Theorem 5.8 (Continuity and differentiability of inverse functions)

Let f be a function whose domain is an interval I. If f has an inverse function, then the following statements are true.

1. *If f is continuous on its domain, then f^{-1} is continuous on its domain.*
2. *If f is increasing on its domain, then f^{-1} is increasing on its domain.*
3. *If f is decreasing on its domain, then f^{-1} is decreasing on its domain.*
4. *If f is differentiable on an interval containing c and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$.*

Theorem 5.9 (The derivative of an inverse function)

Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0.$$

Example 5 (Evaluating the derivative of an inverse function)

Let $f(x) = \frac{1}{4}x^3 + x - 1$.
a. What is the value of $f^{-1}(x)$ when $x = 3$?
b. What is the value of $(f^{-1})'(x)$ when $x = 3$?

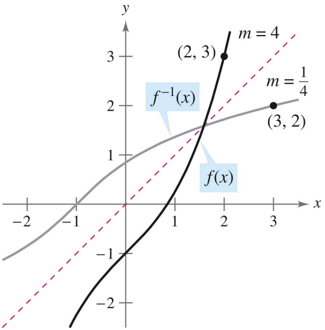


Figure 13: The graphs of the inverse functions f and f^{-1} have reciprocal slopes at points (a, b) and (b, a).

Example 6 (Graphs of inverse functions have reciprocal slopes)

Let $f(x) = x^2$ (for $x \geq 0$) and let $f^{-1}(x) = \sqrt{x}$. Show that the slopes of the graphs of f and f^{-1} are reciprocals at each of the following points.

a. $(2, 4)$ and $(4, 2)$

b. $(3, 9)$ and $(9, 3)$

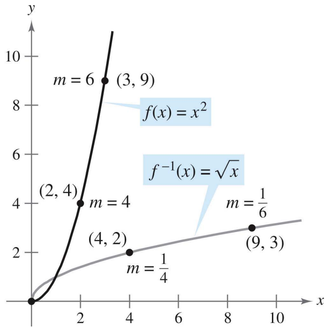


Figure 14: At $(0, 0)$, the derivative of $f(x) = x^2$ is 0, and the derivative of $f^{-1}(x) = \sqrt{x}$ does not exist.

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The natural exponential function

- The function $f(x) = \ln x$ is increasing on its entire domain, and therefore it has an inverse function f^{-1}.
- The domain of f^{-1} is the set of all reals, and the range is the set of positive reals, as shown in Figure 15.

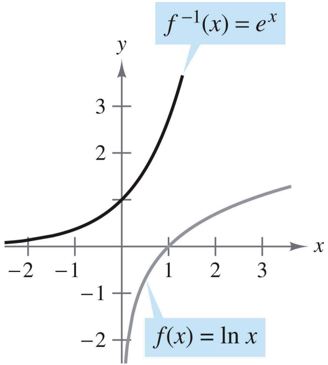


Figure 15: The inverse function of the natural logarithmic function is the natural exponential function.

- So, for any real number x,

$$f(f^{-1}(x)) = \ln[f^{-1}(x)] = x. \qquad x \text{ is any real number}$$

- If x happens to be rational, then

$$\ln(e^x) = x \ln e = x(1) = x. \qquad x \text{ is a rational number}$$

- Because the natural logarithmic function is one-to-one, you can conclude that $f^{-1}(x)$ and e^x agree for rational values of x. The following definition extends to include all real values of x.

Definition 5.4 (The natural exponential function)

The inverse function of the natural logarithmic function $f(x) = \ln x$ is called the natural exponential function and is denoted by

$$f^{-1}(x) = e^x.$$

That is $y = e^x$ if and only if $x = \ln y$.

- The inverse relationship between the natural logarithmic function and the natural exponential function can be summarized as follows.

$$\ln(e^x) = x \quad \text{and} \quad e^{\ln x} = x \qquad \text{Inverse relationship}$$

Example 1 (Solving an exponential equation)

Solve $7 = e^{x+1}$.

Example 2 (Solving a logarithmic equation (exponentiate))

Solve $\ln(2x - 3) = 5$.

Theorem 5.10 (Operations with exponential functions)

Let a and b be any real numbers.

1. $e^a e^b = e^{a+b}$
2. $\frac{e^a}{e^b} = e^{a-b}$

- An inverse function f^{-1} shares many properties with f.
- So, the natural exponential function inherits the following properties from the natural logarithmic function (see Figure 16).

Properties of the natural exponential function

1. The domain of $f(x) = e^x$ is $(-\infty, \infty)$, and the range is $(0, \infty)$.
2. The function $f(x) = e^x$ is continuous, increasing, and one-to-one on its entire domain.
3. The graph of $f(x) = e^x$ is concave upward on its entire domain.
4. $\lim_{x\to-\infty} e^x = 0$ and $\lim_{x\to\infty} e^x = \infty$.

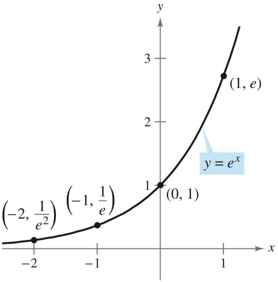


Figure 16: The natural exponential function is increasing, and its graph is concave upward.

Derivatives of exponential functions

- One of the most intriguing (and useful) characteristics of the natural exponential function is that it is its own derivative.

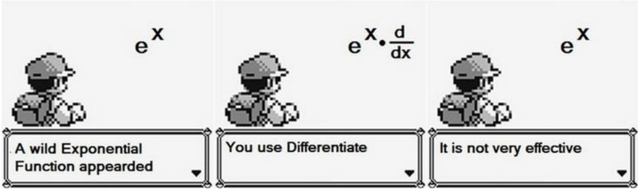


Figure 17: source: https://www.pinterest.com/pin/548454060851043602/

Theorem 5.11 (Derivatives of the natural exponential function)

Let u be a differentiable function of x.

1. $\frac{\mathrm{d}}{\mathrm{d}x}[e^x] = e^x$
2. $\frac{\mathrm{d}}{\mathrm{d}x}[e^u] = e^u \frac{\mathrm{d}u}{\mathrm{d}x}$

Example 3 (Differentiating exponential functions)

a. $\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{2x-1}\right]$

b. $\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{-3/x}\right]$

c. $\frac{\mathrm{d}}{\mathrm{d}x}\left[x^2 e^x\right]$

d. $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{e^{3x}}{e^x+1}\right]$

Example 4 (Locating relative extrema)

Find the relative extrema of $f(x) = xe^x$.

Example 5 (Finding an equation of a tangent line)

Find an equation of the tangent line to the graph of $f(x) = 2 + e^{1-x}$ at the point $(1, 3)$.

Integrals of exponential functions

Theorem 5.12 (Integration rules for exponential functions)

Let u be a differentiable function of x.
1. $\int e^x \, \mathrm{d}x = e^x + C$ *2.* $\int e^u \, \mathrm{d}u = e^u + C$

Example 7 (Integrating exponential functions)

Find $\int e^{3x+1} \, \mathrm{d}x$.

Example 8 (Integrating exponential functions)

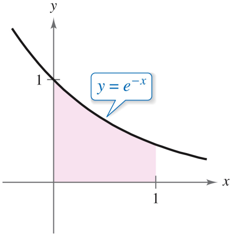
Find $\int 5xe^{-x^2}\,\mathrm{d}x$.

Example 9 (Integrating exponential functions)

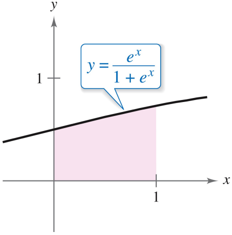
a. $\int \frac{e^{1/x}}{x^2}\,\mathrm{d}x$ **b.** $\int \sin x\, e^{\cos x}\,\mathrm{d}x$

Example 10 (Finding areas bounded by exponential functions)

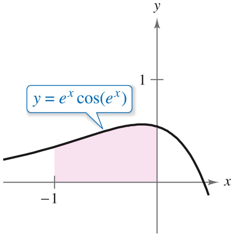
a. $\int_0^1 e^{-x}\,\mathrm{d}x$ **b.** $\int_0^1 \frac{e^x}{1+e^x}\,\mathrm{d}x$ **c.** $\int_{-1}^0 [e^x\cos(e^x)]\,\mathrm{d}x$



(a) $y = e^{-x}$



(b) $y = \frac{e^x}{1+e^x}$



(c) $y = e^x\cos(e^x)$

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- The base of the natural exponential function is e. This "natural" base can be used to assign a meaning to a general base a.

Definition 5.5 (Exponential function to base a)

If a is a positive real number ($a \neq 1$) and x is any real number, then the exponential function to the base a is denoted by a^x and is defined by

$$a^x = e^{(\ln a)x}.$$

If $a = 1$, then $y = 1^x = 1$ is a constant function.

- These functions obey the usual laws of exponents. For instance, here are some familiar properties.
  1. $a^0 = 1$
  2. $a^x a^y = a^{x+y}$
  3. $\frac{a^x}{a^y} = a^{x-y}$
  4. $(a^x)^y = a^{xy}$
- When modeling the half-life of a radioactive sample, it is convenient to use $\frac{1}{2}$ as the base of the exponential model. (Half-life is the number of years required for half of the atoms in a sample of radioactive material to decay.)

Definition 5.6 (Logarithmic function to base a)

If a is a positive real number ($a \neq 1$) and x is any positive real number, then the logarithmic function to the base a is denoted by $\log_a x$ and is defined as

$$\log_a x = \frac{1}{\ln a} \ln x.$$

- Logarithmic functions to the base a have properties similar to those of the natural logarithmic function. $a > 0$, $a \neq 1$, x, $y > 0$
  1. $\log_a 1 = 0$ Log of 1
  2. $\log_a xy = \log_a x + \log_a y$ Log of a product
  3. $\log_a x^n = n \log_a x$ Log of a power
  4. $\log_a \frac{x}{y} = \log_a x - \log_a y$ Log of a quotient

- From the definitions of the exponential and logarithmic functions to the base a, it follows that $f(x) = a^x$ and $g(x) = \log_a x$ are inverse functions of each other.

Properties of inverse functions

1. $y = a^x$ if and only if $x = \log_a y$.
2. $a^{\log_a x} = x$, for $x > 0$.
3. $\log_a a^x = x$, for all x.

- The logarithmic function to the base 10 is called the common logarithmic function. So, for common logarithms, $y = 10^x$ if and only if $x = \log_{10} y$.

Example 2 (Bases other than e)

Solve for x in each equation. **a.** $3^x = \frac{1}{81}$ **b.** $\log_2 x = -4$

Differentiation and integration

- To differentiate exponential and logarithmic functions to other bases, you have three options:
  (1) use the definitions of a^x and $\log_a x$ and differentiate using the rules for the natural exponential and logarithmic functions,
  (2) use logarithmic differentiation, or
  (3) use the following differentiation rules for bases other than e.

Theorem 5.13 (Derivatives for bases other than e)

Let a be a positive real number ($a \neq 1$) and let u be a differentiable function of x.

1. $\frac{\mathrm{d}}{\mathrm{d}x}[a^x] = (\ln a)a^x$
2. $\frac{\mathrm{d}}{\mathrm{d}x}[a^u] = (\ln a)a^u \frac{\mathrm{d}u}{\mathrm{d}x}$
3. $\frac{\mathrm{d}}{\mathrm{d}x}[\log_a x] = \frac{1}{(\ln a)x}$
4. $\frac{\mathrm{d}}{\mathrm{d}x}[\log_a u] = \frac{1}{(\ln a)u} \frac{\mathrm{d}u}{\mathrm{d}x}$

Example 3 (Differentiating functions to other bases)

Find the derivative of each function.

a. $y = 2^x$ **b.** $y = 2^{3x}$ **c.** $y = \log_{10} \cos x$ **d.** $y = \log_3 \frac{\sqrt{x}}{x+5}$

- Occasionally, an integrand involves an exponential function to a base other than e. When this occurs, there are two options:
  1. convert to base e using the formula $a^x = e^{(\ln a)x}$ and then integrate, or
  2. integrate directly, using the integration formula

$$\int a^x \, \mathrm{d}x = \left(\frac{1}{\ln a}\right) a^x + C.$$

Example 4 (Integrating an exponential function to another base)

Find $\int 2^x \, dx$.

Theorem 5.14 (The Power Rule for real exponents)

Let n be any real number and let u be a differentiable function of x.

1. $\frac{d}{dx}[x^n] = nx^{n-1}$
2. $\frac{d}{dx}[u^n] = nu^{n-1}\frac{du}{dx}$

Example 5 (Comparing variables and constants)

a. $\frac{\mathrm{d}}{\mathrm{d}x}\left[e^e\right]$

b. $\frac{\mathrm{d}}{\mathrm{d}x}\left[e^x\right]$

c. $\frac{\mathrm{d}}{\mathrm{d}x}\left[x^e\right]$

d. $y = x^x$

Applications of exponential functions

- Suppose P dollars is deposited in an account at an annual interest rate r (in decimal form). If interest accumulates in the account, what is the balance in the account at the end of 1 year?
- The answer depends on the number of times n the interest is compounded according to the formula

$$A = P\left(1 + \frac{r}{n}\right)^n.$$

- For instance, the result for a deposit of \$1000 at 8% interest compounded n times a year is shown in the table.

n	A
1	\$1080.00
2	\$1081.60
4	\$1082.33
12	\$1083.00
365	\$1083.28

- As n increases, the balance A approaches a limit. To develop this limit, use the following theorem.

Theorem 5.15 (A limit involving e)

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty}\left(\frac{x+1}{x}\right)^x = e$$

- To test the reasonableness of this theorem, try evaluating $[(x+1)/x]^x$ for several values of x, as shown in the table.

x	$\left(\frac{x+1}{x}\right)^x$
10	2.59374
100	2.70481
1,000	2.71692
10,000	2.71815
100,000	2.71827
1,000,000	2.71828

- Now, let's take another look at the formula for the balance A in an account in which the interest is compounded n times per year.
- By taking the limit as n approaches infinity, you obtain

$$A = \lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^n = P\lim_{n\to\infty}\left[\left(1+\frac{1}{n/r}\right)^{n/r}\right]^r$$
$$= P\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right]^r = Pe^r.$$

- This limit produces the balance after 1 year of continuous compounding. So, for a deposit of 1000 at 8% interest compounded continuously, the balance at the end of 1 year would be

$$A = 1000e^{0.08} \approx \$1083.29.$$

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Indeterminate forms

- The forms $0/0$ and ∞/∞ are called indeterminate because they do not guarantee that a limit exists, nor do they indicate what the limit is, if one does exist.
- When you encountered one of these indeterminate forms earlier in the text, you attempted to rewrite the expression by using various algebraic techniques.

Indeterminate forms	*Limit*	*Algebraic technique*
$\frac{0}{0}$	$\lim_{x\to -1} \frac{2x^2-2}{x+1}$ $= \lim_{x\to -1} 2(x-1) = -4$	Divide numerator and denominator by $(x+1)$.
$\frac{\infty}{\infty}$	$\lim_{x\to\infty} \frac{3x^2-1}{2x^2+1}$ $= \lim_{x\to\infty} \frac{3-(1/x^2)}{2+(1/x^2)} = \frac{3}{2}$	Divide numerator and denominator by x^2.

- You can extend these algebraic techniques to find limits of transcendental functions. For instance, the limit

$$\lim_{x \to 0} \frac{e^{2x} - 1}{e^x - 1}$$

produces the indeterminate form $0/0$.

- Factoring and then dividing produces

$$\lim_{x \to 0} \frac{e^{2x} - 1}{e^x - 1} = \lim_{x \to 0} \frac{(e^x + 1)(e^x - 1)}{e^x - 1} = \lim_{x \to 0} (e^x + 1) = 2.$$

- However, not all indeterminate forms can be evaluated by algebraic manipulation. This is often true when both algebraic and transcendental functions are involved. For instance, the limit

$$\lim_{x \to 0} \frac{e^{2x} - 1}{x}$$

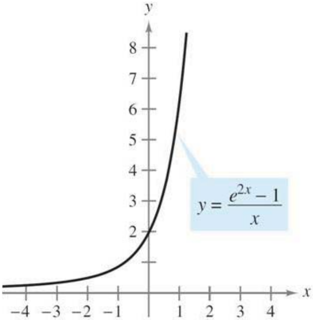
produces the indeterminate form $0/0$.

- Rewriting the expression to obtain

$$\lim_{x\to 0}\left(\frac{e^{2x}}{x}-\frac{1}{x}\right)$$

merely produces another indeterminate form, $\infty - \infty$.
- You could use technology to estimate the limit, as shown below. From the table and the graph, the limit appears to be 2.

x	-1	-0.1	-0.01	-0.001	0	0.001	0.01	0.1	1
$\frac{e^{2x}-1}{x}$	0.865	1.813	1.980	1.998	?	2.002	2.020	2.214	6.389



L'Hôpital's Rule

- To find the limit illustrated above, you can use a theorem called L'Hôpital's Rule. This theorem states that under certain conditions the limit of the quotient $f(x)/g(x)$ is determined by the limit of the quotient of the derivatives $\frac{f'(x)}{g'(x)}$.
- To prove this theorem, you can use a more general result called the Extended Mean Value Theorem.

Theorem 5.16 (The Extended Mean Value Theorem)

If f and g are differentiable on an open interval (a, b) and continuous on $[a, b]$ such that $g'(x) \neq 0$ for any x in (a, b), then there exists a point c in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Theorem 5.17 (L'Hôpital's Rule)

Let f and g be functions that are differentiable on an open interval (a, b) containing c, except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, b), except possibly at c itself. If the limit of $f(x)/g(x)$ as x approaches c produces the indeterminate form $0/0$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite). This result also applies if the limit of $f(x)/g(x)$ as x approaches c produces anyone of the indeterminate forms ∞/∞, $(-\infty)/\infty$, $\infty/(-\infty)$ or $(-\infty)/(-\infty)$.

Example 1 (Indeterminate form 0/0)

Evaluate $\lim_{x \to 0} \frac{e^{2x} - 1}{x}$.

Example 2 (Indeterminate form $\frac{\infty}{\infty}$)

Evaluate $\lim_{x\to\infty} \frac{\ln x}{x}$.

Example 3 (Applying L'Hôpital's Rule more than once)

Evaluate $\lim_{x\to-\infty} \frac{x^2}{e^{-x}}$.

Example 4 (Indeterminate form $0 \cdot \infty$)

Evaluate $\lim_{x\to\infty} e^{-x}\sqrt{x}$.

Example 5 (Indeterminate form 1^∞)

Evaluate $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$.

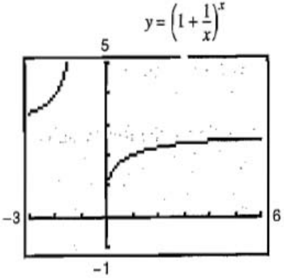


Figure 19: The limit of $[1 + (1/x)]^x$ as x approaches infinity is e.

Example 6 (Indeterminate form 0^0)

Find $\lim_{x\to 0^+}(\sin x)^x$.

Example 7 (Indeterminate form $\infty - \infty$)

Evaluate $\lim_{x\to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right)$.

- The forms $0/0$, ∞/∞, $\infty - \infty$, $0 \cdot \infty$, 0^0, 1^∞, and ∞^0 have been identified as indeterminate. There are similar forms that you should recognize as determinate.

$$\begin{array}{lll} \infty = \infty + \infty & \to \infty & \text{Limit is positive infinity} \\ -\infty - \infty & \to -\infty & \text{Limit is negative infinity} \\ 0^{\infty} & \to 0 & \text{Limit is zero} \\ 0^{-\infty} & \to \infty & \text{Limit is positive infinity} \end{array}$$

- As a final comment, remember that L'Hôpital's Rule can be applied only to quotients leading to the indeterminate forms $0/0$ and ∞/∞.
- For instance, the following application of L'Hôpital's Rule is incorrect.

$$\lim_{x\to 0} \frac{e^x}{x} = \lim_{x\to 0} \frac{e^x}{1} = 1$$

Incorrect use of L'Hôpital's Rule

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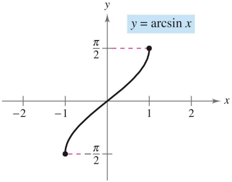
Inverse trigonometric functions

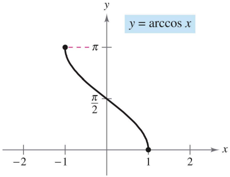
- None of the six basic trigonometric functions has an inverse function. This statement is true because all six trigonometric functions are periodic and therefore are not one-to-one.
- In this section you will examine these six functions to see whether their domains can be redefined in such a way that they will have inverse functions on the restricted domains.
- Under suitable restrictions, each of the six trigonometric functions is one-to-one and so has an inverse function, as shown in the following definition.

Function	Domain	Range
$y = \arcsin x$ iff $\sin y = x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \arccos x$ iff $\cos y = x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \arctan x$ iff $\tan y = x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \operatorname{arccot} x$ iff $\cot y = x$	$-\infty < x < \infty$	$0 < y < \pi$
$y = \operatorname{arcsec} x$ iff $\sec y = x$	$\lvert x \rvert \ge 1$	$0 \le y \le \pi,\ y \ne \frac{\pi}{2}$
$y = \operatorname{arccsc} x$ iff $\csc y = x$	$\lvert x \rvert \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2},\ y \ne 0$

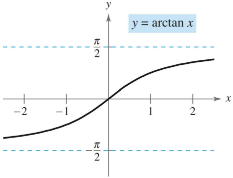
- The graphs of the six inverse trigonometric functions are shown in Figure 20.



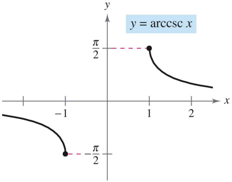
(a) Domain: $[-1, 1]$, Range: $[-\pi/2, \pi/2]$



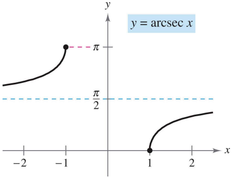
(b) Domain: $[-1, 1]$, Range: $[0, \pi]$



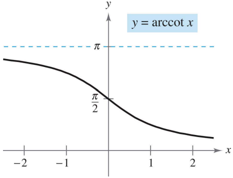
(c) Domain: $(-\infty, \infty)$, Range: $(-\pi/2, \pi/2)$



(d) Domain: $(-\infty, -1] \cup [1, \infty)$, Range: $[-\pi/2, 0) \cup (0, \pi/2]$



(e) Domain: $(-\infty, -1] \cup [1, \infty)$, Range: $[0, \pi/2) \cup (\pi/2, \pi]$



(f) Domain: $(-\infty, \infty)$, Range: $(0, \pi)$

Figure 20: Six inverse trigonometric functions.

Example 1 (Evaluating inverse trigonometric functions)

Evaluate each function.

a. $\arcsin\left(-\frac{1}{2}\right)$ **b.** $\arccos 0$ **c.** $\arctan\sqrt{3}$

- Inverse functions have the properties

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

- When applying these properties to inverse trigonometric functions, remember that the trigonometric functions have inverse functions only in restricted domains.
- For x-values outside these domains, these two properties do not hold.
- For example, $\arcsin(\sin\pi)$ is equal to 0, not π.

Properties of inverse trigonometric functions

If $-1 \leq x \leq 1$ and $-\pi/2 \leq y \leq \pi/2$, then

$$\sin(\arcsin x) = x \quad \text{and} \quad \arcsin(\sin y) = y.$$

If $-\infty < x < \infty$ and $-\pi/2 < y < \pi/2$, then

$$\tan(\arctan x) = x \quad \text{and} \quad \arctan(\tan y) = y.$$

If $|x| \geq 1$ and $0 \leq y < \pi/2$ or $\pi/2 < y \leq \pi$, then

$$\sec(\operatorname{arcsec} x) = x \quad \text{and} \quad \operatorname{arcsec}(\sec y) = y.$$

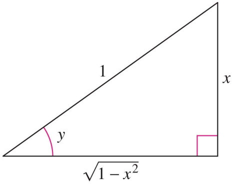
Similar properties hold for the other inverse trigonometric functions.

Example 2 (Solving an equation)

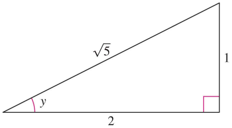
$\arctan(2x - 3) = \frac{\pi}{4}$

Example 3 (Using right triangles)

a. Given $y = \arcsin x$, where $0 < y < \pi/2$, find $\cos y$.
b. Given $y = \operatorname{arcsec}(\sqrt{5}/2)$, find $\tan y$.



(a) $y = \arcsin x$



(b) $y = \operatorname{arcsec}\left(\frac{\sqrt{5}}{2}\right)$

Figure 21: Using right triangles.

Derivatives of inverse trigonometric functions

- The derivative of the transcendental function $f(x) = \ln x$ is the algebraic function $f'(x) = 1/x$.
- You will now see that the derivatives of the inverse trigonometric functions also are algebraic!

Theorem 5.18 (Derivatives of inverse trigonometric functions)

Let u be a differentiable function of x.

$$\frac{\mathrm{d}}{\mathrm{d}x}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\arctan u] = \frac{u'}{1+u^2} \qquad \frac{\mathrm{d}}{\mathrm{d}x}[\operatorname{arccot} u] = \frac{-u'}{1+u^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

Example 4 (Differentiating inverse trigonometric functions)

a. $\frac{\mathrm{d}}{\mathrm{d}x}[\arcsin(2x)]$

b. $\frac{\mathrm{d}}{\mathrm{d}x}[\arctan(3x)]$

c. $\frac{\mathrm{d}}{\mathrm{d}x}[\arcsin\sqrt{x}]$

d. $\frac{\mathrm{d}}{\mathrm{d}x}[\operatorname{arcsec} e^{2x}]$

Example 5 (A derivative that can be simplified)

Find the derivative of $y = \arcsin x + x\sqrt{1 - x^2}$

Review of basic differentiation rules

1. $\frac{d}{dx}[cu] = cu'$	2. $\frac{d}{dx}[u \pm v] = u' \pm v'$	3. $\frac{d}{dx}[uv] = uv' + vu'$
4. $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$	5. $\frac{d}{dx}[c] = 0$	6. $\frac{d}{dx}[u^n] = nu^{n-1}u'$
7. $\frac{d}{dx}[x] = 1$	8. $\frac{d}{dx}[\lvert u\rvert] = \frac{u}{\lvert u\rvert}(u'), \quad u \neq 0$	9. $\frac{d}{dx}[\ln u] = \frac{u'}{u}$
10. $\frac{d}{dx}[e^u] = e^u u'$	11. $\frac{d}{dx}[\log_a u] = \frac{u'}{(\ln a)u}$	12. $\frac{d}{dx}[a^u] = (\ln a)a^u u'$
13. $\frac{d}{dx}[\sin u] = (\cos u)u'$	14. $\frac{d}{dx}[\cos u] = -(\sin u)u'$	15. $\frac{d}{dx}[\tan u] = (\sec^2 u)u'$
16. $\frac{d}{dx}[\cot u] = -(\csc^2 u)u'$	17. $\frac{d}{dx}[\sec u] = (\sec u \tan u)u'$	18. $\frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$
19. $\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$	20. $\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$	21. $\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$
22. $\frac{d}{dx}[\text{arccot}\, u] = \frac{-u'}{1+u^2}$	23. $\frac{d}{dx}[\text{arcsec}\, u] = \frac{u'}{\lvert u\rvert\sqrt{u^2-1}}$	24. $\frac{d}{dx}[\text{arccsc}\, u] = \frac{-u'}{\lvert u\rvert\sqrt{u^2-1}}$

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Integrals involving inverse trigonometric functions

- The derivatives of the six inverse trigonometric functions fall into three pairs. In each pair, the derivative of one function is the negative of the other.
- For example

$$\frac{\mathrm{d}}{\mathrm{d}x}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}[\arccos x] = -\frac{1}{\sqrt{1-x^2}}.$$

- When listing the antiderivative that corresponds to each of the inverse trigonometric functions, you need to use only one member from each pair. It is conventional to use $\arcsin x$ as the antiderivative of $1/\sqrt{1-x^2}$, rather than $-\arccos x$.

Identities involving inverse trigonometric functions

$$\arcsin x + \arccos x = \frac{1}{2}\pi, \quad |x| \le 1$$
$$\arctan x + \operatorname{arccot} x = \frac{1}{2}\pi, \quad |x| \in \mathbb{R}$$
$$\operatorname{arcsec} x + \operatorname{arccsc} x = \frac{1}{2}\pi, \quad |x| \ge 1$$

Theorem 5.19 (Integrals involving inverse trigonometric functions)

Let u be a differentiable function of x, and let $a > 0$.

1. $\int \frac{\mathrm{d}u}{\sqrt{a^2-u^2}} = \arcsin \frac{u}{a} + C$ **2.** $\int \frac{\mathrm{d}u}{a^2+u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$ **3.** $\int \frac{\mathrm{d}u}{u\sqrt{u^2-a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + C$

Example 1 (Integration with inverse trigonometric functions)

a. $\int \frac{\mathrm{d}x}{\sqrt{4-x^2}}$

b. $\int \frac{\mathrm{d}x}{2+9x^2}$

c. $\int \frac{\mathrm{d}x}{x\sqrt{4x^2-9}}$

Example 2 (Integration by substitution)

Find $\int \frac{\mathrm{d}x}{\sqrt{e^{2x}-1}}$.

Example 3 (Rewriting as the sum of two quotients)

Find $\int \frac{x+2}{\sqrt{4-x^2}}\,\mathrm{d}x$.

Completing the square

- Completing the square helps when quadratic functions are involved in the integrand.
- For example, the quadratic $x^2 + bx + c$ can be written as the difference of two squares by adding and subtracting $(b/2)^2$.

$$\begin{aligned} x^2 + bx + c &= x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \\ &= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \end{aligned}$$

Example 4 (Completing the square)

Find $\int \frac{\mathrm{d}x}{x^2-4x+7}$.

Example 5 (Completing the square (negative leading coefficient))

Find the area of the region bounded by the graph of $f(x) = \frac{1}{\sqrt{3x-x^2}}$ the x-axis, and the lines $x = \frac{3}{2}$ and $x = \frac{9}{4}$.

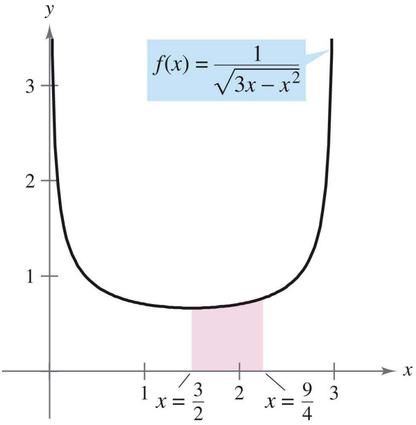

Figure 22: The area of the region bounded by the graph of f, the x-axis, and the lines $x = \frac{3}{2}$ and $x = \frac{9}{4}$ is $\pi/6$.

Review of basic integration rules

Table 2: Basic integration rules ($a > 0$)

1. $\int kf(u)\,\mathrm{d}u = k\int f(u)\,\mathrm{d}u$	2. $\int [f(u) \pm g(u)]\,\mathrm{d}u = \int f(u)\,\mathrm{d}u \pm \int g(u)\,\mathrm{d}u$
3. $\int \mathrm{d}u = u + C$	4. $\int u^n\,\mathrm{d}u = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
5. $\int \frac{\mathrm{d}u}{u} = \ln\lvert u\rvert + C$	6. $\int e^u\,\mathrm{d}u = e^u + C$
7. $\int a^u\,\mathrm{d}u = \left(\frac{1}{\ln a}\right) a^u + C$	8. $\int \sin u\,\mathrm{d}u = -\cos u + C$
9. $\int \cos u\,\mathrm{d}u = \sin u + C$	10. $\int \tan u\,\mathrm{d}u = -\ln\lvert\cos u\rvert + C$
11. $\int \cot u\,\mathrm{d}u = \ln\lvert\sin u\rvert + C$	12. $\int \sec u\,\mathrm{d}u = \ln\lvert\sec u + \tan u\rvert + C$
13. $\int \csc u\,\mathrm{d}u = -\ln\lvert\csc u + \cot u\rvert + C$	14. $\int \sec^2 u\,\mathrm{d}u = \tan u + C$
15. $\int \csc^2 u\,\mathrm{d}u = -\cot u + C$	16. $\int \sec u \tan u\,\mathrm{d}u = \sec u + C$
17. $\int \csc u \cot u\,\mathrm{d}u = -\csc u + C$	18. $\int \frac{\mathrm{d}u}{\sqrt{a^2 - u^2}} = \arcsin\frac{u}{a} + C$
19. $\int \frac{\mathrm{d}u}{a^2 + u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$	20. $\int \frac{\mathrm{d}u}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\,\mathrm{arcsec}\,\frac{\lvert u\rvert}{a} + C$

Example 6 (Comparing integration problems)

Find as many of the following integrals as you can using the formulas and techniques you have studied so far in the text.

a. $\int \frac{dx}{x\sqrt{x^2-1}}$ **b.** $\int \frac{x\,dx}{\sqrt{x^2-1}}$ **c.** $\int \frac{dx}{\sqrt{x^2-1}}$

Example 7 (Comparing integration problems)

Find as many of the following integrals as you can using the formulas and techniques you have studied so far in the text.

a. $\int \frac{\mathrm{d}x}{x \ln x}$ **b.** $\int \frac{\ln x \, \mathrm{d}x}{x}$ **c.** $\int \ln x \, \mathrm{d}x$