# Chapter 3 Applications of Differentiation

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# Extrema of a function

- In calculus, much effort is devoted to determining the behavior of a function $f$ on an interval $I$.

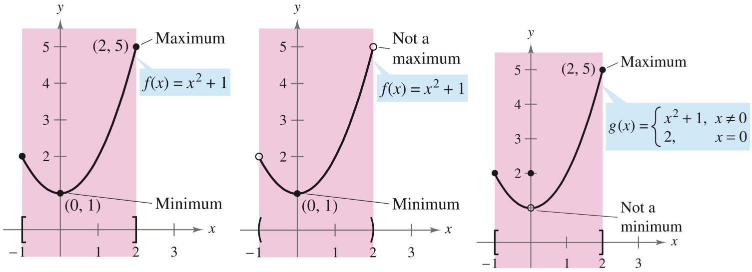
**Definition 3.1 (Extrema)**

Let $f$ be defined on an interval $I$ containing $c$

1. $f(c)$ is the minimum of $f$ on $I$ if $f(c) \leq f(x)$ for all $x$ in $I$.
2. $f(c)$ is the maximum of $f$ on $I$ if $f(c) \geq f(x)$ for all $x$ in $I$.

The minimum and maximum of a function on an interval are the extrema of the function on the interval. The minimum and maximum of a function on an interval are also called the absolute minimum and absolute maximum, or the global minimum and global maximum, on the interval.

- A function need not have a minimum or a maximum on an interval.




(a) $f$ is continuous, $[-1, 2]$ is closed.

(b) $f$ is continuous, $(-1, 2)$ is open.

(c) $g$ is not continuous, $[-1, 2]$ is closed.

Figure 1: Extrema can occur at interior points or endpoints of an interval. Extrema that occur at the endpoints are called endpoint extrema.

## Theorem 3.1 (The Extreme Value Theorem)

*If $f$ is continuous on a closed interval $[a, b]$, then $f$ has both a minimum and a maximum on the interval.*

## Definition 3.2 (Relative extrema)

1. If there is an open interval containing $c$ on which $f(c)$ is a maximum, then $f(c)$ is called a relative maximum of $f$, or you can say that $f$ has a relative maximum at $(c, f(c))$.
2. If there is an open interval containing $c$ on which $f(c)$ is a minimum, then $f(c)$ is called a relative minimum of $f$, or you can say that $f$ has a relative minimum at $(c, f(c))$.

Relative maximum and relative minimum are sometimes called local maximum and local minimum, respectively.

## Example 1 (The value of the derivative at relative extrema)

Find the value of the derivative at each relative extremum shown in Figure 2.

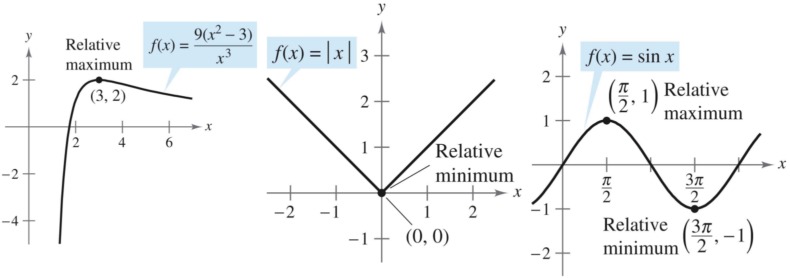



Figure 2: The value of the derivative at relative extrema.

(a)

(b)

(c)

- Note in Example 1 that at each relative extremum, the derivative either is zero or does not exist. The $x$-values at these special points are called critical numbers.
- Figure 3 illustrates the two types of critical numbers. Notice in the definition that the critical number $c$ has to be in the domain of $f$, but $c$ does not have to be in the domain of $f'$.

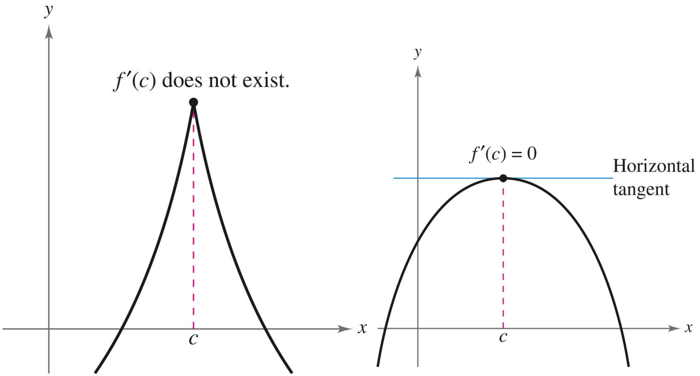



Figure 3: $c$ is a critical number of $f$.

**Definition 3.3 (Critical number)**

Let $f$ be defined at $c$. If $f'(c) = 0$ or if $f$ is not differentiable at $c$, then $c$ is a critical number of $f$.

**Theorem 3.2 (Relative extrema occur only at critical numbers)**

*If $f$ has a relative minimum or relative maximum at $x = c$, then $c$ is a critical number of $f$.*

# Finding extrema on a closed interval

**Guidelines for finding extrema on a closed interval**

1. Find the critical numbers of $f$ in $(a, b)$.
2. Evaluate $f$ at each critical number in $(a, b)$.
3. Evaluate $f$ at each endpoint of $[a, b]$.
4. The least of these values is the minimum. The greatest is the maximum.

## Example 2 (Finding extrema on a closed interval)

Find the extrema of $f(x) = 3x^4 - 4x^3$ on the interval $[-1, 2]$.

## Example 3 (Finding extrema on a closed interval)

Find the extrema of $f(x) = 2x - 3x^{2/3}$ on the interval $[-1, 3]$.

## Example 4 (Finding extrema on a closed interval)

Find the extrema of $f(x) = 2\sin x - \cos 2x$ on the interval $[0, 2\pi]$.

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# Rolle's Theorem

- The Extreme Value Theorem states that a continuous function on a closed interval $[a, b]$ must have both a minimum and a maximum on the interval. Both of these values, however, can occur at the endpoints
- Rolle's Theorem gives conditions that guarantee the existence of an extreme value in the interior of a closed interval.
- A theorem of calculus that ensures the existence of a critical point between any two points on a "nice" function that has the same $y$-value.

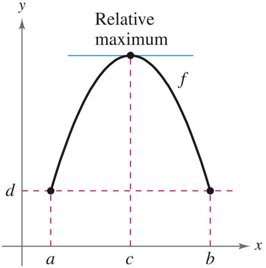
## Theorem 3.3 (Rolle's Theorem)

*Let $f$ be continuous on the closed interval $[a, b]$ and differentiable on the open interval $(a, b)$. If*

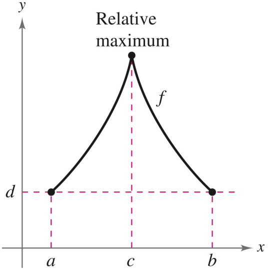
$$f(a) = f(b)$$

*then there is at least one number $c$ in $(a, b)$ such that $f'(c) = 0$.*

- From Rolle's Theorem, if a function $f$ satisfies the properties, there must be at least one $x$-value between $a$ and $b$ at which the graph of $f$ has a horizontal tangent. If the differentiability requirement is dropped, $f$ will still have a critical number in $(a, b)$, but it may not yield a horizontal tangent. Such a case is shown below.




(a) $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$.




(b) $f$ is continuous on $[a, b]$ but not differentiable on $(a, b)$.

## Example 1 (Illustrating Rolle's Theorem)

Find the two $x$-intercepts of

$$f(x) = x^2 - 3x + 2$$

and show that $f'(x) = 0$ at some point between the two $x$-intercepts.

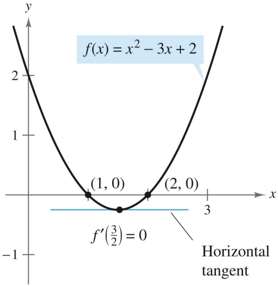



Figure 5: The $x$-value for which $f'(x) = 0$ is between the two $x$-intercepts.

## Example 2 (Illustrating Rolle's Theorem)

Let $f(x) = x^4 - 2x^2$. Find all values of $c$ in the interval $(-2, 2)$ such that $f'(c) = 0$.

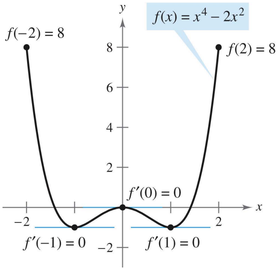



Figure 6: $f'(x) = 0$ for more than one $x$-value in the interval $(-2, 2)$.

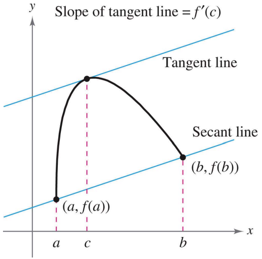
- Rolle's Theorem can be used to prove the Mean Value Theorem. A major theorem of calculus that relates the values of a function to the value of its derivative. Essentially the theorem states that for a "nice" function, there is a tangent line parallel to any secant line.

**Theorem 3.4 (The Mean Value Theorem)**

*If $f$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval $(a, b)$, then there exists a number $c$ in $(a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

- Although the Mean Value Theorem can be used directly in problem-solving, it is used more often to prove other theorems.
- In fact, some people consider this to be the most important theorem in calculus - it is closely related to the Fundamental Theorem of Calculus discussed in Section 4.4.
- Geometrically, the theorem guarantees the existence of a tangent line that is parallel to the secant line through the points $(a, f(a))$ and $(b, f(b))$.



## Example 3 (Finding a tangent line)

Given $f(x) = 5 - (4/x)$, find all values of $c$ in the open interval $(1, 4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}.$$

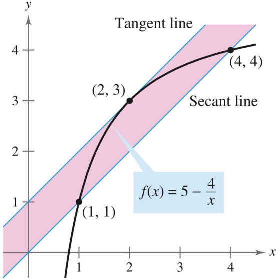



Figure 7: The tangent line at $(2, 3)$ is parallel to the secant line through $(1, 1)$ and $(4, 4)$.

- A useful alternative form of the Mean Value Theorem is as follows: If $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$, then there exists a number $c$ in $(a, b)$ such that alternative form of Mean Value Theorem

$$f(b) = f(a) + (b - a)f'(c).$$

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## Definition 3.4 (Increasing and decreasing functions)

A function $f$ is increasing on an interval if for any two numbers $x_1$ and $x_2$ in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.
A function $f$ is decreasing on an interval if for any two numbers $x_1$ and $x_2$ in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

- The function below is decreasing on the interval $(-\infty, a)$ is constant on the interval $(a, b)$ and is increasing on the interval $(b, \infty)$.

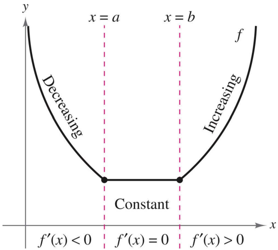



Figure 8: The derivative is related to the slope of a function.

## Theorem 3.5 (Test for increasing and decreasing functions)

*Let $f$ be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval $(a, b)$.*

1. *If $f'(x) > 0$ for all $x$ in $(a, b)$, then $f$ is increasing on $[a, b]$.*
2. *If $f'(x) < 0$ for all $x$ in $(a, b)$, then $f$ is decreasing on $[a, b]$.*
3. *If $f'(x) = 0$ for all $x$ in $(a, b)$, then $f$ is constant on $[a, b]$.*

## Example 1 (Intervals on which $f$ is increasing or decreasing)

Find the open intervals on which $f(x) = x^3 - \frac{3}{2}x^2$ is increasing or decreasing.

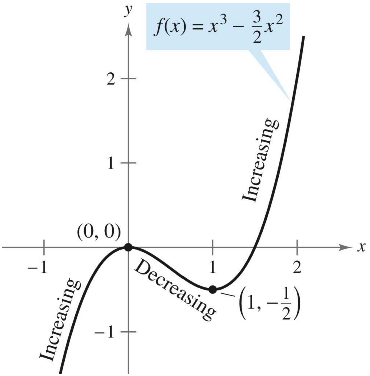



Figure 9: Increasing and decreasing intervals of $f(x) = x^3 - \frac{3}{2}x^2$.

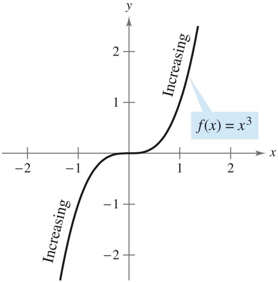
Guidelines for finding intervals on which a function is increasing or decreasing

Let $f$ be continuous on the interval $(a, b)$. To find the open intervals on which $f$ is increasing or decreasing, use the following steps.

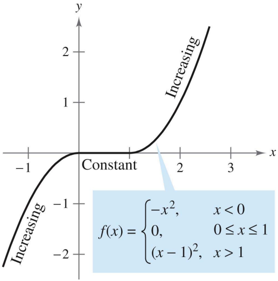
1. Locate the critical numbers of $f$ in $(a, b)$, and use these numbers to determine test intervals.
2. Determine the sign of $f'(x)$ at one test value in each of the intervals.
3. Use Theorem 3.5 to determine whether $f$ is increasing or decreasing on each interval.

These guidelines are also valid if the interval $(a, b)$ is replaced by an interval of the form $(-\infty, b)$, $(a, \infty)$, or $(-\infty, \infty)$.

- A function is strictly monotonic on an interval if it is either increasing on the entire interval or decreasing on the entire interval.




(a) Strictly monotonic function.




(b) Not strictly monotonic function.

# The First Derivative Test

- After finding the intervals on which a function is increasing or decreasing, it is not difficult to locate the relative extrema of the function.
- For instance, in the figure from Example 1, the function

$$f(x) = x^3 - \frac{3}{2}x^2$$

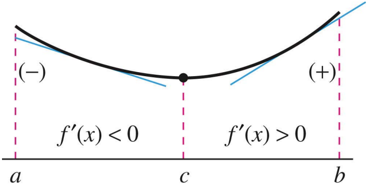
has a relative maximum at the point $(0, 0)$ because $f$ is increasing immediately to the left of $x = 0$ and decreasing immediately to the right of $x = 0$.

- Similarly, $f$ has a relative minimum at the point $\left(1, -\frac{1}{2}\right)$ because $f$ is decreasing immediately to the left of $x = 1$ and increasing immediately to the right of $x = 1$.

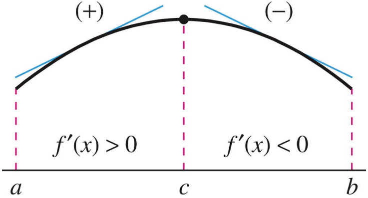
## Theorem 3.6 (The First Derivative Test)

*Let $c$ a critical number of a function $f$ that is continuous on an open interval $I$ containing $c$. If $f$ is differentiable on the interval, except possibly at $c$, then $f(c)$ can be classified as follows.*

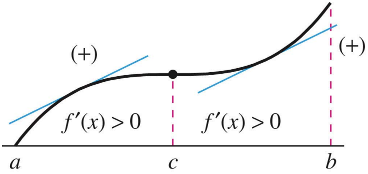
1. *If $f'(x)$ changes from negative to positive at $c$, then $f$ has a relative minimum at $(c, f(c))$.*
2. *If $f'(x)$ changes from positive to negative at $c$, then $f$ has a relative maximum at $(c, f(c))$.*
3. *If $f'(x)$ is positive on both sides of $c$ or negative on both sides of $c$, then $f$ is neither a relative minimum nor a relative maximum.*



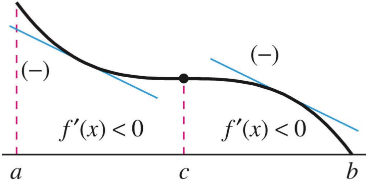

(a) Relative minimum.




(b) Relative maximum.




(c) Neither relative minimum nor relative maximum.




(d) Neither relative minimum nor relative maximum.

## Example 2 (Applying the First Derivative Test)

Find the relative extrema of the function $f(x) = \frac{1}{2}x - \sin x$ in the interval $(0, 2\pi)$.

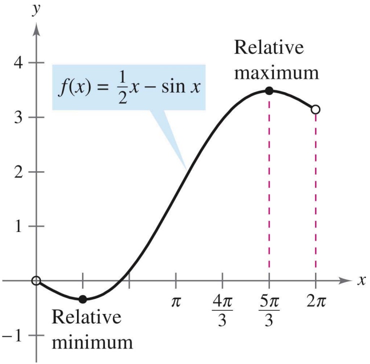



Figure 12: A relative minimum occurs where $f$ changes from decreasing to increasing, and a relative maximum occurs where $f$ changes from increasing to decreasing.

## Example 3 (Applying the First Derivative Test)

Find the relative extrema of

$$f(x) = (x^2 - 4)^{2/3}.$$

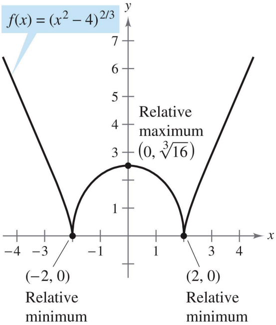



Figure 13: You can apply the First Derivative Test to find relative extrema.

## Example 4 (Applying the First Derivative Test)

Find the relative extrema of

$$f(x) = \frac{x^4 + 1}{x^2}.$$

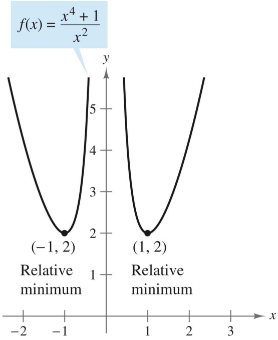



Figure 14: $x$-values that are not in the domain of $f$, as well as critical numbers, determine test intervals for $f'$.

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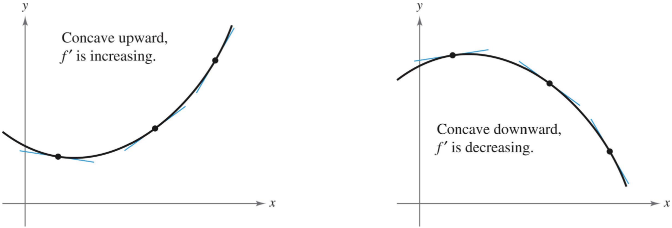
**Definition 3.5 (Concavity)**

Let $f$ be differentiable on an open interval $I$. The graph of $f$ is concave upward on $I$ if $f'$ is increasing on the interval and concave downward on $I$ if $f'$ is decreasing on the interval.

The following graphical interpretation of concavity is useful.

1. Let $f$ be differentiable on an open interval $I$. If the graph of $f$ is concave upward on $I$, then the graph of $f$ lies above all of its tangent lines on $I$. [See Figure 15 (a).]
2. Let $f$ be differentiable on an open interval $I$. If the graph of $f$ is concave downward on $I$, then the graph of $f$ lies below all of its tangent lines on $I$. [See Figure 15 (b).]




(a) Concave upward: The graph of $f$ lies above its tangent lines.

(b) Concave downward: The graph of $f$ lies below its tangent lines.

Figure 15: Concave upward and downward.

## Theorem 3.7 (Test for concavity)

*Let $f$ be a function whose second derivative exists on an open interval $I$.*

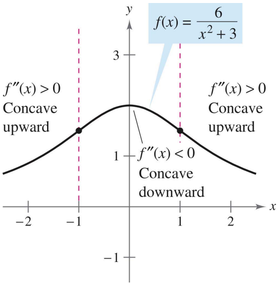
1. *If $f''(x) > 0$ for all $x$ in $I$, then the graph of $f$ is concave upward on $I$.*
2. *If $f''(x) < 0$ for all $x$ in $I$, then the graph of $f$ is concave downward on $I$.*

## Example 1 (Determining concavity)

Determine the open intervals on which the graph of

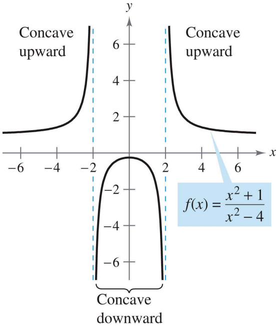
$$f(x) = \frac{6}{x^2 + 3}$$

is concave upward or downward.

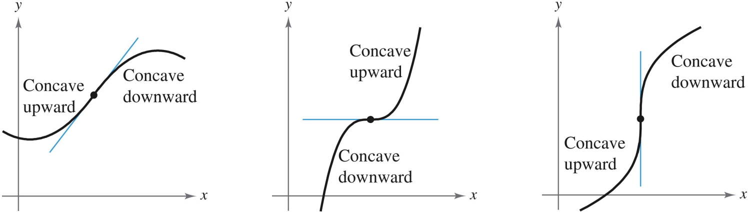

y
$f(x) = \frac{6}{x^2 + 3}$
3
$f''(x) > 0$
Concave
upward
$f''(x) > 0$
Concave
upward
1
$f''(x) < 0$
Concave
downward
x
−2
−1
1
2
−1

## Example 2 (Determining concavity)

Determine the open intervals on which the graph of $f(x) = \frac{x^2+1}{x^2-4}$ is concave upward or concave downward.


y
Concave
upward
Concave
upward
6
4
2
−6
−4
−2
2
4
6
x
−2
−4
−6
$f(x) = \frac{x^2 + 1}{x^2 - 4}$
Concave
downward

- If the tangent line to the graph exists at such a point where the concavity changes, that point is a point of inflection.
- Three types of points of inflection are shown below.




**Definition 3.6 (Point of inflection)**

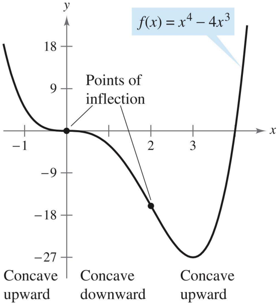
Let $f$ be a function that is continuous on an open interval and let $c$ be a point in the interval. If the graph of $f$ has a tangent line at this point $(c, f(c))$, then this point is a point of inflection of the graph of $f$ if the concavity of $f$ changes from upward to downward (or downward to upward) at the point.

## Theorem 3.8 (Points of inflection)

*If $(c, f(c))$ is a point of inflection of the graph of $f$, then either $f''(c) = 0$ or $f''$ does not exist at $x = c$.*

## Example 3 (Finding points of inflection)

Determine the points of inflection and discuss the concavity of the graph of $f(x) = x^4 - 4x^3$.


y
$f(x) = x^4 - 4x^3$
18
9
Points of inflection
x
−1
2
3
−9
−18
−27
Concave upward
Concave downward
Concave upward

# The Second Derivative Test

A method for determining whether a critical point is a relative minimum or maximum.

## Theorem 3.9 (The Second Derivative Test)

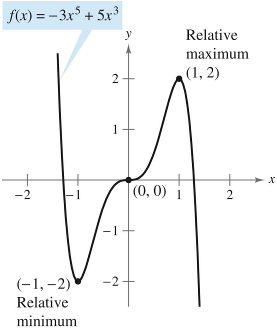
*Let $f$ be a function such that $f'(c) = 0$ and the second derivative of $f$ exists on an open interval containing $c$.*

1. *If $f''(c) > 0$, then $f$ has a relative minimum at $(c, f(c))$.*
2. *If $f''(c) < 0$, then $f$ has a relative maximum at $(c, f(c))$.*

*If $f''(c) = 0$, the test fails. That is, $f$ may have a relative maximum, a relative minimum, or neither. In such cases, you can use the First Derivative Test.*

## Example 4 (Using the Second Derivative Test)

Find the relative extrema for $f(x) = -3x^5 + 5x^3$.



$f(x) = -3x^5 + 5x^3$
y
Relative
maximum
(1, 2)
2
1
−1
−2
x
−2
−1
(0, 0)
1
2
(−1, −2)
Relative
minimum

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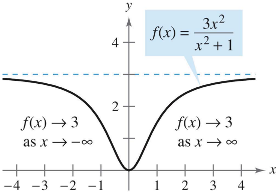
- This section discusses the "end behavior" of a function on an infinite interval. Consider the graph of $f(x) = \frac{3x^2}{x^2+1}$ as shown below. We can see that the values of $f(x)$ appear to approach 3 as $x$ increases without bound or decreases without bound.




- You can come to the same conclusions numerically, as shown below.

x decreases without bound. | x increases without bound.

| $x$ | $-\infty \leftarrow$ | $-100$ | $-10$ | $-1$ | 0 | 1 | 10 | 100 | $\rightarrow \infty$ |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | $3 \leftarrow$ | 2.9997 | 2.97 | 1.5 | 0 | 1.5 | 2.97 | 2.9997 | $\rightarrow 3$ |

$f(x)$ approaches 3. | $f(x)$ approaches 3.

- These limits at infinity are denoted by

$$\lim_{x\to-\infty} f(x) = 3 \qquad \text{Limit at negative infinity}$$

and

$$\lim_{x\to\infty} f(x) = 3. \qquad \text{Limit at positive infinity}$$

- To say that a statement is true as $x$ increases without bound means that for some (large) real number $M$, the statement is true for all $x$ in the interval $\{x : x > M\}$.

## Definition 3.7 (Limits at infinity)

Let $L$ be a real number.

1. The statement $\lim_{x\to\infty} f(x) = L$ means that for each $\epsilon > 0$ there exists an $M > 0$ such that $|f(x) - L| < \epsilon$ whenever $x > M$.
2. The statement $\lim_{x\to-\infty} f(x) = L$ means that for each $\epsilon > 0$ there exists an $N < 0$ such that $|f(x) - L| < \epsilon$ whenever $x < N$.

- Note that for a given positive number $\epsilon$ there exists a positive number $M$ such that, for $x > M$, the graph of $f$ will lie between the horizontal lines given by $y = L + \epsilon$ and $y = L - \epsilon$.

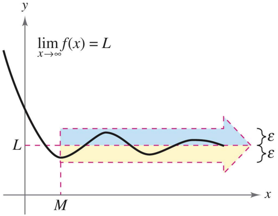



Figure 17: $f(x)$ is within $\epsilon$ units of $L$ as $x \to \infty$.

- The graph of $f$ approaches the line $y = L$ as $x$ increases without bounds. The line $y = L$ is called a horizontal asymptote of $f$.

### Definition 3.8 (Horizontal asymptote)

The line $y = L$ is a horizontal asymptote of the graph of $f$ if

$$\lim_{x \to -\infty} f(x) = L \quad \text{or} \quad \lim_{x \to \infty} f(x) = L.$$

- Many properties hold for limits at infinity. For example, if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ both exist, then

$$\lim_{x\to\infty} [f(x) + g(x)] = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x).$$

  and

$$\lim_{x\to\infty} [f(x)g(x)] = \left[\lim_{x\to\infty} f(x)\right] \left[\lim_{x\to\infty} g(x)\right].$$

  Similar properties hold for limits at $-\infty$.

## Theorem 3.10 (Limits at infinity)

*If $r$ is a positive rational number and $c$ is any real number, then*

$$\lim_{x\to\infty} \frac{c}{x^r} = 0.$$

*Furthermore, if $x^r$ is defined when $x < 0$, then*

$$\lim_{x\to-\infty} \frac{c}{x^r} = 0.$$

## Example 1 (Finding a limit at infinity)

Find the limit:

$$\lim_{x \to \infty} \left( 5 - \frac{2}{x^2} \right).$$

## Example 2 (Finding a limit at infinity)

Find the limit:

$$\lim_{x\to\infty} \frac{2x-1}{x+1}.$$

## Example 3 (A comparison of three rational functions)

Find each limit.

**a.** $\lim_{x\to\infty} \frac{2x+5}{3x^2+1}$ **b.** $\lim_{x\to\infty} \frac{2x^2+5}{3x^2+1}$ **c.** $\lim_{x\to\infty} \frac{2x^3+5}{3x^2+1}$

Guidelines for finding limits at $\pm\infty$ of rational functions

1. If the degree of the numerator is less than the degree of the denominator, then the limit of the rational function is 0.
2. If the degree of the numerator is equal to the degree of the denominator, then the limit of the rational function is the ratio of the leading coefficients.
3. If the degree of the numerator is greater than the degree of the denominator, then the limit of the rational function does not exist.

## Example 4 (A function with two horizontal asymptotes)

Find each limit.

**a.** $\lim_{x\to\infty} \frac{3x-2}{\sqrt{2x^2+1}}$ **b.** $\lim_{x\to-\infty} \frac{3x-2}{\sqrt{2x^2+1}}$

**a.**

## b.

## Example 5 (Limits involving trigonometric functions)

Find each limit.

**a.** $\lim_{x\to\infty} \sin x$ **b.** $\lim_{x\to\infty} \frac{\sin x}{x}$

Many functions don't approach a finite limit as $x$ increases/decreases without bounds. For instance, no polynomial function has a finite limit at infinity.

### Definition 3.9 (Infinite limits at infinity)

1. The statement $\lim_{x\to\infty} f(x) = \infty$ means that for each positive number $M$, there is a corresponding number $N > 0$ such that $f(x) > M$ whenever $x > N$.
2. The statement $\lim_{x\to\infty} f(x) = -\infty$ means that for each negative number $M$, there is a corresponding number $N > 0$ such that $f(x) < M$ whenever $x > N$.

Similar definitions can be given for the statements

$$\lim_{x\to-\infty} f(x) = \infty \text{ and } \lim_{x\to-\infty} f(x) = -\infty.$$

## Example 6 (Finding infinite limits at infinity)

Find each limit. **a.** $\lim_{x\to\infty} x^3$ **b.** $\lim_{x\to-\infty} x^3$

## Example 7 (Finding infinite limits at infinity)

Find each limit.

**a.** $\lim_{x\to\infty} \frac{2x^2-4x}{x+1}$ **b.** $\lim_{x\to-\infty} \frac{2x^2-4x}{x+1}$

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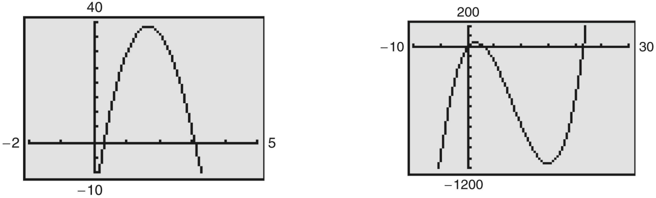
# Analyzing the graph of a function

So far, you have studied several concepts that are useful in analyzing the graph of a function.

| | |
|---|---|
| $x$-intercepts and $y$-intercepts | (Section P.I) |
| Symmetry | (Section P.1) |
| Domain and range | (Section P.1) |
| Continuity | (Section 1.4) |
| Vertical asymptotes | (Section 1.5) |
| Differentiability | (Section 2.1) |
| Relative extrema | (Section 3.1) |
| Concavity | (Section 3.4) |
| Points of inflection | (Section 3.4) |
| Horizontal asymptotes | (Section 3.5) |
| Infinite limits at infinity | (Section 3.5) |

- When we are sketching the graph of a function, remember that normally, you cannot show the entire graph. The decision to which part of the graph you choose to show is often crucial.
- For instance, which of the viewing windows below better represents the graph of $f(x) = x^3 - 25x^2 + 74x - 20$?




- By seeing both views, it is clear that the second viewing window gives a more complete representation of the graph.
- But would a third viewing window reveal other interesting portions of the graph?

- To answer this, you need to use calculus to interpret the first and second derivatives. Here are some guidelines for determining a good viewing window for the graph of a function:

**Guidelines for analyzing the graph of a function**

1. Determine the domain and range of the function.
2. Determine the intercepts, asymptotes, and symmetry of the graph.
3. Locate the $x$-values for which $f'(x)$ and $f''(x)$ either are zero or do not exist. Use the results to determine relative extrema and points of inflection.

Review the properties of odd and even functions:

- Odd function: $f(-x) = -f(x)$, $\forall x \in I$, the graph of $f(x)$ symmetry about the origin $(0, 0)$, $f'(x)$ is an even function.
- Even function: $f(-x) = f(x)$, $\forall x \in I$, the graph of $f(x)$ symmetry about the $y$-axis $(x = 0)$, $f'(x)$ is an odd function.

Table 1: Properties of odd and even functions, $f(x) \neq 0$ and $g(x) \neq 0$

| $(f, g)$ | $f \pm g$ | $f \times g$, $f \div g$ |
|---|---|---|
| (odd, odd) | odd | even |
| (even, even) | even | even |
| (odd, even), (even, odd) | neither | odd |

## Example 1 (Sketching the graph of a rational function)

Analyze and sketch the graph of $f(x) = \frac{2(x^2-9)}{x^2-4}$.

## Example 2 (Sketching the graph of a rational function)

Analyze and sketch the graph of $f(x) = \frac{x^2-2x+4}{x-2}$.

## Definition 3.10 (Slant asymptote)

The line $y = mx + b$ is a slant asymptote of the graph of $f$ if

$$\lim_{x\to\infty} f(x) - (mx + b) = 0 \quad \text{or} \quad \lim_{x\to-\infty} f(x) - (mx + b) = 0.$$

## Theorem 3.11 (Slant asymptote)

*The line $y = mx + b$ is a slant asymptote of the graph of $f$ if*

$$m = \lim_{x\to\infty} \frac{f(x)}{x} \qquad b = \lim_{x\to\infty} f(x) - mx$$

*or*

$$m = \lim_{x\to-\infty} \frac{f(x)}{x} \qquad b = \lim_{x\to-\infty} f(x) - mx.$$

## Example 3 (Sketching the graph of a radical function)

Analyze and sketch the graph of $f(x) = \frac{x}{\sqrt{x^2+2}}$.

## Example 5 (Sketching the graph of a polynomial function)

Analyze and sketch the graph of $f(x) = x^4 - 12x^3 + 48x^2 - 64x$.

## Example 6 (Sketching the graph of a trigonometric function)

Analyze and sketch the graph of $f(x) = \frac{\cos x}{1+\sin x}$.

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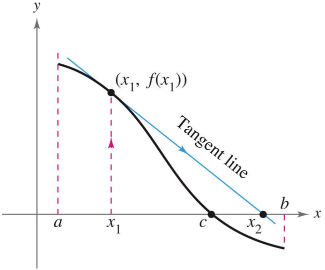
# Newton's Method

- The technique for approximating the real zeros of a function is called Newton's Method, and it uses tangent lines to approximate the graph of the function near its $x$-intercepts.
- Consider a function $f$ that is continuous on the interval $[a, b]$ and differentiable on the interval $(a, b)$. If $f(a)$ and $f(b)$ differ in sign, then, by the Intermediate Value Theorem, $f$ must have at least one zero in the interval $(a, b)$.
- Suppose we estimate this zero to occur at

$$x = x_1 \qquad \text{First estimate}$$

Newton's Method is based on the assumption that the graph of $f$ and the tangent line at $(x_1, f(x_1))$ both cross the $x$-axis at about the same point.




- Because we can easily calculate the $x$-intercept for this tangent line, it can become a second (usually better) estimate of the zero of $f$.
- The tangent line passes through the point $(x_1, f(x_1))$ with a slope of $f'(x_1)$. In point-slope form, the equation of the tangent line is:

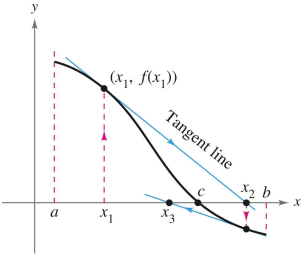
$$y - f(x_1) = f'(x_1)(x - x_1) \qquad y = f'(x_1)(x - x_1) + f(x_1).$$

- Letting $y = 0$ produces

$$x = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

- So, from the initial estimate $x_1$, we obtain a new estimate

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \qquad \text{Second estimate.}$$




- We can improve on $x_2$ and calculate yet a third estimate

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}. \qquad \text{Third estimate}$$

- Repeated application of this process is called Newton's Method.

Newton's method for approximating the zeros of a function

Let $f(c) = 0$, where $f$ is differentiable on an open interval containing $c$. Then, to approximate $c$, use the following steps.

1. Make an initial estimate $x_1$ that is close to $c$. (A graph is helpful.)
2. Determine a new approximation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

3. If $|x_n - x_{n+1}|$ is within the desired accuracy, let $x_{n+1}$ serve as the final approximation. Otherwise, return to Step 2 and calculate a new approximation.

Each successive application of this procedure is called an iteration.

## Example 1 (Using Newton's Method)

Calculate three iterations of Newton's Method to approximate a zero of $f(x) = x^2 - 2$. Use $x_1 = 1$ as the initial guess.

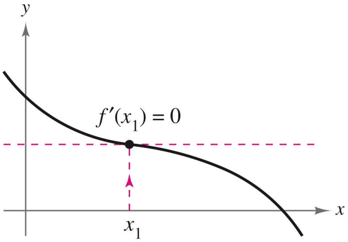
## Example 2 (Using Newton's Method)

Use Newton's Method to approximate the zeros of

$$f(x) = 2x^3 + x^2 - x + 1.$$

Continue the iterations until two successive approximations differ by less than 0.0001.

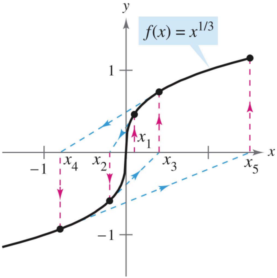
- When the approximations approach a limit, the sequence $x_1$, $x_2$, $x_3$, $\ldots$, $x_n$, $\ldots$ is said to converge. Moreover, if the limit is $c$, it can be shown that $c$ must be a zero of $f$.
- Newton's Method does not always yield a convergent sequence. One way it can fail to do so is shown below.




- Because Newton's Method involves division by $f'(x_n)$, it is clear that the method will fail if the derivative is zero for any $x_n$ in the sequence.
- When you encounter this problem, you can usually overcome it by choosing a different value for $x_1$.

## Example 3 (An example in which Newton's Method fails)

The function $f(x) = x^{1/3}$ is not differentiable at $x = 0$. Show that Newton's Method fails to converge using $x_1 = 0.1$.


y
$f(x) = x^{1/3}$
1
−1
−1
x
$x_1$
$x_2$
$x_3$
$x_4$
$x_5$

- In an application of the fixed-point theorem, one obtains a condition

$$\left|\frac{f(x)f''(x)}{[f'(x)]^2}\right| < 1$$

sufficient to produce convergence of Newton's Method to a zero of $f(x)$ on an open interval containing the zero.

- In example 1, the test would yield

$$\left|\frac{f(x)f''(x)}{[f'(x)]^2}\right| = \left|\frac{1}{2} - \frac{1}{x^2}\right|$$

On the interval $(1, 3)$, this quantity is less than 1 and so the convergence is guaranteed.

- In contrast, the quantity is not less than 1 for any value of $x$ in example 3, so we can't say that the method will converge.

- The zeros of some functions, such as

$$f(x) = x^3 - 2x^2 - x + 2$$

can be found by simple algebraic techniques, such as factoring.
- The zeros of other functions, such as

$$f(x) = x^3 - x + 1$$

cannot be found by elementary algebraic methods.
- By using more advanced algebraic techniques, we can find

$$x = -\sqrt[3]{\frac{3-\sqrt{23/3}}{6}} - \sqrt[3]{\frac{3+\sqrt{23/3}}{6}}.$$

- Because the exact solution is written in terms of square roots and cube roots, it is called a solution by radicals. The numerical solution can be found using Newton's method.