1. Find the following limit. (If the limit does not exist or has an infinite limit, you should point it out. In addition, also remember the definition of definite integral). (20%)

(a) $\lim_{n\to\infty} \frac{1}{\sqrt{2n\cdot 1-1^2}} + \frac{1}{\sqrt{2n\cdot 2-2^2}} + \cdots + \frac{1}{\sqrt{2n\cdot n-n^2}}$

(b) $\lim_{x\to a} \frac{x\int_a^x f(t)dt}{x-a}$

(c) $\lim_{x\to 0^+} \frac{\ln(sinx)}{\ln(tanx)}$

(d) $\lim_{x\to\infty} (1+\frac{3}{x}+\frac{5}{x^2})^x$

(e) $\lim_{x\to 0} \left(\frac{1}{2x} - \frac{1}{1-e^{-2x}}\right)$

Ans:

(a) $\lim_{n\to\infty} \frac{1}{\sqrt{2n\cdot 1-1^2}} + \frac{1}{\sqrt{2n\cdot 2-2^2}} + \cdots + \frac{1}{\sqrt{2n\cdot n-n^2}} = \lim_{n\to\infty} \left(\frac{1}{\sqrt{\frac{2}{n}-(\frac{1}{n})^2}} + \frac{1}{\sqrt{\frac{4}{n}-(\frac{2}{n})^2}} + \cdots + \right.$

$\left. \frac{1}{\sqrt{\frac{2n}{n}-(\frac{n}{n})^2}} \right) \frac{1}{n} = \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{1}{\sqrt{\frac{2i}{n}-(\frac{i}{n})^2}} \right) \frac{1}{n} =$

$\int_0^1 \frac{1}{\sqrt{2x-x^2}} dx = \int_0^1 \frac{1}{\sqrt{1-(1-x)^2}} dx = -\sin^{-1}(1-x)]_0^1 = \frac{\pi}{2}$

(b) $\lim_{x\to a} \frac{x\int_a^x f(t)dt}{x-a} = \lim_{x\to a} \frac{\int_a^x f(t)dt + xf(x)}{1}$ (L' Hôpital' s rule and fundamental theorem of calculus) $= af(a)$

(c) $\lim_{x\to 0^+} \frac{\ln(sinx)}{\ln(tanx)} = \lim_{x\to 0^+} \frac{\frac{cosx}{sinx}}{\frac{sec^2x}{tanx}}$ (L' Hôpital' s rule)

$$= \lim_{x\to 0^+} \frac{tanx \cdot cosx}{sinx \cdot sec^2x} = \lim_{x\to 0^+} \frac{1}{\cdot sec^2x} = 1$$

(d) $y = \lim_{x\to\infty} (1+\frac{3}{x}+\frac{5}{x^2})^x$

$\ln y = \ln \lim_{x\to\infty} \left(1+\frac{3}{x}+\frac{5}{x^2}\right)^x = \lim_{x\to\infty} \ln\left(1+\frac{3}{x}+\frac{5}{x^2}\right)^x = \lim_{x\to\infty} x\ln\left(1+\frac{3}{x}+\right.$

$\left.\frac{5}{x^2}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{3}{x}+\frac{5}{x^2}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{3}{x}+\frac{5}{x^2}}\left(\frac{-3}{x^2}-\frac{10}{x^3}\right)}{\frac{-1}{x^2}}$ (L' Hôpital' s rule)

$$= \lim_{x\to\infty} \frac{1}{1+\frac{3}{x}+\frac{5}{x^2}}(3+\frac{10}{x}) = 3$$

Therefore, $y = e^3$

(e) $\lim_{x\to 0}\left(\frac{1}{2x}-\frac{1}{1-e^{-2x}}\right) = \lim_{x\to 0}\left(\frac{1-e^{-2x}-2x}{2x(1-e^{-2x})}\right)$

$= \lim_{x\to 0}\left(\frac{2e^{-2x}-2}{2(1-e^{-2x})+4xe^{-2x}}\right)$ (L' Hôpital' s rule)

$= \lim_{x\to 0}\left(\frac{e^{-2x}-1}{1-e^{-2x}+2xe^{-2x}}\right)$

$= \lim_{x\to 0}\left(\frac{-2e^{-2x}}{4e^{-2x}-4xe^{-2x}}\right)$ (L' Hôpital' s rule)

$= \frac{-1}{2}$

2. Let $f(x) = 3 + x + e^x$ (9%)
   (a) What is the value of $f^{-1}(x)$ when $x = 4$
   (b) What is the value of $(f^{-1})'(x)$ when $x = 4$
   (c) What is the value of $(f^{-1})''(x)$ when $x = 4$

   Ans:

(a) Note that f is strictly increasing and therefore has an inverse function ($f' = 1 + e^x > 0$)

Because $f(x) = 4$ when $x = 0$, we know that $f^{-1}(4) = 0$

(b) $f'(x) = 1 + e^x$

Because f is differentiable and has an inverse function, we have

$$(f^{-1})'(4) = \frac{1}{f'(0)} = \frac{1}{2}$$

(c) $f''(x) = e^x$

Let $g(x) = f^{-1}(x) \to x = g(y) \to 1 = g'(f(x))f'(x)$ (differentiate both side with respect to x)

$0 = g''(f(x))(f'(x))^2 + g'(f(x))f''(x)$ (differentiate both side with respect to x again!)

Now substitute $x = 0, y = f(x) = 4$, we get

$$0 = g''(4)(f'(0))^2 + g'(4)f''(0) = g''(4)2^2 + \frac{1}{2}\cdot 1$$

$$(f^{-1})''(4) = g''(4) = -\frac{1}{8}$$

3. Use the Mean Value Theorem to prove that $\forall a \geq 0$, we have $\frac{a}{1+a^2} \leq \tan^{-1} a \leq a$. (Hunt: use the theorem in the interval $(0, a)$) (8%)

Ans:

When $a = 0$, the equality holds!

When $a > 0$, let $f(x) = \tan^{-1} x \ \rightarrow f'(x) = \frac{1}{1+x^2}$

According to the mean value theorem, we know that there exist $c \in (0, a)$ such that

$f'(c) = \frac{f(a)-0}{a-0} = \frac{\tan^{-1} a}{a}$

Note that since $f'(x) = \frac{1}{1+x^2}$. Therefore, in this interval for any $c \in (0, a)$, we have

$\frac{1}{1+a^2} \leq f'(c) \leq \frac{1}{1+0^2} = 1$ ($f'(x)$ is strictly decreasing due to $f''(x) = \frac{-2x}{(1+x^2)^2} < 0$)

Therefore, $\frac{1}{1+a^2} \leq \frac{\tan^{-1} a}{a} \leq 1 \rightarrow \frac{a}{1+a^2} \leq \tan^{-1} a \leq a$

4. Evaluate the following integral. (Hint: Try to use change of variables for all the problems) (15%)

(a) $\int x \cdot 10^{x^2} dx$

(b) $\int \sqrt{1 + e^{2x}} dx$

(c) $\int \frac{\sin(x)\cos(x)}{1+\sin^4(x)} dx$

Ans:

(a) $\int x \cdot 10^{x^2} dx = \int x \cdot e^{x^2 \ln 10} dx$ Let $u = x^2 \ln 10 \rightarrow du = 2x \ln 10 \, dx$

$$\int x \cdot e^{x^2 \ln 10} dx = \int \frac{1}{2 \ln 10} e^u \, du = \frac{1}{2 \ln 10} e^u + C = \frac{1}{2 \ln 10} 10^{x^2} + C$$

(b) Let $u = \sqrt{1 + e^{2x}}$, $u^2 = 1 + e^{2x} \rightarrow \ 2u du = 2e^{2x} dx$

$$\int \sqrt{1 + e^{2x}} dx = \int u \frac{u}{e^{2x}} du = \int \frac{u^2}{u^2 - 1} du = \int 1 + \frac{1}{u^2 - 1} du$$

$$= u + \frac{1}{2} \int \frac{1}{u-1} - \frac{1}{u+1} du = u + \frac{1}{2} \ln|u - 1| - \frac{1}{2} \ln|u + 1| + C$$

$$= u + \frac{1}{2} \ln \frac{|u-1|}{|u+1|} + C = \sqrt{1 + e^{2x}} + \frac{1}{2} \ln \left| \frac{\sqrt{1 + e^{2x}} - 1}{\sqrt{1 + e^{2x}} + 1} \right| + C$$

(c) Let $u = \sin^2(x) \rightarrow du = 2\sin(x)\cos(x)dx$

$$\int \frac{\sin(x)\cos(x)}{1 + \sin^4(x)} dx = \frac{1}{2} \int \frac{du}{1 + u^2} = \frac{1}{2} \tan^{-1} u + C = \frac{1}{2} \tan^{-1}(\sin^2(x)) + C$$

5. Find the equation of the tangent line $\tan^{-1}(xy) = \sin^{-1}(x+y)$ at $(0,0)$. (8%)

Ans:

$$\frac{1}{1+(xy)^2}[y+xy'] = \frac{1}{\sqrt{1-(x+y)^2}}[1+y']$$

At (0,0), we have $0 = 1 + y' \rightarrow y' = -1$

The tangent line is $\mathrm{y} = -\mathrm{x}$

6. Evaluate the following integral. (16%)

(a) $\int \frac{\ln x}{x^2} dx$

(b) $\int_0^1 \ln(x^2+1)\, dx$

(c) $\int_0^{\frac{\pi}{4}} tan^3\theta sec^2\theta d\theta$

(d) $\int \frac{1}{x^2\sqrt{x^2+4}} dx$

Ans:

(a) Let $\mathrm{u} = \ln x\,, dv = \frac{dx}{x^2} \rightarrow \mathrm{du} = \frac{1}{x} dx, v = \frac{-1}{x}$

$$\int \frac{\ln x}{x^2} dx = \frac{-1}{x}\ln x + \int \frac{1}{x^2} dx = \frac{-1}{x}\ln x - \frac{1}{x} + C$$

(b) Let $\mathrm{u} = \ln(x^2+1)\,, dv = dx \rightarrow \mathrm{du} = \frac{2x}{x^2+1} dx, v = x$

$$\int \ln(x^2+1)\, dx = x\ln(x^2+1) - \int x \cdot \frac{2x}{x^2+1} dx$$

$$= x\ln(x^2+1) - \int 2 - \frac{2}{x^2+1} dx$$

$$= x\ln(x^2+1) - 2x + 2\tan^{-1} x + C$$

$$\int_0^1 \ln(x^2+1)\, dx = x\ln(x^2+1) - 2x + 2\tan^{-1} x\Big]_0^1 = \ln 2 - 2 + \frac{\pi}{2}$$

(c) Let $\mathrm{u} = \tan\theta \rightarrow du = sec^2\theta d\theta$

$$\int_0^{\frac{\pi}{4}} tan^3\theta sec^2\theta d\theta = \int_0^1 u^3 du = \frac{u^4}{4}\bigg]_0^1 = \frac{1}{4}$$

(d) Let $\mathrm{x} = 2\tan\theta \rightarrow \mathrm{dx} = 2sec^2\theta d\theta$

$$\int \frac{1}{x^2\sqrt{x^2+4}}dx = \int \frac{2sec^2\theta d\theta}{4tan^2\theta \cdot 2sec\theta} = \frac{1}{4}\int \frac{sec\theta}{tan^2\theta}d\theta = \frac{1}{4}\int csc\theta cot\theta \, d\theta$$

$$= \frac{-1}{4}csc\theta + C = -\frac{\sqrt{x^2+4}}{4x} + \mathrm{C}$$

7. Let $f(x) = \frac{-8x^2-7x+3}{(x+1)(x+2)(x^2+1)}$. (9%)

   (a) Solve $\int f(x)dx$

   (b) Solve $\int_0^\infty f(x)dx$

   Ans:

(a) $\frac{-8x^2-7x+3}{(x+1)(x+2)(x^2+1)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{Cx+D}{x^2+1}$

We have $-8x^2 - 7x + 3 = A(x+2)(x^2+1) + B(x+1)(x^2+1) + (\mathrm{Cx} + \mathrm{D})(\mathrm{x}+1)(\mathrm{x}+2)$

When $x = -1$, we get $A = 1$

When $x = -2$, we get $B = 3$

Substitute A and B back we get $-8x^2 - 7x + 3 = (x+2)(x^2+1) + 3(x+1)(x^2+1) + (\mathrm{Cx}+\mathrm{D})(\mathrm{x}+1)(\mathrm{x}+2) = (4+\mathrm{C})x^3 + (5+3C+D)x^2 + (4+2C+3D)x + (5+2D)$

So, $\mathrm{C} = -4, \mathrm{D} = -1$

$$\int f(x)dx = \int \frac{1}{x+1} + \frac{3}{x+2} + \frac{-4x-1}{x^2+1}dx$$

$$= \int \frac{1}{x+1} + \frac{3}{x+2} - \frac{4x}{x^2+1} - \frac{1}{x^2+1}dx$$

$$= \ln|x+1| + 3\ln|x+2| - 2\ln|x^2+1| - \tan^{-1}x + C$$

(b) $\int_0^\infty f(x)dx = \lim_{b\to\infty}\int_0^b f(x)dx = \lim_{b\to\infty} \ln\left|\frac{(x+1)(x+2)^3}{(x^2+1)^2}\right| - \tan^{-1}x\Big]_0^b =$

$\lim_{b\to\infty} \ln\left|\frac{(b+1)(b+2)^3}{(b^2+1)^2}\right| - \tan^{-1}b - \ln 8 = 0 - \frac{\pi}{2} - \ln 8 = -\frac{\pi}{2} - \ln 8$

(Note that $\lim_{b\to\infty}\left|\frac{(b+1)(b+2)^3}{(b^2+1)^2}\right| = 1$)

8. Determine whether the following integral diverges or converges. (9%)

   (a) $\int_1^9 \frac{1}{\sqrt[3]{x-1}}dx$

(b) $\int_1^{\infty} \frac{1}{1+e^x} dx$

(c) $\int_1^{\infty} \frac{\sqrt{1+\frac{1}{x^4}}}{x} dx$

Ans:

(a) $\int_1^9 \frac{1}{\sqrt[3]{x-1}} dx = \lim_{b \to 1^+} \int_b^9 \frac{1}{\sqrt[3]{x-1}} dx + \lim_{b \to 1^+} \int_{b-1}^8 u^{-\frac{1}{3}} du \; (u - x - 1 \to du = dx) =$

$\lim_{b \to 1^+} \frac{3}{2} u^{\frac{2}{3}} \Big]_{b-1}^{8} = \lim_{b \to 1^+} (6 - \frac{3}{2}(b-1)^{\frac{2}{3}}) = 6$. Thereofre it is converge.

(b)

$$\int_1^{\infty} e^{-x}\, dx = \lim_{b \to \infty} \int_1^b e^{-x} dx = \lim_{b \to \infty} -e^{-x} \Big]_1^b = \frac{1}{e}$$

Since $\frac{1}{1+e^x} < \frac{1}{e^x} = e^{-x}$ on $[1, \infty)$ and $\int_1^{\infty} e^{-x}\, dx$ converge, then by the comparison test so does $\int_1^{\infty} \frac{1}{1+e^x} dx$

(c) Since $\int_1^{\infty} \frac{1}{x} dx$ is divergent

And $\frac{\sqrt{1+\frac{1}{x^4}}}{x} > \frac{1}{x}$ on $[1, \infty)$ then by the comparison test $\int_1^{\infty} \frac{\sqrt{1+\frac{1}{x^4}}}{x} dx$ is divergent

9. Find the volume of the solid generated by revolving the region bounded by the graphs of $y \le xe^{-x}, y \ge 0$ and $x \ge 0$ about the x-axis. (6%)

Ans:

$$V = \pi \int_0^{\infty} (xe^{-x})^2 dx = \pi \int_0^{\infty} x^2 e^{-2x}\; dx = \lim_{b \to \infty} -\frac{\pi e^{-2x}}{4}(2x^2 + 2x + 1) \Big]_0^b = \frac{\pi}{4}$$