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Chapter 5

Logarithmic, Exponential, and Other Transcendental Functions

5.1 Summary

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1. **The natural logarithmic function**

The **natural logarithmic function** (自然對數函數) is defined by

$$\ln x = \int_1^x \frac{1}{t}\,\mathrm{d}t, \quad x > 0.$$

(c) $\ln(a^n) = n \ln a$

(d) $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

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4. **e** (**自然對數的底數**) The letter e denotes the positive real number such that

$$\ln e = \int_1^e \frac{1}{t}\,\mathrm{d}t = 1.$$

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5. **Euler's Formula** (**尤拉公式**)

$$e^{ix} = \cos x + i \sin x$$

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6. **Euler's Identity** (**尤拉恆等式**): The most beautiful theorem in math-

9. **Log Rule for integration** (**積分的對數法則**) Let u be a differentiable function of x.

1. $\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C$ **2.** $\int \frac{1}{u}\,\mathrm{d}u = \ln|u| + C$ 31

10. **Guidelines for integration**

(a) Learn a basic list of integration formulas. (Including those given in this section, you now have 12 formulas: the Power Rule, the Log Rule, and ten trigonometric rules. By the end of Section ??, this list will have expanded to 20 basic rules.)

(b) Find an integration formula that resembles all or part of the integrand, and, by trial and error, find a choice of u that will make the integrand conform to the formula.

(c) If you cannot find a u-substitution that works, try altering the integrand. You might try a trigonometric identity, multiplication and division by the same quantity, addition and subtraction of the same quantity, or long division. Be creative.

(d) If you have access to computer software that will find antiderivatives symbolically, use it.

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Table 5.1: Integrals of the six basic trigonometric functions

$\int \sin u \, \mathrm{d}u = -\cos u + C$	$\int \cos u \, \mathrm{d}u = \sin u + C$
$\int \tan u \, \mathrm{d}u = -\ln\lvert \cos u\rvert + C$	$\int \cot u \, \mathrm{d}u = \ln\lvert \sin u\rvert + C$
$\int \sec u \, \mathrm{d}u = \ln\lvert \sec u + \tan u\rvert + C$	$\int \csc u \, \mathrm{d}u = -\ln\lvert \csc u + \cot u\rvert + C$

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(b) Solve for x as a function of $y : x = g(y) = f^{-1}(y)$.

(c) Interchange x and y. The resulting equation is $y = f^{-1}(x)$.

(d) Define the domain of f^{-1} as the range of f.

(e) Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

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17. **Continuity and differentiability of inverse functions** Let f be a function whose domain is an interval I. If f has an inverse function, then the following statements are true.

(a) If f is continuous on its domain, then f^{-1} is continuous on its domain.

(b) If f is increasing on its domain, then f^{-1} is increasing on its domain.

(c) If f is decreasing on its domain, then f^{-1} is decreasing on its domain.

natural logarithmic function $f(x) = \ln x$ is called the **natural exponential function** (**自然指數函數**) and is denoted by

$$f^{-1}(x) = e^x.$$

That is,

$$y = e^x \quad \text{if and only if} \quad x = \ln y.$$

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20. **Operations with exponential functions** Let a and b be any real numbers.

(a) $e^a e^b = e^{a+b}$

(b) $\frac{e^a}{e^b} = e^{a-b}$

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21. **Properties of the natural exponential function**

   (a) The domain of $f(x) = e^x$ is $(-\infty, \infty)$, and the range is $(0, \infty)$.

   (b) The function $f(x) = e^x$ is continuous, increasing, and one-to-one on its entire domain.

   (c) The graph of $f(x) = e^x$ is concave upward on its entire domain.

   (d) $\lim_{x\to-\infty} e^x = 0$ and $\lim_{x\to\infty} e^x = \infty$.

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22. **Derivatives of the natural exponential function** Let u be a differentiable function of x.

   (a) $\frac{\mathrm{d}}{\mathrm{d}x}\left[e^x\right] = e^x$

   (b) $\frac{\mathrm{d}}{\mathrm{d}x}\left[e^u\right] = e^u \frac{\mathrm{d}u}{\mathrm{d}x}$

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to the base a is denoted by $\log_a x$ and is defined as

$$\log_a x = \frac{1}{\ln a} \ln x.$$

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26. **Properties of inverse functions**

(a) $y = a^x$ if and only if $x = \log_a y$.

(b) $a^{\log_a x} = x$, for $x > 0$.

(c) $\log_a a^x = x$, for all x.

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27. **Derivatives for bases other than e** Let a be a positive real number ($a \neq 1$) and let u be a differentiable function of x.

30. **Summary of compound interest formulas (複利公式)**

Let P = amount of deposit, t = number of years, A = balance after t years, r = annual interest rate (decimal form), and n = number of compoundings per year.

(a) Compounded n times per year: $A = P\left(1+\frac{r}{n}\right)^{nt}$

(b) **Compounded continuously (連續複利)**: $A = Pe^{rt}$

31. **The Extended Mean Value Theorem (廣義均值定理)** If f and g are differentiable on an open interval (a, b) and continuous on $[a, b]$ such that $g'(x) \neq 0$ for any x in (a, b), then there exists a point c in (a, b)

such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

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32. **L'Hôpital's Rule (羅比達法則)** Let f and g be functions that are differentiable on an open interval (a, b) containing c, except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, b), except possibly at c itself. If the limit of $f(x)/g(x)$ as x approaches c produces the indeterminate form $0/0$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite). This result also applies if the limit of $f(x)/g(x)$ as x approaches c produces anyone of

1 and $-\pi/2 \le y \le \pi/2$, then

$$\sin(\arcsin x) = x \quad \text{and} \quad \arcsin(\sin y) = y.$$

If $-\infty < x < \infty$ and $-\pi/2 < y < \pi/2$, then

$$\tan(\arctan x) = x \quad \text{and} \quad \arctan(\tan y) = y.$$

If $|x| \ge 1$ and $0 \le y < \pi/2$ or $\pi/2 < y \le \pi$, then

$$\sec(\operatorname{arcsec} x) = x \quad \text{and} \quad \operatorname{arcsec}(\sec y) = y.$$

Similar properties hold for the other inverse trigonometric functions. 146

35. **Derivatives of inverse trigonometric functions** Let u

be a differentiable function of x.

$$\frac{\mathrm{d}}{\mathrm{d}x}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\arctan u] = \frac{u'}{1+u^2} \qquad \frac{\mathrm{d}}{\mathrm{d}x}[\operatorname{arccot} u] = \frac{-u'}{1+u^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

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36. **Elementary function** An **elementary function** (基本函數) is a function from the following list or one that can be formed as the sum, product, quotient, or composition of functions in the list.

Algebraic Functions	Transcendental Functions
Polynomial functions	Logarithmic functions
Rational functions	Exponential functions
Functions involving radicals	Trigonometric functions
	Inverse trigonometric functions

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37. **Basic differentiation rules for elementary functions**

1. $\frac{\mathrm{d}}{\mathrm{d}x}[cu] = cu'$	2. $\frac{\mathrm{d}}{\mathrm{d}x}[u \pm v] = u' \pm v'$	3. $\frac{\mathrm{d}}{\mathrm{d}x}[uv] = uv' + vu'$
4. $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$	5. $\frac{\mathrm{d}}{\mathrm{d}x}[c] = 0$	6. $\frac{\mathrm{d}}{\mathrm{d}x}[u^n] = nu^{n-1}u'$
7. $\frac{\mathrm{d}}{\mathrm{d}x}[x] = 1$	8. $\frac{\mathrm{d}}{\mathrm{d}x}[\lvert u \rvert] = \frac{u}{\lvert u \rvert}(u'), \quad u \neq 0$	9. $\frac{\mathrm{d}}{\mathrm{d}x}[\ln u] = \frac{u'}{u}$
10. $\frac{\mathrm{d}}{\mathrm{d}x}[e^u] = e^u u'$	11. $\frac{\mathrm{d}}{\mathrm{d}x}[\log_a u] = \frac{u'}{(\ln a)u}$	12. $\frac{\mathrm{d}}{\mathrm{d}x}[a^u] = (\ln a)a^u u'$
13. $\frac{\mathrm{d}}{\mathrm{d}x}[\sin u] = (\cos u)u'$	14. $\frac{\mathrm{d}}{\mathrm{d}x}[\cos u] = -(\sin u)u'$	15. $\frac{\mathrm{d}}{\mathrm{d}x}[\tan u] = (\sec^2 u)u'$
16. $\frac{\mathrm{d}}{\mathrm{d}x}[\cot u] = -(\csc^2 u)u'$	17. $\frac{\mathrm{d}}{\mathrm{d}x}[\sec u] = (\sec u \tan u)u'$	18. $\frac{\mathrm{d}}{\mathrm{d}x}[\csc u] = -(\csc u \cot u)u'$
19. $\frac{\mathrm{d}}{\mathrm{d}x}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$	20. $\frac{\mathrm{d}}{\mathrm{d}x}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$	21. $\frac{\mathrm{d}}{\mathrm{d}x}[\arctan u] = \frac{u'}{1+u^2}$
22. $\frac{\mathrm{d}}{\mathrm{d}x}[\operatorname{arccot} u] = -u'$	23. $\frac{\mathrm{d}}{\mathrm{d}x}[\operatorname{arcsec} u] = u'$	24. $\frac{\mathrm{d}}{\mathrm{d}x}[\operatorname{arccsc} u] = -u'$

40. Basic integration rules ($a > 0$)

1. $\int kf(u)\,\mathrm{d}u = k\int f(u)\,\mathrm{d}u$

2. $\int [f(u) \pm g(u)]\,\mathrm{d}u = \int f(u)\,\mathrm{d}u \pm \int g(u)\,\mathrm{d}u$

3. $\int \mathrm{d}u = u + C$

4. $\int u^n\,\mathrm{d}u = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$

5. $\int \frac{\mathrm{d}u}{u} = \ln|u| + C$

6. $\int e^u\,\mathrm{d}u = e^u + C$

7. $\int a^u\,\mathrm{d}u = \left(\frac{1}{\ln a}\right) a^u + C$

8. $\int \sin u\,\mathrm{d}u = -\cos u + C$

9. $\int \cos u\,\mathrm{d}u = \sin u + C$

10. $\int \tan u\,\mathrm{d}u = -\ln|\cos u| + C$

11. $\int \cot u\,\mathrm{d}u = \ln|\sin u| + C$

12. $\int \sec u\,\mathrm{d}u = \ln|\sec u + \tan u| + C$ .169

13. $\int \csc u\,\mathrm{d}u = -\ln|\csc u + \cot u| + C$

14. $\int \sec^2 u\,\mathrm{d}u = \tan u + C$

15. $\int \csc^2 u\,\mathrm{d}u = -\cot u + C$

16. $\int \sec u \tan u\,\mathrm{d}u = \sec u + C$

17. $\int \csc u \cot u\,\mathrm{d}u = -\csc u + C$

18. $\int \frac{\mathrm{d}u}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C$

41. The hyperbolic functions (雙曲函數)

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}, \quad x \neq 0$$
$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$
$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{1}{\tanh x}, \quad x \neq 0$$

42. Hyperbolic identities (雙曲函數恆等式)

$$\cosh^2 x - \sinh^2 x = 1 \qquad \sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\tanh^2 x + \operatorname{sech}^2 x = 1 \qquad \sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$$
$$\coth^2 x - \operatorname{csch}^2 x = 1 \qquad \cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

$$\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$$

$$\sinh^2 x = \frac{-1+\cosh 2x}{2} \quad \cosh^2 x = \frac{1+\cosh 2x}{2}$$

$$\sinh 2x = 2\sinh x \cosh x \quad \cosh 2x = \cosh^2 x + \sinh^2 x$$

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43. **Derivatives and integrals of hyperbolic functions** Let u be a differentiable function of x.

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sinh u] = (\cosh u)u' \qquad \int \cosh u \,\mathrm{d}u = \sinh u + C$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\cosh u] = (\sinh u)u' \qquad \int \sinh u \,\mathrm{d}u = \cosh u + C$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\tanh u] = (\operatorname{sech}^2 u)u' \qquad \int \operatorname{sech}^2 u \,\mathrm{d}u = \tanh u + C$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\coth u] = -(\operatorname{csch}^2 u)u' \qquad \int \operatorname{csch}^2 u \,\mathrm{d}u = -\coth u + C$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\operatorname{sech} u] = -(\operatorname{sech} u \tanh u)u' \qquad \int \operatorname{sech} u \tanh u \,\mathrm{d}u = -\operatorname{sech} u + C$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\operatorname{csch} u] = -(\operatorname{csch} u \coth u)u' \qquad \int \operatorname{csch} u \coth u \,\mathrm{d}u = -\operatorname{csch} u + C$$

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44. **Inverse hyperbolic functions (反雙曲函數)**

Function	Domain
$\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$	$(-\infty, \infty)$
$\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$	$[1, \infty)$
$\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}$	$(-1, 1)$
$\coth^{-1} x = \frac{1}{2}\ln\frac{x+1}{x-1}$	$(-\infty, -1) \cup (1, \infty)$
$\operatorname{sech}^{-1} x = \ln\frac{1+\sqrt{1-x^2}}{x}$	$(0, 1]$
$\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{\lvert x\rvert}\right)$	$(-\infty, 0) \cup (0, \infty)$

45. Differentiation and integration involving inverse hyperbolic functio

Let u be a differentiable function of x.

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sinh^{-1} u] = \frac{u'}{\sqrt{u^2+1}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}[\cosh^{-1} u] = \frac{u'}{\sqrt{u^2-1}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\tanh^{-1} u] = \frac{u'}{1-u^2} \qquad \frac{\mathrm{d}}{\mathrm{d}x}[\coth^{-1} u] = \frac{u'}{1-u^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\operatorname{sech}^{-1} u] = \frac{-u'}{u\sqrt{1-u^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}[\operatorname{csch}^{-1} u] = \frac{-u'}{|u|\sqrt{1+u^2}}$$

$$\int \frac{\mathrm{d}u}{\sqrt{u^2 \pm a^2}} = \ln(u + \sqrt{u^2 \pm a^2}) + C$$

$$\int \frac{\mathrm{d}u}{a^2 - u^2} = \frac{1}{2a} \ln \left| \frac{a+u}{a-u} \right| + C$$

$$\int \frac{\mathrm{d}u}{u\sqrt{a^2 \pm u^2}} = -\frac{1}{a} \ln \frac{a + \sqrt{a^2 \pm u^2}}{|u|} + C$$

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