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Chapter 4

INTEGRATION

4.1 Summary

1. **Antiderivative** A function F is an **antiderivative** (反導數) of f on an interval I if $F'(x) = f(x)$ for all x in I. 4

2. **Representation of antiderivatives** (反導數表示法) If F is an antiderivative of f on an interval I, then G is an antiderivative of f on

the interval I if and only if G is of the form $G(x) = F(x) + C$, for all x in I where C is a constant. 4

3. **Basic integration rules** (基本積分法則)

Differentiation Formula	Integration Formula
$\frac{\mathrm{d}}{\mathrm{d}x}[C] = 0$	$\int 0 \, \mathrm{d}x = C$
$\frac{\mathrm{d}}{\mathrm{d}x}[kx] = k$	$\int k \, \mathrm{d}x = kx + C$
$\frac{\mathrm{d}}{\mathrm{d}x}[kf(x)] = kf'(x)$	$\int kf(x) \, \mathrm{d}x = k \int f(x) \, \mathrm{d}x$
$\frac{\mathrm{d}}{\mathrm{d}x}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	$\int [f(x) \pm g(x)] \, \mathrm{d}x = \int f(x) \, \mathrm{d}x \pm \int g(x) \, \mathrm{d}x$
$\frac{\mathrm{d}}{\mathrm{d}x}[x^n] = nx^{n-1}$	$\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + C, \ n \neq -$ **Power Rule** (幂法则)
$\frac{\mathrm{d}}{\mathrm{d}x}[\sin x] = \cos x$	$\int \cos x \, \mathrm{d}x = \sin x + C$
$\frac{\mathrm{d}}{\mathrm{d}x}[\cos x] = -\sin x$	$\int \sin x \, \mathrm{d}x = -\cos x + C$

1. $\sum_{i=1}^{n} c = cn$ **2.** $\sum_{i=1}^{n} i = \dfrac{n(n+1)}{2}$

3. $\sum_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$ **4.** $\sum_{i=1}^{n} i^3 = (\sum_{i=1}^{n} i)^2 = \dfrac{n^2(n+1)^2}{4}$

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6. **Power sums** (**幂次和**) Let $S_k = \sum_{x=1}^{n} x^k = 1^k + 2^k + \cdots + n^k$, $k \in \mathbb{N}$. Then

$$S_k = \frac{1}{k+1}\left((n+1)^{k+1} - 1^{k+1} - \left(\binom{k+1}{2} S_{k-1} + \cdots + \binom{k+1}{k+1} S_0\right)\right)$$

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7. **Limits of the lower and upper sums** Let f be continuous and nonnegative on the interval $[a, b]$. The limits as $n \to \infty$ of both the

lower and upper sums exist and are equal to each other. That is,

$$\lim_{n\to\infty} s(n) = \lim_{n\to\infty} \sum_{i=1}^{n} f(m_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(M_i)\Delta x = \lim_{n\to\infty} S(n)$$

9. **Riemann sum** Let f be defined on the closed interval $[a, b]$, and let Δ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b \qquad (\text{分割})$$

where Δx_i is the width of the ith subinterval. If c_i (取樣) is any point in the ith subinterval $[x_{i-1}, x_i]$, then the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i, \quad x_{i-1} \leq c_i \leq x_i \qquad (\text{求和})$$

is called a **Riemann sum** (黎曼和) of f for the partition Δ. 72

10. **Definite integral** If f is defined on the closed interval $[a, b]$

and the limit of Riemann sums over partitions Δ

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

exists (as described above), then f is said to be **integrable** (**可積的**) on $[a, b]$ and the limit is denoted by

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)\,\mathrm{d}x.$$

The limit is called the **definite integral** (**定積分**) of f from a to b. The number a is the **lower limit** (**下限**) of integration. 77

11. Four steps of finding the definite integral $\int_a^b f(x)\,\mathrm{d}x$ using **Riemann sum** (**黎曼和**)

(a) **partition** (分割): $a = x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots < x_n = b$

(b) **sampling** (取樣): $c_i \in [x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$

(c) **summation** (求和): $\sum_{i=1}^{n} f(c_i)\Delta x_i$

(d) **limit** (求極限): $\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)\,\mathrm{d}x$

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12. **Continuity implies integrability** (連續隱含可積) If a function f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$. That is, $\int_a^b f(x)\,\mathrm{d}x$ exists. 78

13. **The definite integral as the area of a region** If f is continuous and nonnegative on the closed interval $[a, b]$, then the **area** (面積) of the region bounded by the graph of f, the x-axis, and the vertical

lines $x = a$ and $x = b$ is given by

$$\text{Area} = \int_a^b f(x)\,\mathrm{d}x.$$

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14. **Two special definite integrals** (**二個特殊定積分**)

1. If f is defined at $x = a$, then we define $\int_a^a f(x)\,\mathrm{d}x = 0$.

2. If f is integrable on $[a, b]$, then we define $\int_b^a f(x)\,\mathrm{d}x = -\int_a^b f(x)\,\mathrm{d}x$.

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15. **Additive interval property** (**區間加法性質**) If f is integrable on the three closed intervals determined by a, b, and c, then

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$$

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16. **Properties of definite integrals** If f and g are integrable on $[a, b]$ and k is a constant, then the functions kf and $f \pm g$ are integrable on $[a, b]$, and

1. $\int_a^b kf(x)\,\mathrm{d}x = k\int_a^b f(x)\,\mathrm{d}x$.

2. $\int_a^b [f(x) \pm g(x)]\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x$.

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17. **Preservation of inequality** (不等關係的保留)

1. If f is integrable and nonnegative on the closed interval $[a, b]$, then

$$0 \le \int_a^b f(x)\,\mathrm{d}x.$$

19. Guidelines for using the Fundamental Theorem of Calculus (微積分

(a) Provided you can find an antiderivative of f, you now have a way to evaluate a definite integral without having to use the limit of a sum.

(b) When applying the Fundamental Theorem of Calculus, the following notation is convenient.

$$\int_a^b f(x)\,\mathrm{d}x = F(x)\Big]_a^b = F(b) - F(a)$$

For instance, to evaluate $\int_1^3 x^3\,\mathrm{d}x$, you can write

$$\int_1^3 x^3\,\mathrm{d}x = \frac{x^4}{4}\Bigg]_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = \frac{81}{4} - \frac{1}{4} = 20.$$

(c) It is not necessary to include a constant of integration C in the antiderivative because

$$\int_a^b f(x)\,\mathrm{d}x = [F(x)+C]_a^b = [F(b)+C] - [F(a)+C] = F(b) - F(a).$$

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20. **Mean Value Theorem for Integrals** (积分形式的均值定理) If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

$$\int_a^b f(x)\,\mathrm{d}x = f(c)(b-a).$$

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21. **The average value of a function on an interval** If f is integrable on the closed interval $[a, b]$, then the **average value** (平均值) of

f on the interval is

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x.$$

22. **The Second Fundamental Theorem of Calculus** (微積分第二基本定理

If f is continuous on an open interval I containing a, then, for every x in the interval,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^x f(t)\,\mathrm{d}t\right] = f(x).$$

23. **Leibniz Integral Rule (來布尼茲積分法則)** Let $G(x) = \int_{a(x)}^{b(x)} f(x,t)\,\mathrm{d}$

is differentiable, then

$$G'(x) = f(x, b(x))b'(x) - f(x, a(x))a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,\mathrm{d}t.$$

24. **The Net Change Theorem** (**淨變化定理**) The definite integral of the rate of change of a quantity $F'(x)$ gives the total change, or **net change** (**淨變化**), in that quantity on the interval $[a, b]$.

$$\int_a^b F'(x)\,\mathrm{d}x = F(b) - F(a) \qquad \textbf{Net change of } F$$

25. **Antidifferentiation of a composite function** (**合成函數的反微分**)

Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then

$$\int f(g(x))g'(x)\,\mathrm{d}x = F(g(x)) + C.$$

Letting $u = g(x)$ gives $\mathrm{d}u = g'(x)\,\mathrm{d}x$ and

$$\int f(u)\,\mathrm{d}u = F(u) + C.$$

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26. **Guidelines for making a change of variables** (**變數變換導引**)

(a) Choose a substitution $u = g(x)$. Usually, it is best to choose the inner part of a composite function, such as a quantity raised to a power.

(b) Compute $\mathrm{d}u = g'(x)\,\mathrm{d}x$.

(c) Rewrite the integral in terms of the variable u.

(d) Find the resulting integral in terms of u.

(e) Replace u by $g(x)$ to obtain an antiderivative in terms of x.

(f) Check your answer by differentiating.

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27. **The General Power Rule for Integration** (**廣義積分冪法則**)

If g is a differentiable function of x, then

$$\int [g(x)]^n g'(x)\,\mathrm{d}x = \frac{[g(x)]^{n+1}}{n+1} + C, \quad n \neq -1.$$

Equivalently, if $u = g(x)$, then

$$\int u^n\,\mathrm{d}u = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1.$$

28. **Change of variables for definite integrals** (**定積分的變數變換**)

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u.$$

29. **Integration of even and odd functions** Let f be integrable on the closed interval $[-a, a]$.

1. If f is an **even function** (**偶函數**), then $\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x$.

2. If f is an **odd function** (**奇函數**), then $\int_{-a}^{a} f(x)\,\mathrm{d}x = 0$.

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