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 Chapter **4**

INTEGRATION

4.1 Summary

Section 4.1 Antiderivatives and indefinite integration 2

1. **Antiderivative** A function F is an **antiderivative** (反導數) of f on an interval I if $F'(x) = f(x)$ for all x in I 4
2. **Representation of antiderivatives** (反導數表示法) If F is an antiderivative of f on an interval I , then G is an antiderivative of f on

the interval I if and only if G is of the form $G(x) = F(x) + C$, for all x in I where C is a constant. 4

3. Basic integration rules (基本積分法則)

Differentiation Formula

$$\frac{d}{dx} [C] = 0$$

$$\frac{d}{dx} [kx] = k$$

$$\frac{d}{dx} [kf(x)] = kf'(x)$$

$$\frac{d}{dx} [f(x) \pm g(x)] = f'(x) \pm g'(x)$$

$$\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$$

$$\frac{d}{dx} [x^n] = nx^{n-1}$$

$$\frac{d}{dx} [\sin x] = \cos x$$

$$\frac{d}{dx} [\cos x] = -\sin x$$

Integration Formula

$$\int 0 dx = C$$

$$\int k dx = kx + C$$

$$\int kf(x) dx = k \int f(x) dx$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

Power Rule (幂法則)

$$\int \cos x dx = \sin x + C$$

$$\int \sin x dx = -\cos x + C$$

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Section 4.2 Area 25

4. **Sigma notation** (和符號) The sum of n terms $a_1, a_2, a_3, \dots, a_n$ is written as

$$\sum_{i=1}^n a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

where i is the **index of summation** (和的序號), a_i is the **i th term** of the sum, and the

upper and lower bounds of summation (和的上、下界) are n and 1..... 26

5. **Summation formulas** (和公式)

$$1. \sum_{i=1}^n c = cn$$

$$2. \sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$3. \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$4. \sum_{i=1}^n i^3 = \left(\sum_{i=1}^n i\right)^2 = \frac{n^2(n+1)^2}{4}$$

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6. **Power sums** (幂次和) Let $S_k = \sum_{x=1}^n x^k = 1^k + 2^k + \cdots + n^k$,

$k \in \mathbb{N}$. Then

$$S_k = \frac{1}{k+1} \left((n+1)^{k+1} - 1^{k+1} - \left(\binom{k+1}{2} S_{k-1} + \cdots + \binom{k+1}{k+1} S_0 \right) \right)$$

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7. **Limits of the lower and upper sums**

Let f be continuous

and nonnegative on the interval $[a, b]$. The limits as $n \rightarrow \infty$ of both the

lower and upper sums exist and are equal to each other. That is,

$$\lim_{n \rightarrow \infty} s(n) = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(m_i) \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(M_i) \Delta x = \lim_{n \rightarrow \infty} S(n)$$

where $\Delta x = (b - a)/n$ and $f(m_i)$ and $f(M_i)$ are the minimum and maximum values of f on the subinterval.....57

8. The area of a region in the plane

Let f be continuous and

nonnegative on the interval $[a, b]$. The area of the region bounded by the graph of f , the x -axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x, \quad x_{i-1} \leq c_i \leq x_i$$

where $\Delta x = (b - a)/n$58

Section 4.3 Riemann sums and definite integrals..... 66

9. **Riemann sum** Let f be defined on the closed interval $[a, b]$, and let Δ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b \quad (\text{分割})$$

where Δx_i is the width of the i th subinterval. If c_i (取樣) is any point in the i th subinterval $[x_{i-1}, x_i]$, then the sum

$$\sum_{i=1}^n f(c_i) \Delta x_i, \quad x_{i-1} \leq c_i \leq x_i \quad (\text{求和})$$

is called a **Riemann sum** (黎曼和) of f for the partition Δ 72

10. **Definite integral**

If f is defined on the closed interval $[a, b]$

and the limit of Riemann sums over partitions Δ

$$\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x_i$$

exists (as described above), then f is said to be **integrable** (可積的) on $[a, b]$ and the limit is denoted by

$$\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x_i = \int_a^b f(x) dx.$$

The limit is called the **definite integral** (定積分) of f from a to b . The number a is the **lower limit** (下限) of integration.....77

11. Four steps of finding the definite integral $\int_a^b f(x) dx$ using **Riemann sum** (黎曼和)

(a) **partition** (分割): $a = x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots < x_n = b$

(b) **sampling** (取樣): $c_i \in [x_{i-1}, x_i], i = 1, 2, \dots, n$

(c) **summation** (求和): $\sum_{i=1}^n f(c_i) \Delta x_i$

(d) **limit** (求極限): $\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x_i = \int_a^b f(x) dx$

..... 78

12. **Continuity implies integrability** (連續隱含可積) If a function f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$. That is, $\int_a^b f(x) dx$ exists. 78

13. **The definite integral as the area of a region** If f is continuous and nonnegative on the closed interval $[a, b]$, then the **area** (面積) of the region bounded by the graph of f , the x -axis, and the vertical

lines $x = a$ and $x = b$ is given by

$$\text{Area} = \int_a^b f(x) \, dx.$$

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14. **Two special definite integrals** (二個特殊定積分)

1. If f is defined at $x = a$, then we define $\int_a^a f(x) \, dx = 0$.

2. If f is integrable on $[a, b]$, then we define $\int_b^a f(x) \, dx = -\int_a^b f(x) \, dx$.

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15. **Additive interval property** (區間加法性質) If f is integrable on the three closed intervals determined by a , b , and c , then

$$\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.$$

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16. **Properties of definite integrals** If f and g are integrable on $[a, b]$ and k is a constant, then the functions kf and $f \pm g$ are integrable on $[a, b]$, and

1. $\int_a^b kf(x) dx = k \int_a^b f(x) dx.$

2. $\int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx.$

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17. **Preservation of inequality** (不等關係的保留)

1. If f is integrable and nonnegative on the closed interval $[a, b]$, then

$$0 \leq \int_a^b f(x) dx.$$

2. If f and g are integrable on the closed interval $[a, b]$ and $f(x) \leq g(x)$ for every x in $[a, b]$, then

$$\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx.$$

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Section 4.4 The Fundamental Theorem of Calculus.....100

18. **The Fundamental Theorem of Calculus** (微積分基本定理) If a function f is continuous on the closed interval $[a, b]$ and F is an anti-derivative of f on the interval $[a, b]$, then

$$\int_a^b f(x) \, dx = F(b) - F(a).$$

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19. Guidelines for using the Fundamental Theorem of Calculus (微積分)

- (a) Provided you can find an antiderivative of f , you now have a way to evaluate a definite integral without having to use the limit of a sum.
- (b) When applying the Fundamental Theorem of Calculus, the following notation is convenient.

$$\int_a^b f(x) \, dx = F(x) \Big|_a^b = F(b) - F(a)$$

For instance, to evaluate $\int_1^3 x^3 \, dx$, you can write

$$\int_1^3 x^3 \, dx = \left. \frac{x^4}{4} \right|_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = \frac{81}{4} - \frac{1}{4} = 20.$$

- (c) It is not necessary to include a constant of integration C in the anti-

derivative because

$$\int_a^b f(x) \, dx = [F(x) + C]_a^b = [F(b) + C] - [F(a) + C] = F(b) - F(a).$$

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20. **Mean Value Theorem for Integrals** (積分形式的均值定理) If

f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

$$\int_a^b f(x) \, dx = f(c)(b - a).$$

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21. **The average value of a function on an interval** If f is integrable on the closed interval $[a, b]$, then the **average value** (平均值) of

f on the interval is

$$\frac{1}{b-a} \int_a^b f(x) dx.$$

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22. The Second Fundamental Theorem of Calculus (微積分第二基本定理)

If f is continuous on an open interval I containing a , then, for every x in the interval,

$$\frac{d}{dx} \left[\int_a^x f(t) dt \right] = f(x).$$

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23. Leibniz Integral Rule (來布尼茲積分法則)

$$\text{Let } G(x) = \int_{a(x)}^{b(x)} f(x, t) dt$$

is differentiable, then

$$G'(x) = f(x, b(x))b'(x) - f(x, a(x))a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x, t) dt.$$

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24. **The Net Change Theorem** (**淨變化定理**) The definite integral of the rate of change of a quantity $F'(x)$ gives the total change, or **net change** (**淨變化**), in that quantity on the interval $[a, b]$.

$$\int_a^b F'(x) dx = F(b) - F(a) \quad \text{Net change of } F$$

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Section 4.5 Integration by substitution 142

25. **Antidifferentiation of a composite function** (**合成函數的反微分**)

Let g be a function whose range is an interval I , and let f be a function that is continuous on I . If g is differentiable on its domain and F is an antiderivative of f on I , then

$$\int f(g(x))g'(x) \, dx = F(g(x)) + C.$$

Letting $u = g(x)$ gives $du = g'(x) \, dx$ and

$$\int f(u) \, du = F(u) + C.$$

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26. **Guidelines for making a change of variables** (變數變換導引)

- (a) Choose a substitution $u = g(x)$. Usually, it is best to choose the inner part of a composite function, such as a quantity raised to a power.
- (b) Compute $du = g'(x) \, dx$.

- (c) Rewrite the integral in terms of the variable u .
- (d) Find the resulting integral in terms of u .
- (e) Replace u by $g(x)$ to obtain an antiderivative in terms of x .
- (f) Check your answer by differentiating.

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27. The General Power Rule for Integration (廣義積分冪法則)

If g is a differentiable function of x , then

$$\int [g(x)]^n g'(x) dx = \frac{[g(x)]^{n+1}}{n+1} + C, \quad n \neq -1.$$

Equivalently, if $u = g(x)$, then

$$\int u^n du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1.$$

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28. **Change of variables for definite integrals** (**定積分的變數變換**)
- If the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$ and f is continuous on the range of g , then
- $$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du.$$
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29. **Integration of even and odd functions** Let f be integrable on the closed interval $[-a, a]$.
1. If f is an **even function** (**偶函數**), then $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$.
 2. If f is an **odd function** (**奇函數**), then $\int_{-a}^a f(x) dx = 0$.
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