Contents

2 Differentiation **1**

2.1 Summary 1

Index **14**

List of Tables

List of Figures

Chapter 2

DIFFERENTIATION

2.1 Summary

Section 2.1 The derivative and the tangent line problem 2

1. **Tangent line with slope m** If f is defined on an open interval containing c, and if the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = m$$

exists, then the line passing through $(c, f(c))$ with slope m is the **tangent line**

(切線) to the graph of f at the point $(c, f(c))$. 6

2. If f is continuous at c and

$$\lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = \infty \quad \text{or} \quad \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = -\infty$$

the vertical line $x = c$ passing through $(c, f(c))$ is a **vertical tangent line** (垂直切線) to the graph of f. 11

3. If the domain of f is the closed interval $[a, b]$, you can extend the definition of a vertical tangent line to include the endpoints by considering continuity and limits from the right (for $x = a$) and from the left (for $x = b$). 11

4. **The derivative of a function** The **derivative** (導數) of f at x

is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

provided the limit exists. For all x for which this limit exists, f' is a function of x. 13

5. **alternative limit form of the derivative**

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$ **Alternative form of derivative**

..... 20

6. f is **differentiable on the closed interval** $[a, b]$ (**在閉區間** $[a, b]$) if it is differentiable on (a, b) and if the derivative from the right at a and the derivative from the left at b both exist. 22

7. **Differentiability implies continuity** If f is differentiable at $x = c$, then f is continuous at $x = c$. 26

8. **Relationship between continuity and differentiability**

   (a) If a function is differentiable at $x = c$, then it is continuous at $x = c$.

   (b) It is possible for a function to be continuous at $x = c$ and not be differentiable at $x = c$.

   28

Section 2.2 Basic differentiation rules and rates of change . 28

9. **The Constant Rule (常數法則)** The derivative of a constant function is 0. That is, if c is a real number, then

$$\frac{\mathrm{d}}{\mathrm{d}x}[c] = 0.$$

. 30

10. **The Power Rule** (簡單冪法則) If n is a rational number, then the function $f(x) = x^n$ is differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^n] = nx^{n-1}.$$

For f to be differentiable at $x = 0$, n must be a number such that x^{n-1} is defined on an interval containing 0. 33

11. **The Constant Multiple Rule** (常數乘積法則) If f is a differentiable function and c is a real number, then cf is also differentiable and $\frac{\mathrm{d}}{\mathrm{d}x}[cf(x)] = cf'(x)$. 40

12. **The Sum and Difference Rules** (和差法則) The sum (or difference) of two differentiable functions f and g is itself differentiable.

Moreover, the derivative of $f+g$ (or $f-g$) is the sum (or difference) of the derivatives of f and g.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)+g(x)\right]=f'(x)+g'(x) \qquad \textbf{Sum Rule}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)-g(x)\right]=f'(x)-g'(x) \qquad \textbf{Difference Rule}$$

. 43

13. **Derivatives of the sine and cosine functions** (正弦和餘弦函數的導數)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin x\right]=\cos x \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left[\cos x\right]=-\sin x$$

. 45

Section 2.3 Product and Quotient Rules and higher-order deriv-

atives 56

14. **The Product Rule** (積法則) The product of two differentiable functions f and g is itself differentiable. Moreover, the derivative of fg is the first function times the derivative of the second, plus the second function times the derivative of the first.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)g(x)\right] = f(x)g'(x) + g(x)f'(x)$$

. 57

15. If f, g, and h are differentiable functions of x, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)g(x)h(x)\right] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).$$

$$(f_1 f_2 f_3 \cdots f_n)'$$

$$= f_1' f_2 \cdots f_{n-1} f_n + f_1 f_2' \cdots f_{n-1} f_n + \cdots$$
$$+ f_1 f_2 \cdots f_{n-1}' f_n + f_1 f_2 \cdots f_{n-1} f_n'$$

. 60

16. **The Quotient Rule** (商法則) The quotient f/g of two differentiable functions f and g is itself differentiable at all values of x for which $g(x) \neq 0$. Moreover, the derivative of f/g is given by the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

. 62

17. **Derivatives of trigonometric functions** (三角函數的導數)

$$\frac{\mathrm{d}}{\mathrm{d}x}[\tan x] = \sec^2 x \qquad \frac{\mathrm{d}}{\mathrm{d}x}[\cot x] = -\csc^2 x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[\sec x] = \sec x \tan x \qquad \frac{\mathrm{d}}{\mathrm{d}x}[\csc x] = -\csc x \cot x$$

..... 68

18. **Binomial Theorem of nth derivative for product**

$$(f(x)g(x))^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x), \qquad n = 1, 2, \ldots$$

..... 75

19. $\cos(x + n\pi/2) + i\sin(x + n\pi/2) = \cos^{(n)} x + i\sin^{(n)} x$ 75

Section 2.4 The Chain Rule 76

20. **The Chain Rule** (連鎖律) If $y = f(u)$ is a differentiable function

of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$$

or, equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f(g(x))\right] = f'(g(x))g'(x).$$

A method for finding the derivative of a composition of functions. . . . 82

21. **The General Power Rule** (**廣義冪法則**) If $y = [u(x)]^n$, where u is a differentiable function of x and n is a rational number, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = n[u(x)]^{n-1}\frac{\mathrm{d}u}{\mathrm{d}x}$$

or, equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[u^n\right] = nu^{n-1}u'.$$

. 86

22. "Chain Rule versions" of the derivatives of the six trigonometric functions:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin u\right] = (\cos u)u' \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left[\cos u\right] = -(\sin u)u'$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\tan u\right] = (\sec^2 u)u' \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left[\cot u\right] = -(\csc^2 u)u'$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sec u\right] = (\sec u \tan u)u' \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left[\csc u\right] = -(\csc u \cot u)u'$$

. 92

Section 2.5 Implicit differentiation 97

23. **Guidelines for implicit differentiation**

(a) Differentiate both sides of the equation with respect to x.

(b) Collect all terms involving $\mathrm{d}y/\mathrm{d}x$ on the left side of the equation and move all other terms to the right side of the equation.

(c) Factor $\mathrm{d}y/\mathrm{d}x$ out of the left side of the equation.

(d) Solve for $\mathrm{d}y/\mathrm{d}x$.

..... 101

Section 2.6 Related rates 112

24. **Guidelines for solving related-rate problems** (解相關變率問題導引)

(a) Identify all given quantities and quantities to be determined. Make a sketch and label the quantities.

(b) Write an equation involving the variables whose rates of change either are given or are to be determined.

(c) Using the Chain Rule, implicitly differentiate both sides of the equation with respect to time t.

(d) After completing step 3, substitute into the resulting equation all known value for the variables and their rates of change. Then solve for the required rate of change.

. 118

Index

alternative form 另一型式
  of the derivative 導數, 3

Chain Rule 連鎖律, 9
Constant 常數
  Rule 法則, 4
constant 常數
  Multiple Rule 乘積法則, 5
continuity 連續
  differentiability implies 可微分隱含, 4
cosecant function 餘割函數
  derivative of 導數, 9
cosine function 餘弦函數
  derivative of 導數, 6
cotangent function 餘切函數
  derivative of 導數, 9

derivative(s) 導數
  alternative form 另一型式, 3

Chain Rule 連鎖律, 9
Constant Multiple Rule 常數乘積法則, 5
Difference Rule 差法則, 5
General Power Rule 廣義冪法則, 10
of a function 函數, 2
of cosecant function 餘割函數, 9
of cosine function 餘弦函數, 6
of cotangent function 餘切函數, 9
of secant function 正割函數, 9
of sine function 正弦函數, 6
of tangent function 正切函數, 9
of trigonometric functions 三角函數, 9
Power Rule 冪法則, 5
Product Rule 積法則, 7
Quotient Rule 商法則, 8
Simple Power Rule 簡單的冪法則, 5
Sum Rule 和法則, 5
derivative 導數
Constant Rule 常數法則, 4
of trigonometric functions 三角函數, 11
Difference Rule 差法則, 5

differentiability 可微分
  implies continuity 隱含連續, 4
differentiable function 可微函數
  on the closed interval $[a, b]$ 在閉區間 $[a, b]$, 3
differentiation rules 微分法則
  chain 連鎖, 9
  constant multiple 常數乘積, 5
  constant 常數, 4
  cosecant function 餘割函數, 9
  cosine function 餘弦函數, 6
  cotangent function 餘切函數, 9
  difference 差, 5
  general power 廣義冪, 10
  power 冪, 5
  product 積, 7
  quotient 商, 8
  secant function 正割函數, 9
  Simple Power 簡單冪, 5
  sine function 正弦函數, 6
  sum 和, 5
  tangent function 正切函數, 9
differentiation 微分
  implicit 隱
    guidelines for 導引, 11
function(s) 函數

derivative of 導數, 2
differentiable 可微, 3

General Power Rule 廣義冪法則
for differentiation 微分, 10
guidelines 導引
for implicit differentiation 隱函數微分, 11
for solving related-rate problems 解相關變率問題, 12

implicit differentiation 隱函數微分
guidelines for 導引, 11

line(s) 直線
tangent 切線, 1
vertical 垂直, 2
with slope m 斜率 m, 1

power rule 冪法則
for differentiation 微分, 5
Product Rule 積法則, 7

Quotient Rule 商法則, 8

related-rate problems, guidelines for solving 相關變率問題，解導引, 12
Rule 法則
Constant 常數, 4

secant function 正割函數
  derivative of 導數, 9
Simple Power Rule 簡單冪法則, 5
sine function 正弦函數
  derivative of 導數, 6
slope(s) 斜率
  of a tangent line 切線, 1
Sum Rule 和法則, 5
sum(s) 和
  rule 法則, 5

tangent function 正切函數
  derivative of 導數, 9
tangent line(s) 切線, 1
  slope of 斜率, 1
  vertical 垂直, 2
  with slope m 斜率 m, 1
trigonometric function(s) 三角函數
  derivative of 導數, 9

vertical tangent line 垂直切線, 2

切線 tangent line(s), 1
  垂直 vertical, 2
  斜率 m with slope m, 1
  斜率 slope of, 1
三角函數 trigonometric function(s)
  導數 derivative of, 9
可微分 differentiability

隱含連續 implies continuity, 4
可微函數 differentiable function
在閉區間 $[a, b]$ on the closed interval $[a, b]$, 3
另一型式 alternative form
導數 of the derivative, 3
正切函數 tangent function
導數 derivative of, 9
正弦函數 sine function
導數 derivative of, 6
正割函數 secant function
導數 derivative of, 9
函數 function(s)
可微 differentiable, 3
導數 derivative of, 2
和 sum(s)
法則 rule, 5
和法則 Sum Rule, 5
法則 Rule
常數 Constant, 4
直線 line(s)
切線 tangent, 1
垂直 vertical, 2
斜率 m with slope m, 1
垂直切線 vertical tangent line, 2
相關變率問題，解導引 related-rate

problems, guidelines for solving, 12
差法則 Difference Rule, 5
商法則 Quotient Rule, 8
斜率 slope(s)
  切線 of a tangent line, 1
常數 Constant
  法則 Rule, 4
常數 constant
  乘積法則 Multiple Rule, 5
連鎖律 Chain Rule, 9
連續 continuity
  可微分隱含 differentiability implies, 4
微分 differentiation
  隱 implicit
    導引 guidelines for, 11
微分法則 differentiation rules
  正切函數 tangent function, 9
  正弦函數 sine function, 6
  正割函數 secant function, 9
  和 sum, 5
  差 difference, 5
  商 quotient, 8
  常數 constant, 4
  常數乘積 constant multiple, 5

連鎖 chain, 9
廣義冪 general power, 10
餘切函數 cotangent function, 9
餘弦函數 cosine function, 6
餘割函數 cosecant function, 9
冪 power, 5
積 product, 7
簡單冪 Simple Power, 5
廣義冪法則 General Power Rule
微分 for differentiation, 10
餘切函數 cotangent function
導數 derivative of, 9
餘弦函數 cosine function
導數 derivative of, 6
餘割函數 cosecant function
導數 derivative of, 9
冪法則 power rule
微分 for differentiation, 5
導引 guidelines
解相關變率問題 for solving related-rate problems, 12
隱函數微分 for implicit differentiation, 11
導數 derivative
三角函數 of trigonometric functions, 11

常數法則 Constant Rule, 4
導數 derivative(s)
三角函數 of trigonometric functions, 9
另一型式 alternative form, 3
正切函數 of tangent function, 9
正弦函數 of sine function, 6
正割函數 of secant function, 9
函數 of a function, 2
和法則 Sum Rule, 5
差法則 Difference Rule, 5
商法則 Quotient Rule, 8
常數乘積法則 Constant Multiple Rule, 5
連鎖律 Chain Rule, 9
廣義冪法則 General Power Rule, 10
餘切函數 of cotangent function, 9
餘弦函數 of cosine function, 6
餘割函數 of cosecant function, 9
冪法則 Power Rule, 5
積法則 Product Rule, 7
簡單的冪法則 Simple Power Rule, 5
積法則 Product Rule, 7
隱函數微分 implicit differentiation

導引 guidelines for, 11
簡單冪法則 Simple Power Rule, 5