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Chapter **1**

LIMITS AND THEIR PROPERTIES

1.1 Summary

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1. Common types of behavior associated with nonexistence of a limit

(a) $f(x)$ approaches a different number from the right side of c than it approaches from the left side.

(b) $f(x)$ increases or decreases without bound as x approaches c .

(c) $f(x)$ oscillates between two fixed values as x approaches c .

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2. **Limit (極限)** Let f be a function defined on an open interval containing c (except possibly at c) and let L be a real number. The statement

$$\lim_{x \rightarrow c} f(x) = L$$

means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$$0 < |x - c| < \delta, \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

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Section 1.3 Evaluating limits analytically 41

3. **Some basic limits** (一些基本極限) Let b and c be real numbers and let n be a positive integer. **1.** $\lim_{x \rightarrow c} b = b$ **2.** $\lim_{x \rightarrow c} x = c$
3. $\lim_{x \rightarrow c} x^n = c^n$ 42

4. **Properties of limits** (極限的性質) Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x \rightarrow c} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = K$$

1. Scalar multiple: $\lim_{x \rightarrow c} [bf(x)] = bL$
2. Sum or difference: $\lim_{x \rightarrow c} [f(x) \pm g(x)] = L \pm K$
3. Product: $\lim_{x \rightarrow c} [f(x)g(x)] = LK$ 45
4. Quotient: $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{K}$, provided $K \neq 0$
5. Power: $\lim_{x \rightarrow c} [f(x)]^n = L^n$

5. Limits of polynomial and rational functions

If p is a poly-

nomial function and c is a real number, then

$$\lim_{x \rightarrow c} p(x) = p(c).$$

If r is a rational function given by $r(x) = p(x)/q(x)$ and c is a real

number such that $q(c) \neq 0$, then

$$\lim_{x \rightarrow c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

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6. **The limit of a function involving a radical** Let n be a positive integer. The following limit is valid for all c if n is odd, and is valid for $c > 0$ if n is even.

$$\lim_{x \rightarrow c} \sqrt[n]{x} = \sqrt[n]{c}$$

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7. **The limit of a composite function (合成函数)** If f and g are

functions such that $\lim_{x \rightarrow c} g(x) = L$ and $\lim_{x \rightarrow L} f(x) = f(L)$, then

$$\lim_{x \rightarrow c} f(g(x)) = f\left(\lim_{x \rightarrow c} g(x)\right) = f(L).$$

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8. Limits of trigonometric functions (三角函数) Let c be a real number in the domain of the given trigonometric function.

1. $\lim_{x \rightarrow c} \sin x = \sin c$ 2. $\lim_{x \rightarrow c} \cos x = \cos c$ 3. $\lim_{x \rightarrow c} \tan x = \tan c$
 4. $\lim_{x \rightarrow c} \cot x = \cot c$ 5. $\lim_{x \rightarrow c} \sec x = \sec c$ 6. $\lim_{x \rightarrow c} \csc x = \csc c$

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9. Functions that agree at all but one point Let c be a real number and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c . If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x)$ 60

10. A strategy for finding limits (求極限的策略)

(a) Learn to recognize which limits can be evaluated by direct substitution.
(These limits are listed in Theorems 1.1 through 1.6.)

(b) If the limit of $f(x)$ as x approaches c **cannot** be evaluated by direct substitution, try to find a function g that agrees with f for all x other than $x = c$.

(c) Apply Theorem 1.7 to conclude **analytically** that

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = g(c).$$

(d) Use a **graph** or **table** to reinforce your conclusion.

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11. The Squeeze Theorem (夾擠定理) If $h(x) \leq f(x) \leq g(x)$ for

all x in an open interval containing c , except possibly at c itself, and if $\lim_{x \rightarrow c} h(x) = L = \lim_{x \rightarrow c} g(x)$ then $\lim_{x \rightarrow c} f(x)$ exists and is equal to L .

A theorem which allows the computation of the limit of an expression by trapping the expression between two other expressions which have limits that are easier to compute..... 73

12.

$$\mathbf{1.} \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad \mathbf{2.} \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0 \dots\dots\dots 75$$

Section 1.4 Continuity and one-sided limits..... 81

13. Continuity at $x = c$ can be destroyed by any one of the following conditions.

1. The function is not defined at $x = c$. **2.** The limit of $f(x)$ does

not exist at $x = c$. **3.** The limit of $f(x)$ exists at $x = c$, but it is not equal to $f(c)$ 84

14. **Continuity** Continuity at a point: A function f is **continuous** (**連續**) at c if the following three conditions are met.

1. $f(c)$ is defined. **2.** $\lim_{x \rightarrow c} f(x)$ exists. **3.** $\lim_{x \rightarrow c} f(x) = f(c)$

Continuity on an open interval: A function is **continuous on an open interval** (a, b) if it is continuous at each point in the interval. Continuity on \mathbb{R} : A function that is continuous on the entire real line $(-\infty, \infty)$ is **everywhere continuous** (**到處連續**). 85

15. Discontinuities fall into two categories: **removable** (**可移除**) and **nonremovable** (**不可移除**).

A discontinuity at c is called **removable** (**可移除**) if f can be made continuous by appropriately defining (or redefining $f(c)$). 86

16. **Greatest integer function** (最大整數函數) $\lfloor x \rfloor$
 $\lfloor x \rfloor =$ greatest integer n such that $n \leq x$ 97
17. **The existence of a limit** Let f be a function and let c and L be real numbers. The limit of $f(x)$ as x approaches c is L if and only if $\lim_{x \rightarrow c^-} f(x) = L$ and $\lim_{x \rightarrow c^+} f(x) = L$ 97
18. **Continuity on a closed interval** A function f is **continuous on the closed interval $[a, b]$** if it is continuous on the open interval (a, b) and
- $$\lim_{x \rightarrow a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x \rightarrow b^-} f(x) = f(b).$$
- The function f is **continuous from the right** (右連續) at a and **continuous from the left** (左連續) at b 98

19. Properties of continuity

If b is a real number and f and g are

continuous at $x = c$, then the following functions are also continuous at c .

(a) Scalar multiple: bf

(b) Sum or difference: $f \pm g$

(c) Product: fg

(d) Quotient: $\frac{f}{g}$, if $g(c) \neq 0$

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20. The following types of functions are **continuous at every point in their domains**.

(a) Polynomial: $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

(b) Rational: $r(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$

(c) Radical: $f(x) = \sqrt[n]{x}$

(d) Trigonometric: $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$

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21. **Continuity of a composite function (合成函数)** If g is continuous at c and f is continuous at $g(c)$, then the composite function given by $(f \circ g)(x) = f(g(x))$ is continuous at c 108

22. **The Intermediate Value Theorem (中間值定理)** If f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that

$$f(c) = k.$$

A theorem verifying that the graph of a continuous function is connected.

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Section 1.5 Infinite limits 122

23. **Infinite limit** (無窮極限) Let f be a function that is defined at every real number in some open interval containing c (except possibly at c itself). The statement

$$\lim_{x \rightarrow c} f(x) = \infty$$

means that for each $M > 0$ there exists a $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - c| < \delta$. Similarly, the statement

$$\lim_{x \rightarrow c} f(x) = -\infty$$

means that for each $N < 0$ there exists a $\delta > 0$ such that $f(x) < N$

whenever $0 < |x - c| < \delta$.

To define the **infinite limit from the left** (無窮左極限), replace $0 < |x - c| < \delta$ by $c - \delta < x < c$. To define the **infinite limit from the right** (無窮右極限), replace $0 < |x - c| < \delta$ by $c < x < c + \delta$ 126

24. **Vertical asymptote** If $f(x)$ approaches infinity (or negative infinity) as x approaches c from the right or the left, then the line $x = c$ is a **vertical asymptote** (垂直漸近線) of the graph of f 131

25. **Vertical asymptotes** Let f and g be continuous on an open interval containing c . If $f(c) \neq 0$, $g(c) = 0$, and there exists an open interval containing c such that $g(x) \neq 0$ for all $x \neq c$ in the interval, then the graph of the function given by

$$h(x) = \frac{f(x)}{g(x)}$$

has a **vertical asymptote** (垂直漸近線) at $x = c$132

26. Properties of infinite limits

Let c and L be real numbers and

let f and g be functions such that

$$\lim_{x \rightarrow c} f(x) = \infty \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = L.$$

(a) Sum or difference: $\lim_{x \rightarrow c} [f(x) \pm g(x)] = \infty$

(b) Product:

$$\lim_{x \rightarrow c} [f(x)g(x)] = \infty, \quad L > 0$$

$$\lim_{x \rightarrow c} [f(x)g(x)] = -\infty, \quad L < 0$$

(c) Quotient: $\lim_{x \rightarrow c} \frac{g(x)}{f(x)} = 0$

Similar properties hold for one-sided limits and for functions for which

the limit of $f(x)$ as x approaches c is $-\infty$ 142

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