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# Chapter 1

# Limits and Their Properties

## 1.1 Summary



1. Common types of behavior associated with nonexistence of a limit

(a) $f(x)$ approaches a different number from the right side of $c$ than it approaches from the left side.

(b) $f(x)$ increases or decreases without bound as $x$ approaches $c$.

(c) $f(x)$ oscillates between two fixed values as $x$ approaches $c$.

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2. **Limit** (**極限**) Let $f$ be a function defined on an open interval containing $c$ (except possibly at $c$) and let $L$ be a real number. The statement

$$\lim_{x \to c} f(x) = L$$

means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$$0 < |x - c| < \delta, \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

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3. **Some basic limits** (**一些基本極限**) Let $b$ and $c$ be real numbers and let $n$ be a positive integer. **1.** $\lim_{x\to c} b = b$ **2.** $\lim_{x\to c} x = c$ **3.** $\lim_{x\to c} x^n = c^n$ . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 42

4. **Properties of limits** (**極限的性質**) Let $b$ and $c$ be real numbers, let $n$ be a positive integer, and let $f$ and $g$ be functions with the following limits.

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = K$$

1. Scalar multiple: $\lim_{x\to c}[bf(x)] = bL$
2. Sum or difference: $\lim_{x\to c}[f(x) \pm g(x)] = L \pm K$
3. Product: $\lim_{x\to c}[f(x)g(x)] = LK$
4. Quotient: $\lim_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{L}{K}$, provided $K \neq 0$
5. Power: $\lim_{x\to c}[f(x)]^n = L^n$

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5. **Limits of polynomial and rational functions** If $p$ is a polynomial function and $c$ is a real number, then

$$\lim_{x\to c} p(x) = p(c).$$

If $r$ is a rational function given by $r(x) = p(x)/q(x)$ and $c$ is a real number such that $q(c) \neq 0$, then

$$\lim_{x \to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

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6. **The limit of a function involving a radical** Let $n$ be a positive integer. The following limit is valid for all $c$ if $n$ is odd, and is valid for $c > 0$ if $n$ is even.

$$\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$$

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7. **The limit of a composite function (合成函數)** If $f$ and $g$ are

10. **A strategy for finding limits** (**求極限的策略**)

(a) Learn to recognize which limits can be evaluated by direct substitution. (These limits are listed in Theorems 1.1 through 1.6.)

(b) If the limit of $f(x)$ as $x$ approaches $c$ **cannot** be evaluated by direct substitution, try to find a function $g$ that agrees with $f$ for all $x$ other than $x = c$.

(c) Apply Theorem 1.7 to conclude **analytically** that

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = g(c).$$

(d) Use a **graph** or **table** to reinforce your conclusion.

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11. **The Squeeze Theorem** (**夾擠定理**) If $h(x) \leq f(x) \leq g(x)$ for

all $x$ in an open interval containing $c$, except possibly at $c$ itself, and if $\lim_{x\to c} h(x) = L = \lim_{x\to c} g(x)$ then $\lim_{x\to c} f(x)$ exists and is equal to $L$.



12.





13. Continuity at $x = c$ can be destroyed by any one of the following conditions.

**1.** The function is not defined at $x = c$. **2.** The limit of $f(x)$ does

16. **Greatest integer function** (**最大整數函數**) $\lfloor x \rfloor$

    $\lfloor x \rfloor$ = greatest integer $n$ such that $n \le x$. . . . . . . . . . . . . . . . . . . . . . . . . . . . 97

17. **The existence of a limit** Let $f$ be a function and let $c$ and $L$ be real numbers. The limit of $f(x)$ as $x$ approaches $c$ is $L$ if and only if $\lim_{x\to c^-} f(x) = L$ and $\lim_{x\to c^+} f(x) = L$. . . . . . . . . . . . . . . . . . . . . . . 97

18. **Continuity on a closed interval** A function $f$ is **continuous on the closed interval** $[a, b]$ if it is continuous on the open interval $(a, b)$ and

$$\lim_{x\to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x\to b^-} f(x) = f(b).$$

    The function $f$ is **continuous from the right** (**右連續**) at $a$ and **continuous from the left** (**左連續**) at $b$. . . . . . . . . . . . . . . . . . . . . . . 98

19. **Properties of continuity** If $b$ is a real number and $f$ and $g$ are continuous at $x = c$, then the following functions are also continuous at $c$.

    (a) Scalar multiple: $bf$

    (b) Sum or difference: $f \pm g$

    (c) Product: $fg$

    (d) Quotient: $\dfrac{f}{g}$, if $g(c) \neq 0$

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20. The following types of functions are **continuous at every point in their domains**.

    (a) Polynomial: $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

(b) Rational: $r(x) = \dfrac{p(x)}{q(x)}$, $q(x) \neq 0$

(c) Radical: $f(x) = \sqrt[n]{x}$

(d) Trigonometric: $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$

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21. **Continuity of a composite function (合成函數)** If $g$ is continuous at $c$ and $f$ is continuous at $g(c)$, then the composite function given by $(f \circ g)(x) = f(g(x))$ is continuous at $c$. . . . . . . . . . . . . . . . . . . 108

22. **The Intermediate Value Theorem (中間值定理)** If $f$ is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and $k$ is any number between $f(a)$ and $f(b)$, then there is at least one number $c$ in $[a, b]$ such that

$$f(c) = k.$$

A theorem verifying that the graph of a continuous function is connected. 115



23. **Infinite limit** (無窮極限) Let $f$ be a function that is defined at every real number in some open interval containing $c$ (except possibly at $c$ itself). The statement

$$\lim_{x \to c} f(x) = \infty$$

means that for each $M > 0$ there exists a $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - c| < \delta$. Similarly, the statement

$$\lim_{x \to c} f(x) = -\infty$$

means that for each $N < 0$ there exists a $\delta > 0$ such that $f(x) < N$

whenever $0 < |x - c| < \delta$.

To define the **infinite limit from the left** (**無窮左極限**), replace $0 < |x-c| < \delta$ by $c-\delta < x < c$. To define the **infinite limit from the right** (**無窮右極限**), replace $0 < |x - c| < \delta$ by $c < x < c + \delta$. . . . . . . . . . . 126

24. **Vertical asymptote** If $f(x)$ approaches infinity (or negative infinity) as $x$ approaches $c$ from the right or the left, then the line $x = c$ is a **vertical asymptote** (**垂直漸近線**) of the graph of $f$. . . . . . . . . . . . . . 131

25. **Vertical asymptotes** Let $f$ and $g$ be continuous on an open interval containing $c$. If $f(c) \neq 0$, $g(c) = 0$, and there exists an open interval containing $c$ such that $g(x) \neq 0$ for all $x \neq c$ in the interval, then the graph of the function given by

$$h(x) = \frac{f(x)}{g(x)}$$

has a **vertical asymptote** (垂直漸近線) at $x = c$. . . . . . . . . . . . . . . . . 132

26. **Properties of infinite limits** Let $c$ and $L$ be real numbers and let $f$ and $g$ be functions such that

$$\lim_{x \to c} f(x) = \infty \quad \text{and} \quad \lim_{x \to c} g(x) = L.$$

(a) Sum or difference: $\lim_{x \to c} [f(x) \pm g(x)] = \infty$

(b) Product:

$$\lim_{x \to c} [f(x)g(x)] = \infty, \quad L > 0$$
$$\lim_{x \to c} [f(x)g(x)] = -\infty, \quad L < 0$$

(c) Quotient: $\lim_{x \to c} \dfrac{g(x)}{f(x)} = 0$

Similar properties hold for one-sided limits and for functions for which

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