Chapter 4 Integration

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Antiderivatives

- To find a function F whose derivative is $f(x) = 3x^2$, you might use your knowledge of derivatives to conclude that

$$F(x) = x^3 \quad \text{because} \quad \frac{\mathrm{d}}{\mathrm{d}x}[x^3] = 3x^2.$$

The function F is an <u>antiderivative</u> of f.

Definition 4.1 (Antiderivative)

A function F is <u>an</u> antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

Theorem 4.1 (Representation of antiderivatives)

If F is an antiderivative of f on an interval I, then G is an antiderivative of f on the interval I if and only if G is of the form $G(x) = F(x) + C$, for all x in I where C is a constant.

- You can represent the entire family of antiderivatives of a function by adding a constant to a known antiderivative.
- For example, knowing that $D_x[x^2] = 2x$, you can represent the family of all antiderivatives of $f(x) = 2x$ by

$$G(x) = x^2 + C \qquad \text{Family of all antiderivatives of } f(x) = 2x$$

where C is a constant. The constant C is called the constant of integration.

- The family of functions represented by G is the general antiderivative of f, and $G(x) = x^2 + C$ is the general solution of the differential equation

$$G'(x) = 2x. \qquad \text{Differential equation}$$

Example 1 (Solving a differential equation)

Find the general solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$.

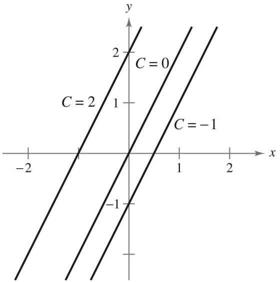


Figure 1: $y' = 2$: $y = 2x + C$, $C = -1, 0, 2$.

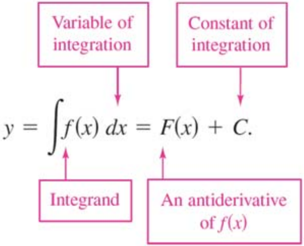
- When solving a differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$

it is convenient to write it in the equivalent differential form

$$\mathrm{d}y = f(x)\,\mathrm{d}x.$$

- The operation of finding all solutions of this equation is called antidifferentiation (or indefinite integration).
- The general solution is denoted by antiderivative.



- The expression $\int f(x)\,\mathrm{d}x$ is read as "the antiderivative of f with respect to x". So, the differential $\mathrm{d}x$ serves to identify x as the variable of integration.

Basic integration rules

- The inverse nature of integration and differentiation can be verified by substituting $F'(x)$ for $f(x)$ in the indefinite integration definition to obtain

$$\int F'(x)\,\mathrm{d}x = F(x) + C.$$

Integration is the "inverse" of differentiation

- Moreover, if $\int f(x)\,\mathrm{d}x = F(x) + C$, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int f(x)\,\mathrm{d}x\right] = f(x).$$

Differentiation is the "inverse" of integration

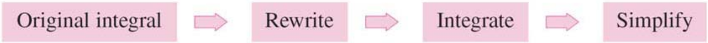
- These two equations allow you to obtain integration formulas directly from differentiation formulas!
- For instance, check out `https://www.mathdoubts.com/integral-sum-rule-proof/` for the sum rule

Differentiation Formula	Integration Formula
$\frac{\mathrm{d}}{\mathrm{d}x}[C] = 0$	$\int 0\,\mathrm{d}x = C$
$\frac{\mathrm{d}}{\mathrm{d}x}[kx] = k$	$\int k\,\mathrm{d}x = kx + C$
$\frac{\mathrm{d}}{\mathrm{d}x}[kf(x)] = kf'(x)$	$\int kf(x)\,\mathrm{d}x = k\int f(x)\,\mathrm{d}x$
$\frac{\mathrm{d}}{\mathrm{d}x}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	$\int [f(x) \pm g(x)]\,\mathrm{d}x = \int f(x)\,\mathrm{d}x \pm \int g(x)\,\mathrm{d}x$
$\frac{\mathrm{d}}{\mathrm{d}x}[x^n] = nx^{n-1}$	$\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$
$\frac{\mathrm{d}}{\mathrm{d}x}[\sin x] = \cos x$	$\int \cos x\,\mathrm{d}x = \sin x + C$
$\frac{\mathrm{d}}{\mathrm{d}x}[\cos x] = -\sin x$	$\int \sin x\,\mathrm{d}x = -\cos x + C$
$\frac{\mathrm{d}}{\mathrm{d}x}[\tan x] = \sec^2 x$	$\int \sec^2 x\,\mathrm{d}x = \tan x + C$
$\frac{\mathrm{d}}{\mathrm{d}x}[\sec x] = \sec x \tan x$	$\int \sec x \tan x\,\mathrm{d}x = \sec x + C$
$\frac{\mathrm{d}}{\mathrm{d}x}[\cot x] = -\csc^2 x$	$\int \csc^2 x\,\mathrm{d}x = -\cot x + C$
$\frac{\mathrm{d}}{\mathrm{d}x}[\csc x] = -\csc x \cot x$	$\int \csc x \cot x\,\mathrm{d}x = -\csc x + C$

Example 2 (Applying the basic integration rules)

Describe the antiderivatives of $3x$.

- Note that the general pattern of integration is similar to that of differentiation.



Example 3 (Rewriting before integrating)

Original Integral	Rewrite	Integrate	Simplify
a. $\int \frac{1}{x^3}\,\mathrm{d}x$			
b. $\int \sqrt{x}\,\mathrm{d}x$			
c. $\int 2\sin x\,\mathrm{d}x$			

Example 4 (Integrating polynomial functions)

a. $\displaystyle\int x\,dx$

b. $\displaystyle\int (x+2)\,dx$

c. $\displaystyle\int (3x^4 - 5x^2 + x)\,dx$

Example 5 (Rewriting before integrating)

$$\int \frac{x+1}{\sqrt{x}}\,\mathrm{d}x$$

Example 6 (Rewriting before integrating)

$$\int \frac{\sin x}{\cos^2 x}\,\mathrm{d}x$$

Example 7 (Rewriting before integrating)

Original Integral	Rewrite	Integrate	Simplify
a. $\int \frac{2}{\sqrt{x}}\,\mathrm{d}x$			
b. $\int (t^2+1)^2\,\mathrm{d}t$			
c. $\int \frac{x^3+3}{x^2}\,\mathrm{d}x$			
d. $\int \sqrt[3]{x}(x-4)\,\mathrm{d}x$			

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Sigma notation

- This section begins by introducing a concise notation for sums. This notation is called sigma notation because it uses the uppercase Greek letter sigma, written as $\sum$.

Definition 4.2 (Sigma notation)

The sum of n terms a_1, a_2, a_3, ..., a_n is written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

where i is the index of summation, a_i is the ith term of the sum, and the upper and lower bounds of summation are n and 1.

Example 1 (Examples of sigma notation)

a. $\sum_{i=1}^{6} i = 1 + 2 + 3 + 4 + 5 + 6$

b. $\sum_{i=0}^{5}(i+1) = 1 + 2 + 3 + 4 + 5 + 6$

c. $\sum_{j=3}^{7} j^2 = 3^2 + 4^2 + 5^2 + 6^2 + 7^2$

d. $\sum_{j=1}^{5} \frac{1}{\sqrt{j}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}}$

e. $\sum_{k=1}^{n} \frac{1}{n}(k^2+1) = \frac{1}{n}(1^2+1) + \frac{1}{n}(2^2+1) + \cdots + \frac{1}{n}(n^2+1)$

f. $\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$ ■

The following summation properties can be derived using the associative and commutative properties of addition and the distributive property of addition over multiplication.

1. $\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$

2. $\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

Theorem 4.2 (Summation formulas)

1. $\sum_{i=1}^{n} c = cn, c \text{ is constant}$
2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\sum_{i=1}^{n} i^3 = (\sum_{i=1}^{n} i)^2 = \frac{n^2(n+1)^2}{4}$

Example 2 (Evaluating a sum)

Evaluate $\sum_{i=1}^{n} \frac{i+1}{n^2}$ for $n = 10$, 100, 1000 and 10000.

n	$\sum_{i=1}^{n} \frac{i+1}{n^2} = \frac{n+3}{2n}$
10	0.65000
100	0.51500
1000	0.50150
10000	0.50015

- In Euclidean geometry, the simplest type of plane region is a rectangle. Although people often say that the formula for the area of a rectangle is $A = bh$, it is actually more proper to say that this is the definition of the area of a rectangle.
- From this definition, you can develop formulas for the areas of many other plane regions. For example, to determine the area of a triangle, you can form a rectangle whose area is twice that of the triangle!

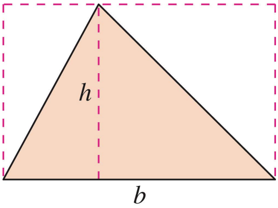
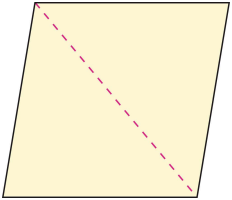
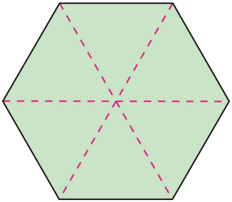


Figure 2: Area of triangle: $A = \frac{1}{2}bh$

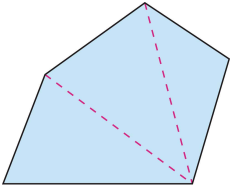
- Once you know how to find the area of a triangle, you can determine the area of any polygon by subdividing the polygon into triangular regions, as shown below.



(a) Parallelogram

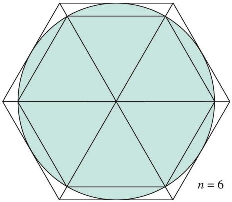


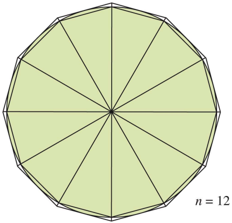
(b) Hexagon



(c) Polygon

- Finding the areas of regions other than polygons is more difficult. The ancient Greeks were able to determine formulas for the areas of some general regions (principally those bounded by conics) by the exhaustion method.
- The clearest description of this method was given by Archimedes (287-212 B.C.) Essentially, the method is a limiting process in which the area is squeezed between two polygons—one inscribed in the region and one circumscribed about the region.
- For instance, in the Figure below, the area of a circular region is approximated by an n-sided inscribed polygon and an n-sided circumscribed polygon.

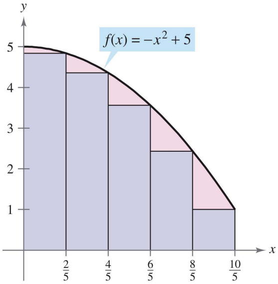


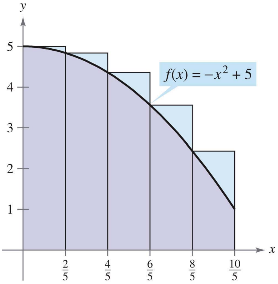


- For each n, the area of the inscribed polygon is less than the area of the circle, and the area of the circumscribed polygon is greater.
- Moreover, as n increases, the areas of both polygons become better and better approximations of the area of the circle!

Example 3 (Approximating the area of a plane region)

Use the five rectangles below to find two approximations of the area of the region lying between the graph of $f(x) = -x^2 + 5$ and the x-axis between $x = 0$ and $x = 2$.





- Consider a plane region bounded above by the graph of a nonnegative, continuous function $y = f(x)$, as shown in Figure 6.
- The region is bounded below by the x-axis, and the left and right boundaries of the region are the vertical lines $x = a$ and $x = b$.

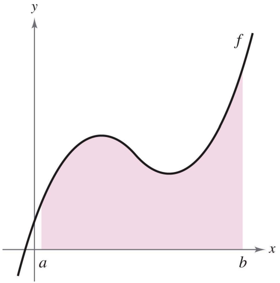
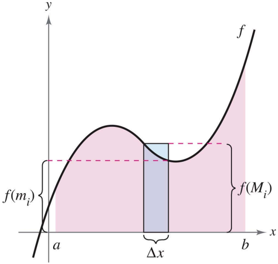


Figure 6: The region under a curve.

- To approximate the area of the region, begin by subdividing the interval $[a, b]$ into n subintervals, each of width $\Delta x = (b - a)/n$, as shown below.



- The endpoints of the intervals are as follows.

$$\overbrace{a + 0(\Delta x)}^{a = x_0} < \overbrace{a + 1(\Delta x)}^{x_1} < \overbrace{a + 2(\Delta x)}^{x_2} < \cdots < \overbrace{a + n(\Delta x)}^{x_n = b}$$

- Because f is continuous, the Extreme Value Theorem guarantees the existence of a minimum and a maximum value of $f(x)$ in each subinterval.

$$f(m_i) = \text{Minimum value of } f(x) \text{ in } i\text{th subinterval}$$
$$f(M_i) = \text{Maximum value of } f(x) \text{ in } i\text{th subinterval}$$

- The height of the ith inscribed rectangle is $f(m_i)$ and the height of the ith circumscribed rectangle is $f(M_i)$.
- For each i, the area of the inscribed rectangle is less than or equal to the area of the circumscribed rectangle.

$$\begin{pmatrix}\text{Area of inscribed}\\ \text{rectangle}\end{pmatrix} = f(m_i)\Delta x \le f(M_i)\Delta x = \begin{pmatrix}\text{Area of circumscribed}\\ \text{rectangle}\end{pmatrix}$$

- The sum of the areas of the inscribed rectangles is called a lower sum, and the sum of the areas of the circumscribed rectangles is called an upper sum.

$$\text{Lower sum} = s(n) = \sum_{i=1}^{n} f(m_i)\Delta x \quad \text{Area of inscribed rectangles}$$

$$\text{Upper sum} = S(n) = \sum_{i=1}^{n} f(M_i)\Delta x \quad \text{Area of circumscribed rectangles}$$

- You can see that the lower sum $s(n)$ is less than or equal to the upper sum $S(n)$. Moreover, the actual area of the region lies between these two sums.

$$s(n) \leq (\text{Area of region}) \leq S(n)$$

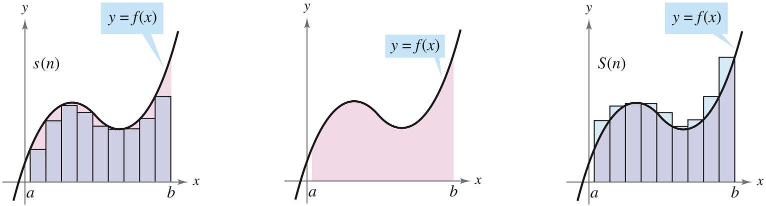
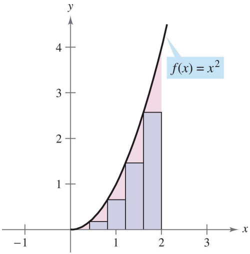


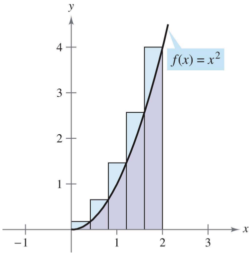
Figure 7: Upper and lower sums for a region.

Example 4 (Finding upper and lower sums for a region)

Find the upper and lower sums for the region bounded by the graph of $f(x) = x^2$ and the x-axis between $x = 0$ and $x = 2$.



(a) Inscribed rectangles.



(b) Circumscribed rectangles

Theorem 4.3 (Limits of the lower and upper sums)

Let f be continuous and nonnegative on the interval $[a, b]$. The limits as $n \to \infty$ of both the lower and upper sums exist and are equal to each other. That is,

$$\lim_{n\to\infty} s(n) = \lim_{n\to\infty} \sum_{i=1}^{n} f(m_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(M_i)\Delta x = \lim_{n\to\infty} S(n)$$

where $\Delta x = (b - a)/n$ and $f(m_i)$ and $f(M_i)$ are the minimum and maximum values of f on the subinterval.

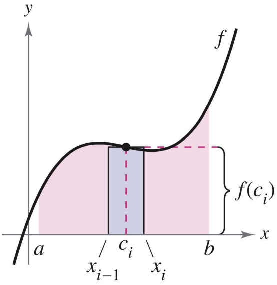
- You are free to choose an arbitrary x-value in the ith subinterval, as in the following definition of the area of a region in the plane.

Definition 4.3 (The area of a region in the plane)

Let f be continuous and nonnegative on the interval $[a, b]$. The area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is

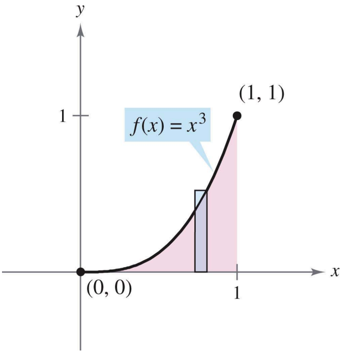
$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x, \quad x_{i-1} \le c_i \le x_i$$

where $\Delta x = (b - a)/n$.



Example 5 (Finding area by the limit definition)

Find the area of the region bounded by the graph $f(x) = x^3$, the x-axis, and the vertical lines $x = 0$ and $x = 1$ as shown below.



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- In the previous section, we partition the axis using equal width. In fact, we can use a partition having subintervals of unequal widths, as shown in Figure 9. This strategy also gave the proper area because as n increases, the width of the largest subinterval approaches zero.
- This is a key feature of the development of definite integrals.

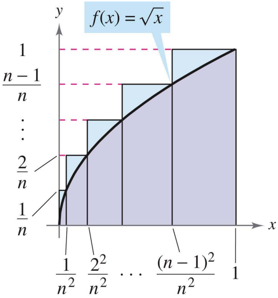


Figure 9: A partition with subintervals of unequal widths.

Definition 4.4 (Riemann sum)

Let f be defined on the closed interval $[a, b]$, and let Δ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

where Δx_i is the width of the ith subinterval. If c_i is any point in the ith subinterval $[x_{i-1}, x_i]$, then the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i, \quad x_{i-1} \leq c_i \leq x_i$$

is called a Riemann sum of f for the partition Δ.

- The width of the largest subinterval of a partition Δ is the norm of the partition and is denoted by $\|\Delta\|$.

- If every subinterval is of equal width, the partition is regular and the norm is denoted by

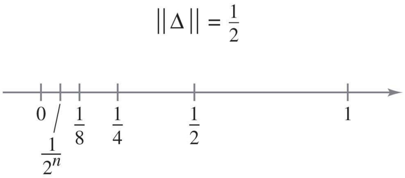
$$\|\Delta\| = \Delta x = \frac{b-a}{n}. \qquad \text{regular partition}$$

- For a general partition, the norm is related to the number of subintervals of $[a, b]$ in the following way.

$$\frac{b-a}{\|\Delta\|} \leq n \qquad \text{general partition}$$

- So, the number of subintervals in a partition approaches infinity as the norm of the partition approaches 0. That is, $\|\Delta\| \to 0$ implies that $n \to \infty$. The converse of this statement is not true. For example, let Δ_n be the partition of the interval $[0, 1]$ given by

$$0 < \frac{1}{2^n} < \frac{1}{2^{n-1}} < \cdots < \frac{1}{8} < \frac{1}{4} < \frac{1}{2} < 1.$$



- As shown above, for any positive value of n, the norm of the partition Δ_n is $\frac{1}{2}$. So, letting n approach infinity does not force $\|\Delta\|$ to approach 0. In a regular partition, however, the statements $\|\Delta\| \to 0$ and $n \to \infty$ are equivalent.

- Now we are ready to define the definite integral, consider the following limit.

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = L$$

Definition 4.5 (Definite integral)

If f is defined on the closed interval $[a, b]$ and the limit of Riemann sums over partitions Δ

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

exists (as described above), then f is said to be integrable on $[a, b]$ and the limit is denoted by

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)\,\mathrm{d}x.$$

The limit is called the definite integral of f from a to b. The number a is the lower limit of integration, and the number b is the upper limit of integration.

Four steps of finding the definite integral $\int_a^b f(x)\,\mathrm{d}x$ using Riemann sum

1. Partition: $a = x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots < x_n = b$
2. Sampling: $c_i \in [x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$
3. Summation: $\sum_{i=1}^{n} f(c_i)\Delta x_i$
4. Limit: $\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)\,\mathrm{d}x$

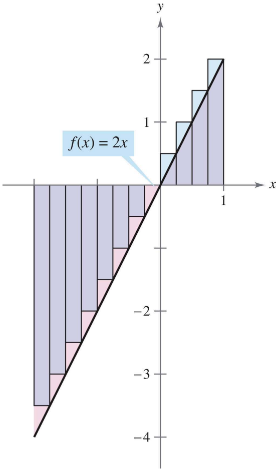
Theorem 4.4 (Continuity implies integrability)

If a function f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$. That is, $\int_a^b f(x)\,\mathrm{d}x$ exists.

Example 2 (Evaluating a definite integral as a limit)

Evaluate the definite integral $\int_{-2}^{1} 2x\,\mathrm{d}x$.

- Because the definite integral above is negative, it does not represent the area of the region.



- Definite integrals can be positive, negative, or zero. For a definite integral to be interpreted as an area, the function f must be continuous and nonnegative on $[a, b]$.

Theorem 4.5 (The definite integral as the area of a region)

If f is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by (See Figure 10)

$$\text{Area} = \int_a^b f(x)\,\mathrm{d}x.$$

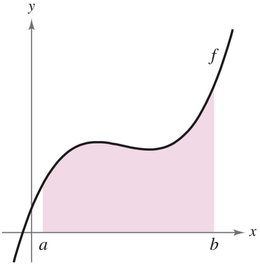
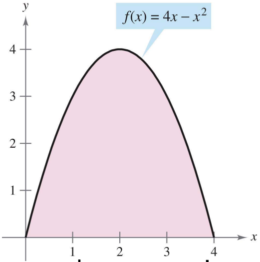


Figure 10: You can use a definite integral to find the area of the region bounded by the graph of f, the x-axis, $x = a$, and $x = b$.

- As an example of Theorem 4.5, consider the region bounded by the graph of $f(x) = 4x - x^2$ and the x-axis, as shown below:



- Because f is continuous and nonnegative on the closed interval $[0, 4]$, the area of the region is

$$\text{Area} = \int_0^4 (4x - x^2)\,\mathrm{d}x.$$

- You can evaluate a definite integral in two ways—you can use the limit definition or you can check to see whether the definite integral represents the area of a common geometric region such as a rectangle, triangle, or semicircle.

Example 3 (Areas of common geometric figures)

Sketch the region corresponding to each definite integral. Then evaluate each integral using a geometric formula.

a. $\int_1^3 4\,\mathrm{d}x$ **b.** $\int_0^3 (x+2)\,\mathrm{d}x$ **c.** $\int_{-2}^2 \sqrt{4-x^2}\,\mathrm{d}x$

a.

b.

c.

- The variable of integration in a definite integral is sometimes called a dummy variable because it can be replaced by any other variable without changing the value of the integral. For instance, the definite integrals

$$\int_0^3 (x+2)\,\mathrm{d}x \quad \text{and} \quad \int_0^3 (t+2)\,\mathrm{d}t$$

have the same value.

- The definition of the definite integral of f on the interval $[a, b]$ specifies that $a < b$. However, it is convenient to extend the definition to cover cases in which $a = b$ or $a > b$.
- Geometrically, the following two definitions seem reasonable. For instance, it makes sense to define the area of a region of zero width and finite height to be 0.

Definition 4.6 (Two special definite integrals)

1. If f is defined at $x = a$, then we define $\int_a^a f(x)\,\mathrm{d}x = 0$.
2. If f is integrable on $[a, b]$, then we define $\int_b^a f(x)\,\mathrm{d}x = -\int_a^b f(x)\,\mathrm{d}x$.

Example 4 (Evaluating definite integrals)

Evaluate each definite integral. **a.** $\int_\pi^\pi \sin x\,\mathrm{d}x$ **b.** $\int_3^0 (x+2)\,\mathrm{d}x$

In Figure 11, the larger region can be divided at $x = c$ into two subregions whose intersection is a line segment. Because the line segment has zero area, it follows that the area of the larger region is equal to the sum of the areas of the two smaller regions.

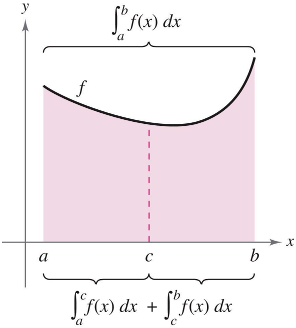


Figure 11: Additive interval property.

Theorem 4.6 (Additive interval property)

If f is integrable on the three closed intervals determined by a, b, and c, then

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x. \qquad \textit{See Figure11}$$

Example 5 (Using the additive interval property)

$$\int_{-1}^{1} |x| \,\mathrm{d}x$$

Theorem 4.7 (Properties of definite integrals)

If f and g are integrable on $[a, b]$ and k is a constant, then the functions kf and $f \pm g$ are integrable on $[a, b]$, and

1. $\int_a^b kf(x)\,\mathrm{d}x = k\int_a^b f(x)\,\mathrm{d}x.$
2. $\int_a^b [f(x) \pm g(x)]\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x.$

- Note that Property 2 of Theorem 4.7 can be extended to cover any finite number of functions. For example,

$$\int_a^b [f(x) + g(x) + h(x)]\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x + \int_a^b h(x)\,\mathrm{d}x.$$

Example 6 (Evaluation of a definite integral)

Evaluate $\int_1^3 (-x^2 + 4x - 3)\,\mathrm{d}x$ using each of the following values.

$$\int_1^3 x^2\,\mathrm{d}x = \frac{26}{3}, \quad \int_1^3 x\,\mathrm{d}x = 4, \quad \int_1^3 \mathrm{d}x = 2$$

- If f and g are continuous on the closed interval $[a, b]$ and

$$0 \leq f(x) \leq g(x)$$

for $a \leq x \leq b$, the following properties are true.
  - First, the area of the region bounded by the graph of f and the x-axis (between a and b) must be nonnegative.
  - Second, this area must be less than or equal to the area of the region bounded by the graph of g and the x-axis (between a and b), as shown in Figure 12.
- These two properties are generalized in Theorem 4.8.

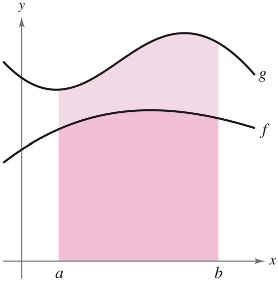


Figure 12: If $f(x) \leq g(x)$, then $\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x$.

Theorem 4.8 (Preservation of inequality)

1. *If f is integrable and nonnegative on the closed interval $[a, b]$, then*

$$0 \leq \int_a^b f(x)\,\mathrm{d}x.$$

2. *If f and g are integrable on the closed interval $[a, b]$ and $f(x) \leq g(x)$ for every x in $[a, b]$, then*

$$\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x.$$

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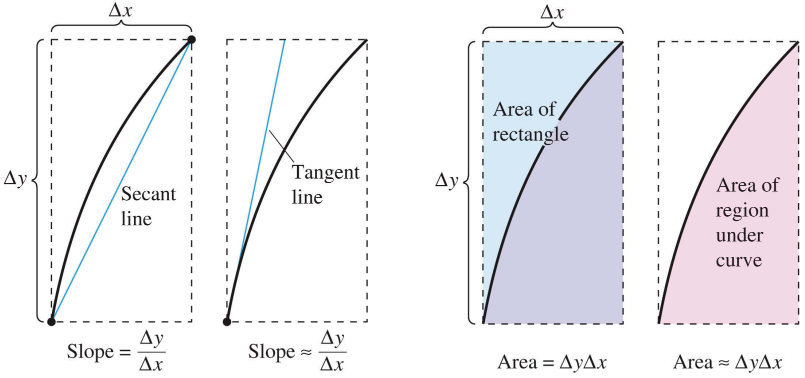
- The two major branches of calculus: differential calculus and integral calculus. The close connection was discovered independently by Isaac Newton and Gottfried Leibniz and is stated in a theorem that is appropriately called the Fundamental Theorem of Calculus.
- Informally, the theorem states that differentiation and (definite) integration are inverse operations, in the same sense that division and multiplication are inverse operations.



(a) Differentiation (b) Definite integration

- The slope of the tangent line was defined using the quotient $\Delta y/\Delta x$. Similarly, the area of a region under a curve was defined using the product $\Delta y \Delta x$.
- So, at least in the primitive approximation stage, the operations of differentiation and definite integration appear to have an inverse relationship in the same sense that division and multiplication are inverse operations.
- The Fundamental Theorem of Calculus states that <u>the limit processes preserve this inverse relationship</u>.

Theorem 4.9 (The Fundamental Theorem of Calculus)

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

Guidelines for using the Fundamental Theorem of Calculus

1. Provided you can find an antiderivative of f, you now have a way to evaluate a definite integral without having to use the limit of a sum.
2. When applying the Fundamental Theorem of Calculus, the following notation is convenient.

$$\int_a^b f(x)\,\mathrm{d}x = F(x)\Big]_a^b = F(b) - F(a)$$

For instance, to evaluate $\int_1^3 x^3\,\mathrm{d}x$, you can write

$$\int_1^3 x^3\,\mathrm{d}x = \frac{x^4}{4}\Big]_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = \frac{81}{4} - \frac{1}{4} = 20.$$

3. It is <u>not necessary to include a constant of integration</u> C because

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x &= [F(x) + C]_a^b = [F(b) + C] - [F(a) + C]\\ &= F(b) - F(a).\end{aligned}$$

Example 1 (Evaluating a definite integral)

Evaluate each definite integral.

a. $\int_1^2 (x^2 - 3)\,\mathrm{d}x$ **b.** $\int_1^4 3\sqrt{x}\,\mathrm{d}x$ **c.** $\int_0^{\pi/4} \sec^2 x\,\mathrm{d}x$

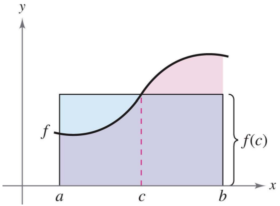
Example 2 (Definite integral involving absolute value)

Evaluate $\int_0^2 |2x - 1| \, \mathrm{d}x$.

Example 3 (Using the Fundamental Theorem to find area)

Find the area of the region bounded by the graph of $y = 2x^2 - 3x + 2$, the x-axis, and the vertical lines $x = 0$ and $x = 2$

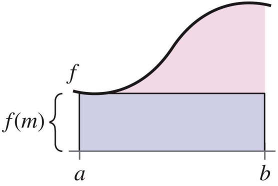
- The area of a region under a curve is greater than the area of an inscribed rectangle and less than the area of a circumscribed rectangle.
- The Mean Value Theorem for integrals states that somewhere "between" the inscribed and circumscribed rectangles there is a rectangle whose area is precisely equal to the area of the region under the curve, as shown below



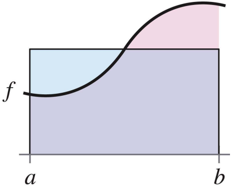
Theorem 4.10 (Mean Value Theorem for Integrals)

If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

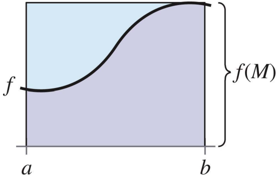
$$\int_a^b f(x)\,\mathrm{d}x = f(c)(b-a).$$



(a) Inscribed rectangle (less than actual area).



(b) Mean value rectangle (equal to actual area).



(c) Circumscribed rectangle (greater than actual area).

- The value of $f(c)$ given in the Mean Value Theorem for integrals is called the average value of f on the interval $[a, b]$.

Definition 4.7 (The average value of a function on an interval)

If f is integrable on the closed interval $[a, b]$, then the average value of f on the interval is

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x. \qquad \text{See Figure 15}$$

- In Figure 15 the area of the region under the graph of f is equal to the area of the rectangle whose height is the average value.
- To see why the average value of f is defined in this way, suppose that you partition $[a, b]$ into n subintervals of equal width $\Delta x = (b-a)/n$.

- If c_i is any point in the ith subinterval, the arithmetic average (or mean) of the function values at the c_i's is given by

$$a_n = \frac{1}{n}\left[f(c_1) + f(c_2) + \cdots + f(c_n)\right]. \quad \text{Average of } f(c_1), \ldots, f(c_n)$$

- By multiplying and dividing by $(b-a)$ you can write the average as

$$\begin{aligned} a_n &= \frac{1}{n}\sum_{i=1}^{n} f(c_i)\left(\frac{b-a}{b-a}\right) \\ &= \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\left(\frac{b-a}{n}\right) = \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\Delta x. \end{aligned}$$

- Finally, taking the limit as $n \to \infty$ produces the average value of f on the interval $[a, b]$ as given in the definition above.

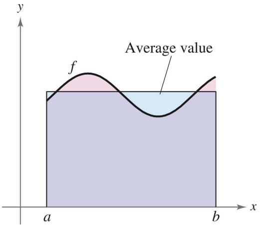


Figure 15: Average value $= \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x$.

Example 4 (Finding the average value of a function)

Find the average value of $f(x) = 3x^2 - 2x$ on the interval $[1, 4]$.

- The definite integral of f on the interval $[a, b]$ is defined using the constant b as the upper limit of integration and x as the variable of integration.
- A slightly different situation may arise in which the variable x is used in the upper limit of integration.
- To avoid the confusion of using x in two different ways, t is temporarily used as the variable of integration.

The Definite Integral as a Number	The Definite Integral as a Function of x
$\int_a^b f(x)\,\mathrm{d}x$	$F(x) = \int_a^x f(t)\,\mathrm{d}t$
a: Constant, b: Constant, f: function of x	a: Constant, F: function of x, f: function of t

Example 6 (The definite integral as a function)

Evaluate the function

$$F(x) = \int_0^x \cos t \, \mathrm{d}t$$

at $x = 0$, $\pi/6$, $\pi/4$, $\pi/3$ and $\pi/2$.

- Now, using $F(x) = \sin x$, you can obtain the results shown in Figure 16.

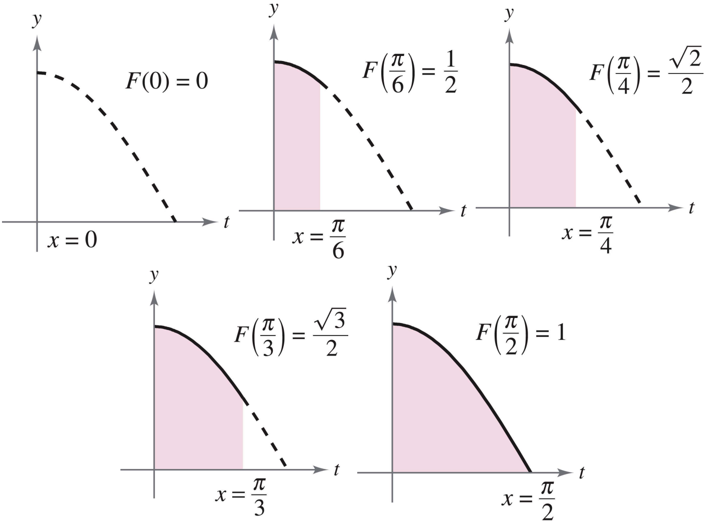


Figure 16: $F(x) = \int_0^x \cos t \, \mathrm{d}t$ is the area under the curve $f(t) = \cos t$ from 0 to x.

- The function $F(x)$ as accumulating the area under the curve $f(t) = \cos t$ from $t = 0$ to $t = x$. For $x = 0$, the area is 0 and $F(0) = 0$. For $x = \pi/2$, $F(\pi/2) = 1$ gives the accumulated area under the cosine curve on the entire interval $[0, \pi/2]$.
- This interpretation of an integral as an accumulation function is used often in applications of integration.
- The derivative of F is the original integrand. That is,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[F(x)\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left[\sin x\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left[\int_0^x \cos t \,\mathrm{d}t\right] = \cos x.$$

- This result is generalized in the following theorem, called the Second Fundamental Theorem of Calculus.

Theorem 4.11 (The Second Fundamental Theorem of Calculus)

If f is continuous on an open interval I containing a, then, for every x in the interval,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^x f(t)\,\mathrm{d}t\right] = f(x).$$

- Using the area model for definite integrals, you can view the approximation

$$f(x)\Delta x \approx \int_x^{x+\Delta x} f(t)\,\mathrm{d}t$$

as saying that the area of the rectangle of height $f(x)$ and width Δx is approximately equal to the area of the region lying between the graph of f and the x-axis on the interval $[x, x+\Delta x]$, as shown in Figure 17.

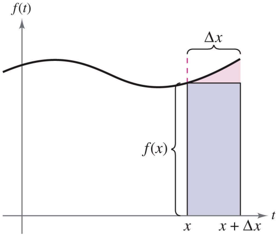


Figure 17: $f(x)\Delta x \approx \int_x^{x+\Delta x} f(t)\,\mathrm{d}t$.

Example 7 (Using the Second Fundamental Theorem of Calculus)

Evaluate $\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_0^x \sqrt{t^2+1}\,\mathrm{d}t\right]$.

Example 8 (Using the Second Fundamental Theorem of Calculus)

Find the derivative $F(x) = \int_{\pi/2}^{x^3} \cos t \, \mathrm{d}t$.

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- In this section, you will study techniques for integrating composite functions. The discussion is split into two parts—pattern recognition and change of variables. Both techniques involve a u-substitution.
  - With pattern recognition you perform the substitution mentally, and with change of variables you write the substitution steps.
  - The role of substitution in integration is comparable to the role of the Chain Rule in differentiation.
- Recall that for differentiable functions given by $y = F(u)$ and $u = g(x)$, the Chain Rule states that

$$\frac{\mathrm{d}}{\mathrm{d}x}\,[F(g(x))] = F'(g(x))g'(x).$$

- From the definition of an antiderivative, it follows that

$$\int F'(g(x))g'(x)\,\mathrm{d}x = F(g(x)) + C.$$

Theorem 4.12 (Antidifferentiation of a composite function)

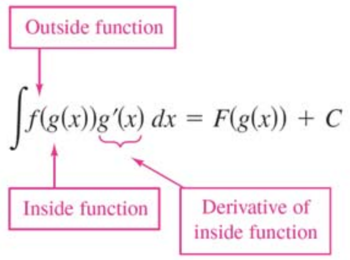
Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then

$$\int f(g(x))g'(x)\,\mathrm{d}x = F(g(x)) + C.$$

Letting $u = g(x)$ gives $\mathrm{d}u = g'(x)\,\mathrm{d}x$ and

$$\int f(u)\,\mathrm{d}u = F(u) + C.$$

- Example 1 and 2 show how to apply Theorem 4.12 directly, by recognizing the presence of $f(g(x))$ and $g'(x)$.
- Note that the composite function in the integrand has an outside function f and an inside function g. Moreover, the derivative $g'(x)$ is present as a factor of the integrated.



Example 1 (Recognizing the $f(g(x))g'(x)$ pattern)

Find $\int (x^2+1)^2(2x)\,\mathrm{d}x$.

Example 2 (Recognizing the $f(g(x))g'(x)$ pattern)

Find $\int 5\cos 5x \, \mathrm{d}x$.

- The integrands in Example 1 and Example 2 fit the $f(g(x))g'(x)$ pattern exactly—you only had to recognize the pattern.
- You can extend this technique considerably with the Constant Multiple Rule
$$\int kf(x)\,\mathrm{d}x = k\int f(x)\,\mathrm{d}x.$$
- Many integrands contain the essential part (the variable part) of $g'(x)$ but are missing a constant multiple.
- In such cases, you can multiply and divide by the necessary constant multiple, as shown in Example 3.

Example 3 (Multiplying and dividing by a constant)

Find $\int x(x^2+1)^2 \, \mathrm{d}x$.

Change of variables for indefinite integrals

- With a formal change of variables, you completely rewrite the integral in terms of u and $\mathrm{d}u$ (or any other convenient variable).
- Although this procedure can involve more written steps than the pattern recognition illustrated in Examples 1 and 3, it is useful for complicated integrands.
- The change of variables technique uses the Leibniz notation for the differential. That is, if $u = g(x)$, then $\mathrm{d}u = g'(x)\,\mathrm{d}x$, and the integral in Theorem 4.12 takes the form

$$\int f(g(x))g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u = F(u) + C.$$

Example 4 (Change of variables)

Find $\int \sqrt{2x-1}\,\mathrm{d}x$.

Example 6 (Change of variables)

Find $\int \sin^2 3x \cos 3x \, \mathrm{d}x$.

Guidelines for making a change of variables

1. Choose a substitution $u = g(x)$. Usually, it is best to choose the inner part of a composite function, such as a quantity raised to a power.
2. Compute $\mathrm{d}u = g'(x)\,\mathrm{d}x$.
3. Rewrite the integral in terms of the variable u.
4. Find the resulting integral in terms of u.
5. Replace u by $g(x)$ to obtain an antiderivative in terms of x.
6. Check your answer by differentiating.

The General Power Rule for integration

- One of the most common u-substitutions involves quantities in the integrand that are raised to a power.
- Because of the importance of this type of substitution, it is given a special name— the General Power Rule for Integration.

Theorem 4.13 (The General Power Rule for Integration)

If g is a differentiable function of x, then

$$\int [g(x)]^n g'(x)\,\mathrm{d}x = \frac{[g(x)]^{n+1}}{n+1} + C, \quad n \neq -1.$$

Equivalently, if $u = g(x)$, then

$$\int u^n\,\mathrm{d}u = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1.$$

Example 7 (Substitution and the General Power Rule)

a. $\int 3(3x-1)^4 \, \mathrm{d}x$

b. $\int (2x+1)(x^2+x) \, \mathrm{d}x$

c. $\int 3x^2\sqrt{x^3-2} \, \mathrm{d}x$

d. $\int \frac{-4x}{(1-2x^2)^2} \, \mathrm{d}x$

e. $\int \cos^2 x \sin x \, \mathrm{d}x$

Change of variables for definite integrals

- When using u-substitution with a definite integral, it is often convenient to determine the limits of integration for the variable u rather than to convert the antiderivative back to the variable x and evaluate at the original limits.

Theorem 4.14 (Change of variables for definite integrals)

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u.$$

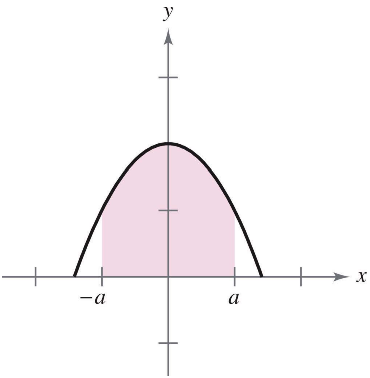
Example 8 (Change of variables)

Evaluate $\int_0^1 x(x^2+1)^3\,\mathrm{d}x$.

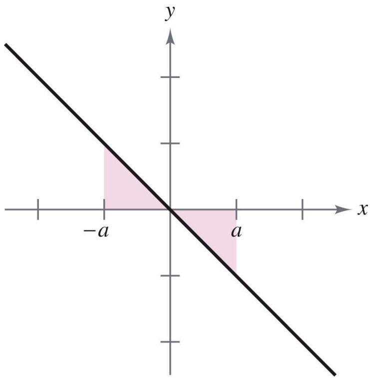
Example 9 (Change of variables)

Evaluate $\int_1^5 \frac{x}{\sqrt{2x-1}}\,\mathrm{d}x$.

- Even with a change of variables, integration can be difficult.
- Occasionally, you can simplify the evaluation of a definite integral over an interval that is symmetric about the y-axis or about the origin by recognizing the integrand to be an even or odd function.



(a) Even function



(b) Odd function

Theorem 4.15 (Integration of even and odd functions)

Let f be integrable on the closed interval $[-a, a]$.

1. *If f is an even function, then $\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_{0}^{a} f(x)\,\mathrm{d}x$.*
2. *If f is an odd function, then $\int_{-a}^{a} f(x)\,\mathrm{d}x = 0$.*

Example 10 (Integration of an odd function)

Evaluate $\int_{-\pi/2}^{\pi/2} (\sin^3 x \cos x + \sin x \cos x) \, \mathrm{d}x$.

- From Figure 19, we can see that the two regions on either side of the y-axis have the same area.
- However, because one lies below the x-axis and one lies above it, integration produces a cancellation effect. (More will be said about areas below the x-axis in Section 7.1.)

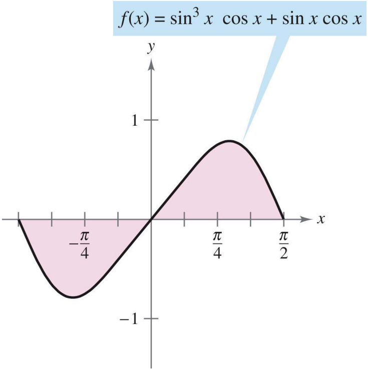


Figure 19: Integration of an odd function $f(x) = \sin^3 x \cos x + \sin x \cos x$, $-\pi/2 < x < \pi/2$.