If the limit does not exist or has an infinite limit, you should point it out. In addition, do not use the L'Hôpital's rule to solve the limit problem.

1. (16%) Find the following limit

   (a) $\lim_{x\to 1} \frac{x^3-3x+2}{x^3-x^2-x+1}$

   (b) $\lim_{x\to\infty} x(\sqrt{x^2+1}-x)$

   (c) $\lim_{\theta\to 0} \frac{1-\cos(\theta)}{\theta^2}$

   (d) $\lim_{x\to 3} \frac{\sqrt{3x+1}}{x-3}$

Ans:

(a) $\lim_{x\to 1} \frac{x^3-3x+2}{x^3-x^2-x+1} = \lim_{x\to 1} \frac{(x-1)^2(x+2)}{(x-1)^2(x+1)} = \lim_{x\to 1} \frac{(x+2)}{(x+1)} = \frac{(1+2)}{(1+1)} = \frac{3}{2}$

(b) $\lim_{x\to\infty} x(\sqrt{x^2+1}-x) = \lim_{x\to\infty} \frac{x(\sqrt{x^2+1}-x)(\sqrt{x^2+1}+x)}{(\sqrt{x^2+1}+x)} = \lim_{x\to\infty} \frac{x}{(\sqrt{x^2+1}+x)} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}+1} = \frac{1}{2}$

(c) $\lim_{\theta\to 0} \frac{1-\cos(\theta)}{\theta^2} = \lim_{\theta\to 0} \frac{(1-\cos(\theta))(1+\cos(\theta))}{\theta^2(1+\cos(\theta))} = \lim_{\theta\to 0} \frac{1-cos^2(\theta)}{\theta^2(1+\cos(\theta))} = \lim_{\theta\to 0} \frac{sin^2(\theta)}{\theta^2(1+\cos(\theta))} =$

$\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta}\frac{\sin(\theta)}{\theta}\frac{1}{(1+\cos(\theta))} = 1\times 1\times\frac{1}{2} = \frac{1}{2}$ (Note $\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = 1$ from theorem 1.9)

(d) $\lim_{x\to 3^+} \frac{\sqrt{3x+1}}{x-3} = \infty$ and $\lim_{x\to 3^-} \frac{\sqrt{3x+1}}{x-3} = -\infty$. Therefore, the limit does not exist!

2. (8%) Considering the following function.

$$f(x) = \begin{cases} |x|\sin(\frac{1}{x}), & \mathrm{x} \neq 0 \\ 0, & \mathrm{x} = 0 \end{cases}$$

(a) Is $f(x)$ continuous at $x = 0$? Explain your answer.
(b) Is $f(x)$ differentiable at $x = 0$? Explain your answer.
Ans:

(a) Since $-1 \le \sin(\frac{1}{x}) \le 1$ for all $x \neq 0$, $-|x| \le \sin(\frac{1}{x}) \le |x|$ for all $x \neq 0$

Furthermore $\lim_{x\to 0}|x| = 0$ ($\lim_{x\to 0^+}|x| = \lim_{x\to 0^+} x = 0$ and $\lim_{x\to 0^-}|x| = \lim_{x\to 0^-} -x = 0$) and

$\lim_{x\to 0} -|x| = 0$

According to the squeeze therorem $\lim_{x\to 0}|x|\sin(\frac{1}{x}) = 0$, we have

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}|x|\sin(\frac{1}{x}) = 0 = f(x)$$

So $f(x)$ is continuous at $x = 0$

(b) Considering the alternative form of derivative:

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+}\frac{|x|\sin\left(\frac{1}{x}\right)-0}{x}\lim_{x\to 0^+}\frac{x\sin\left(\frac{1}{x}\right)}{x} = \lim_{x\to 0^+}\sin\left(\frac{1}{x}\right)$$

Since $\lim_{x\to 0^+}\sin\left(\frac{1}{x}\right)$ doest not exist (Oscillation), $\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$ doest not exist!

We can conclude that the function is not differentiable at $x = 0$.

3. (8%) Proof that there is only one intersect point between $f(x) = 2x - 2$ and $g(x) = \cos x$. (Hint: use the mean value theorem)

Ans:

Let $F(x) = f(x) - g(x) = 2x - 2 - \cos x$, since $F(\pi) > 0$ and $F(0) < 0$. By the intermediate value theorem, it has at least one real root between π and 0 (which means there exists at least one intersecting point).

Using proof by contradiction, assume there exists a and b such that $F(a) = F(b) = 0, a \neq b$. According to the Mean value theorem (or Rolle's theorem), $\exists\ c \in (a,b)$ such that $F'(c) = \frac{F(a)-F(b)}{a-b} = 0$, contradict. ($F'(x) = 2 - (-\sin x) = 2 + \sin x > 0$).

Therefore, there is only one intersecting point!

4. (15%) Remember that you can solve the derivative using the definition or the differentiation rule for the following question.

(a) Find the following limit. $\lim_{x\to 0}\frac{cos(\pi+x)\ +\ 1}{x}$

(b) Find the derivative of $f(x) = \frac{x^3+3x-1}{x+1}$

(c) Let $f(x) = xcos(x) - tan(x) + 2\pi$, find $f''(x)$

Ans:

(a) Let $f(x) = cos(x)$, then the limit is the derivative of $f(x)$ at $x = \pi$ ($f'(\pi) = \lim_{\Delta x \to 0} \frac{cos(\pi+\Delta x)\ -\ cos(\pi)}{\Delta x} = \lim_{\Delta x \to 0} \frac{cos(\pi+\Delta x)+1}{\Delta x}$). Which is $f'(x) = -sin(x)$ therefore $f'(\pi) = -\sin(\pi) = 0$.

(b) $f'(x) = \frac{(x+1)(3x^2+3)-(x^3+3x-1)}{(x+1)^2} = \frac{2x^3+3x^2+4}{(x+1)^2}$

(c) $f' = cos(x) - xsin(x) - (sec\, x)^2 \rightarrow f'' = -sin(x) - xcos(x) - sin(x) - 2\, sec\, x\, sec\, x\, tan\, x = -2sin(x) - xcos(x) - 2sec^2 x\, tan\, x$

5. (8%) Given $x^2 + \frac{y^2}{4} = 1$, find all the tangent lines of the graph that pass the point (3,0) (Note (3,0) is not on the graph).

Ans:

$$\left(x^2 + \frac{y^2}{4}\right)' = (1)'$$

$$2x + 2\frac{y}{4}y' = 0$$

$$y' = \frac{-4x}{y}$$

Let (x, y) be the point on the graph that is on the tangent line pass through (3,2)

$$y' = \frac{-4x}{y} = \frac{y-0}{x-3} \rightarrow y^2 = -4x^2 + 12x \rightarrow 12x = y^2 + 4x^2$$

According to the original equation, $y^2 + 4x^2 = 4$, therefore, we have $12x = 4 \rightarrow$ $\mathrm{x} = \frac{1}{3}$.

Substitute back to the original equation, we have $(\frac{1}{3})^2 + \frac{y^2}{4} = 1 \rightarrow y = \pm\frac{\sqrt{32}}{3} = \pm\frac{4\sqrt{2}}{3}$

$$y' = \frac{-4x}{y} = \frac{-\frac{4}{3}}{\pm\frac{4\sqrt{2}}{3}} = \mp\frac{1}{\sqrt{2}}$$

Two tangent lines are at $(\frac{1}{3}, \frac{4\sqrt{2}}{3})$: $\mathrm{y} - 0 = -\frac{1}{\sqrt{2}}(x-3) \rightarrow y = -\frac{1}{\sqrt{2}}(x-3)$

At $(\frac{1}{3}, \frac{-4\sqrt{2}}{3})$: $\mathrm{y} - 0 = \frac{1}{\sqrt{2}}(x-3) \rightarrow y = \frac{1}{\sqrt{2}}(x-3)$

6. (15%) Let $f(x) = \frac{x^3}{(x+2)^2}$

(a) Find the critical numbers and the possible points of inflection of $f(x)$

(b) Find the open intervals on which f is increasing or decreasing

(c) Find the open intervals of concavity

(d) Find all the asymptotes (Vertical/horizontal/Slant)

(e) Sketch the graph of $f(x)$ (Label any intercepts, relative extrema, points of inflection,and asymptotes)

Ans: Note that the original function is undefined at $x=-2$, therefore we should include it in the following table.

$$f(x)=\frac{x^3}{(x+2)^2}, f'(x)=\frac{(x+2)^2 3x^2-x^3 2(x+2)}{(x+2)^4}=\frac{x^3+6x^2}{(x+2)^3}=\frac{x^2(x+6)}{(x+2)^3}$$

$$f''(x)=\frac{(x+2)^3(3x^2+12x)-(x^3+6x^2)3(x+2)^2}{(x+2)^6}=\frac{24x}{(x+2)^4}$$

	$(-\infty,-\mathbf{6})$	$(-\mathbf{6},-\mathbf{2})$	$(-\mathbf{2},\mathbf{0})$	$(\mathbf{0},\infty)$
測試值	-7	-3	-1	1
f'的正負號	+	-	+	+
f''的正負號	-	-	-	+
結論	遞增/向下凹	遞減/向下凹	遞增/向下凹	遞增/向上凹

(a) The critical numbers are $x=0,-6$

Possible points of inflection: $x=0$

(b) Increasing $(-\infty,-6),(-2,\infty)$. Decreasing $(-6,-2)$.

(c) Upward: $(0,\infty)$. Downward $(-\infty,-2)$ and $(-2,0)$

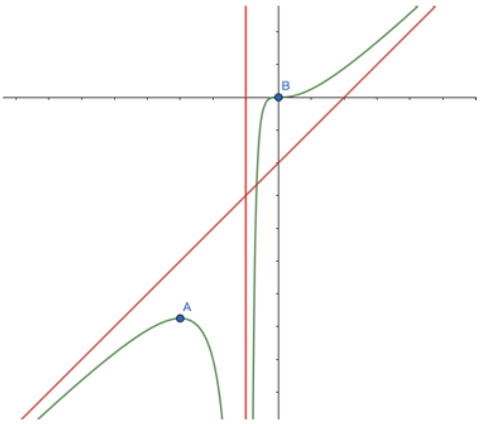
(d) Since $\lim_{x\to\pm\infty} f(x)=\pm\infty \rightarrow$ No horizontal asymptote

Since $\lim_{x\to-2^+} f(x)=-\infty$ and $\lim_{x\to-2^-} f(x)=-\infty$ vertical asymptote at $x=-2$

$$\frac{x^3}{(x+2)^2}=x-4+\frac{12x-16}{(x+2)^2} \text{ (Using long division)}$$

$$\lim_{x\to\pm\infty} f(x)-(x-4+\frac{12x-16}{(x+2)^2})=0 \rightarrow y=x-4 \text{ is a slant asymptote}$$

(e) Graph



There is a local maximum at $x = -6$ and an inflection point at (0,0)

7. (15%) Remember the meaning and the definition of definite integral when solving the following question

(a) $\int \frac{2+t+t^3}{\sqrt{t}} dt$

(b) $\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (t^3 + t^6 \tan(t))\, dt$

(c) $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} (\frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n+2}} + \ldots + \frac{1}{\sqrt{n+n}})$

Ans:

(a) $\int \frac{2+t+t^3}{\sqrt{t}} dt = \int 2t^{-\frac{1}{2}} + t^{\frac{1}{2}} + t^{\frac{5}{2}} dt = 4\sqrt{t} + \frac{2t^{3/2}}{3} + \frac{2t^{7/2}}{7} + C$

(b) Since $\tan(t)$ is an odd functions and t^6 is an even function. We know that $t^6 \tan(t)$ is an odd function. Moreover, since t^3 is also an odd function, therefore $t^3 + t^6 \tan(t)$ is an odd function. We have $\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (t^3 + t^6 \tan(t))\, dt = 0$

(c) $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} (\frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n+2}} + \ldots + \frac{1}{\sqrt{n+n}}) = \lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} \frac{1}{\sqrt{n}} (\frac{\sqrt{n}}{\sqrt{n+1}} + \frac{\sqrt{n}}{\sqrt{n+2}} + \ldots + \frac{\sqrt{n}}{\sqrt{n+n}}) =$
$\lim\limits_{n\to\infty} \frac{1}{n} (\frac{1}{\sqrt{1+\frac{1}{n}}} + \frac{1}{\sqrt{1+\frac{2}{n}}} + \frac{1}{\sqrt{1+\frac{3}{n}}} + \ldots + \frac{1}{\sqrt{1+\frac{n}{n}}}) = \int_0^1 \frac{1}{\sqrt{1+x}} dx = 2(x+1)^{\frac{1}{2}} |_0^1 = 2\sqrt{2} - 2$

8. (9%) Considering the function $f(x) = cos(x) + 2cos(2x) + \cdots + ncos(nx)$. Proof that there exist at least one root between $(0, \pi)$ (Hint: Let $F(x) = \int_0^x f(t)dt$ and use the fundamental theorem of calculus as well as Rolle's theorem.)

Ans: Let $F(x) = \int_0^x f(t)dt$, since $f(t)$ is continuous on all real value, by the

fundamental theorem of calculus, $F'(x) = f(x)$.
On the other hand, since $F(x)$ is differentiable on all real value ($F'(x) = f(x)$) and $F(0) = 0, F(\pi) = \int_0^\pi f(t)dt = \sin(t) + \sin(2t) + \cdots + \sin(nt)\Big|_0^\pi = 0$. By Rolle's theroerm, there is at least one number c in $(0, \pi)$ such that $F'(c) = 0$.
From above, we know that there is at least one number c in $(0, \pi)$ such that $f(c) = 0$ which concludes the proof.

9. (6%) Evaluate $\int_{\frac{1}{4}}^{1} \frac{\sqrt{1-\sqrt{x}}}{\sqrt{x}} dx$

Ans:

Let $u = 1 - \sqrt{x}, du = \frac{-1}{2\sqrt{x}} dx$

$$\int_{\frac{1}{4}}^{1} \frac{\sqrt{1-\sqrt{x}}}{\sqrt{x}} dx = -\int_{\frac{1}{2}}^{0} 2\sqrt{u}\, du = \frac{4}{3} u^{\frac{3}{2}} \Big|_0^{\frac{1}{2}} = \frac{\sqrt{2}}{3}$$