If the limit does not exist or has an infinite limit, you should point it out. In addition, do not use the L'Hôpital's rule to solve the limit problem.

1. (20%) Find the following limit. (If the limit does not exist or has an infinite limit, you should point it out)
   (a) $\lim_{x\to -2}\frac{x^2-2x-8}{x^2+3x+2}$
   (b) $\lim_{x\to 0}\frac{2\sin(x^2)}{1-\cos(x)}$
   (c) $\lim_{x\to \infty}\frac{1}{x}\left(\sin(x)+\sin(\frac{2}{x})\right)$
   (d) $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{|x^2+x|}$

   Ans:

(a) $\lim_{x\to -2}\frac{x^2-2x-8}{x^2+3x+2} = \lim_{x\to -2}\frac{(x+2)(x-4)}{(x+2)(x+1)} = \lim_{x\to -2}\frac{(x-4)}{(x+1)} = 6$

(b) $\lim_{x\to 0}\frac{2sin\ (x^2)}{1-cos\ (x)} = \lim_{x\to 0}\frac{2sin\ (x^2)\ (1+cos\ (x)\ )}{(1-cos\ (x)\ )(1+cos\ (x)\ )} = \lim_{x\to 0}\frac{(x^2)\ (1+cos\ (x)\ )}{sin^2 x} =$
$\lim_{x\to 0}\frac{(x^2)\ x^2(1+cos\ (x)\ )}{x^2 sin^2 x} = \lim_{x\to 0}\frac{(x^2)}{x^2}\frac{x}{sin\ (x)}\frac{x}{sin\ (x)}\ (1+cos\ (x)\ ) =$
$2\lim_{t\to 0}\frac{sin\ (t)}{t}\lim_{x\to 0}\frac{x}{sin\ (x)}\frac{x}{sin\ (x)}\ (1+cos\ (x)\ )$ (Let $t = x^2$) $= 4$

(c) For any $x>0, -2\le \sin(x)+\sin(\frac{2}{x})\le 2 \Rightarrow -\frac{2}{x}\le\frac{1}{x}\left(\sin(x)+\sin(\frac{2}{x})\right)\le\frac{2}{x}$,
In addition, $\lim_{x\to\infty} -\frac{2}{x} = 0$ and $\lim_{x\to\infty}\frac{2}{x} = 0$
According to Squeeze theorem, $\lim_{x\to\infty}\frac{1}{x}\left(\sin(x)+\sin(\frac{2}{x})\right)= 0$

(d) $\lim_{x\to 0^+}\frac{\sqrt{1+x}-1}{|x^2+x|} = \lim_{x\to 0^+}\frac{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}{(x^2+x)(\sqrt{1+x}+1)} = \lim_{x\to 0^+}\frac{x}{(x^2+x)(\sqrt{1+x}+1)} =$
$\lim_{x\to 0^+}\frac{1}{(x+1)(\sqrt{1+x}+1)} = \frac{1}{2}$

$$\lim_{x\to 0^-}\frac{\sqrt{1+x}-1}{|x^2+x|} = \lim_{x\to 0^-}\frac{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}{-(x^2+x)(\sqrt{1+x}+1)}$$
$$= \lim_{x\to 0^-}\frac{x}{-(x^2+x)(\sqrt{1+x}+1)} = \lim_{x\to 0^-}\frac{1}{-(x+1)(\sqrt{1+x}+1)}$$
$$= -\frac{1}{2}$$

Therefore, the limit does not exist!

2. (8%)
Suppose $f(x) = \begin{cases} -ax^2-x-a & \text{if } x<-1 \\ ax^2+bx+6 & \text{if } -1\le x<2 \\ 3x^2-bx-b & \text{if } x\ge 2 \end{cases}$ is a continuous function on $(-\infty,\infty)$. What are the values of a and b?

Ans:
(a)

Since f is continuous at -1, we know $\lim_{x\to-1^-} f(x) = \lim_{x\to-1^+} f(x)$. Therefore, $\lim_{x\to-1^-} -ax^2 - x - a = \lim_{x\to-1^+} ax^2 + bx + 6 \to -a + 1 - a = a - b + 6 \to 3a - b = -5$.

On the other hand, f is continuous at 2, we know $\lim_{x\to2^-} f(x) = \lim_{x\to2^+} f(x)$. Therefore, $\lim_{x\to2^-} ax^2 + bx + 6 = \lim_{x\to2^+} 3x^2 - bx - b \to 4a + 2b + 6 = 12 - 2b - b \to 4a + 5b = 6$.

Solving the two equations we get $a = -1, b = 2$

3. (8%) Proof that $f(x) = 3x^3 + 2x - sin(x)$ has exactly one real root (Hint: use the mean value theorem)

Ans:

$f(1) > 0, f(-1) < 0$ by the intermediate value theorem, it has at least one real root between -1 and 1.

Assume the real root is a and there is a second real root b. Then, by the mean value theorem, there is a c such that $f'(c) = \frac{f(b)-f(a)}{b-a} = 0$. However, $f'(x) = 9x^2 + 2 - cos(x) > 0$. Contradict, therefore, there is only one real root.

4. (15%) Remember that you can solve the derivative using the definition or the differentiation rule for the following question.

(a) Find the derivative of $f(x) = \sqrt{1 + \cot(x^2)}$

(b) Given $f(x) = \frac{x^2}{(0-x)(1-x)(2-x)\dots(2023-x)}$, what is the value of $f'(0)$?

(c) Let $f(\mathrm{x}) = \begin{cases} \cos(2x) & if\ x \le 0 \\ ax & if\ x > 0 \end{cases}$, where a is a constant. Find the value of a makes $f(x)$ differentiable at 0.

Ans:

(a) $f(x) = \sqrt{1 + cot(x^2)} = (1 + cot(x^2))^{\frac{1}{2}} \to f'(x) = \frac{1}{2}(1 + cot\ (x^2)\ )^{\frac{-1}{2}}(-csc^2(x^2))2x = -\frac{csc^2(x^2)\cdot x}{\sqrt{1+cot(x^2)}}$

(b) $f'(0) = \lim_{\Delta x\to0} \frac{f(0+\Delta x)-f(0)}{\Delta x} = \lim_{\Delta x\to0} \frac{\frac{(\Delta x)^2}{-\Delta x(1-\Delta x)(2-\Delta x)\dots(2023-\Delta x)}}{\Delta x} = \lim_{\Delta x\to0} \frac{-1}{(1-\Delta x)(2-\Delta x)\dots(2023-\Delta x)} = \frac{-1}{2023!}$

(c) Since $f(x)$ is not continuous at 0, there is no value of a that can make it differentiable.

5. (8%) Given the graph $x^2 + xy + y^2 = 12$.
   (a) Express y' in terms of x and y
   (b) Find the extrema of the graph by checking the critical number
   Ans:
   (a)

$$\frac{d}{dx}(x^2 + xy + y^2) = \frac{d}{dx}(12)$$
$$2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$$
$$(x + 2y)\frac{dy}{dx} = -2x - y$$
$$\frac{dy}{dx} = \frac{-2x - y}{(x + 2y)}$$

   (b) The critical number occurs at $\frac{dy}{dx} = 0$ or $\frac{dy}{dx}$ does not exist

   When $\frac{dy}{dx} = 0 \rightarrow y = -2x$, substitute back to the original equation we get $x = \pm 2,\ y = \mp 4$

   When $\frac{dy}{dx}$ does not exist, $x = -2y$, substitute back to the original equation we get $x = \mp 4,\ y = \pm 2$

   Therefore, the graph has maximum at $(-2,4)$ at minimum at $(2,-4)$

6. (20%) Let $f(x) = \frac{-x^2-4x-7}{x+3}$
   (a) Find the open intervals on which f is increasing or decreasing. Indicates the extreme values
   (b) Find the open intervals on which f is concave upward or concave downward. Indicates the points of inflection
   (c) Find all the asymptotes (Vertical/horizontal/Slant)
   (d) Sketch the graph of $f(x)$
   (e) What is the domain and range of $f(x)$?

Ans: Note that the original function is undefined at $x = -3$, therefore we should include it in the following table.

(a)

(b) $f(x) = \frac{-x^2-4x-7}{x+3}, f'(x) = \frac{-(x+1)(x+5)}{(x+3)^2}, f''(x) = \frac{-8}{(x+3)^3}$

	$(-\infty, \mathbf{-5})$	$(\mathbf{-5}, \mathbf{-3})$	$(\mathbf{-3}, \mathbf{-1})$	$(\mathbf{-1}, \infty)$
測試值	-6	-4	-2	0
f'的正負號	-	+	+	-
f''的正負號	+	+	-	-
結論	遞減/向上凹	遞增/向上凹	遞增/向下凹	遞減/向下凹

The critical numbers are $x = -1, -5$. f is increasing on $(-5, -3)$ and $(-3, -1)$ since $f'(x) > 0$, f is decreasing on $(-\infty, -5)$ and $(-1, \infty)$ since $f'(x) < 0$.
Local (global) maxima is $(-5,6)$ and local (global) minima is $(-1, -2)$.
There are no possible points of inflection. f is concave downward on $(-3, \infty)$ since $f''(x) < 0$, f is concave upward on $(-\infty, -3)$ since $f''(x) > 0$.

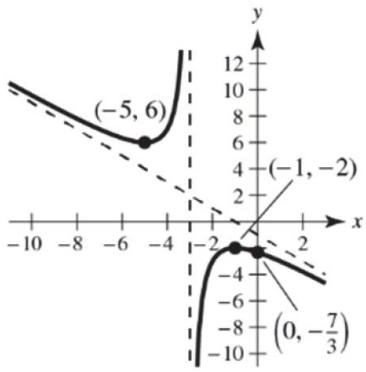
(c) Since $\lim_{x\to\pm\infty} f(x) = \pm\infty \rightarrow$ No horizontal asymptote

Since $\lim_{x\to -3^+} f(x) = -\infty$ and $\lim_{x\to -3^-} f(x) = \infty$ vertical asymptote at $x = -3$

$$\frac{-x^2-4x-7}{x+3} = -x - 1 - \frac{4}{x+3} \text{ (Using long division)}$$

$$\lim_{x\to\pm\infty} f(x) - (-x - 1 - \frac{4}{x+3}) = 0 \rightarrow y = -x - 1 \text{ is a slant asymptote}$$

(d)



(e) Domain is entire real line except -3. Range is $(-\infty, -2] \cup [6, \infty)$.

7. (15%) Evaluate the following expression. Remember the meaning and the definition of definite integral when solving the following question

(a) $\int \; 3x - \frac{6}{x^3} + 5\, sec\,(x)\, tan(x)\, dx$

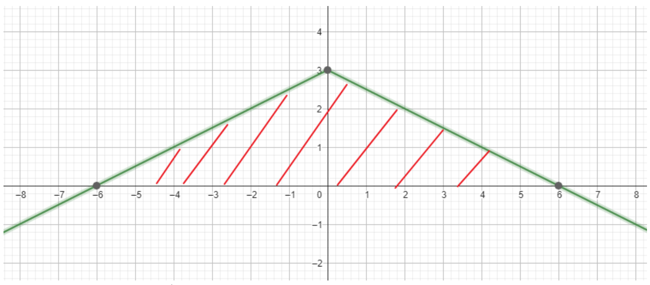
(b) $\int_{-6}^{6} 3 - |\frac{x}{2}| dx$

(c) $\lim_{n\to\infty} \frac{2^5}{n^5}(1^4 + 2^4 + 3^4 + \cdots + (2n)^4)$

Ans:

(a) $\frac{3x^2}{2} + \frac{3}{x^2} + 5\, sec\,(x) \; + C$

(b) $\int_{-6}^{6} 3 - |\frac{x}{2}| dx$ can be considered as the area in the following graph colored with red slash



Therefore, $\int_{-6}^{6} 3 - |\frac{x}{2}| dx = \frac{1}{2} 12 \times 3 = 18$

(c) $2^5 \lim_{n\to\infty} \frac{1}{n}\left(\frac{1^4+2^4+3^4 \ldots + (2n)^4}{n^4}\right) = 2^5 \lim_{n\to\infty} \left(\sum_{i=1}^{n} \left(\frac{i}{n}\right)^4 + \frac{1}{n}\sum_{i=n+1}^{2n} \left(\frac{i}{n}\right)^4\right) =$
$2^5\left(\int_0^1 x^4 dx + \int_1^2 x^4 dx\right) = 2^5 \frac{1}{5} x^5\Big]_0^2 = \frac{2^{10}}{5}$

8. (8%) Find $\frac{d}{dx}\int_{x}^{x^2}\sqrt{1+t^2}dt$. (Hint: Let $F(x)=\int_{1}^{x}\sqrt{1+t^2}dt$ and use the fundamental theorem of calculus)

Ans: Let $F(x)=\int_{1}^{x}\sqrt{1+t^2}dt$, since $\sqrt{1+t^2}$ is continuous, by the fundamental theorem of calculus, $F'(x)=\sqrt{1+x^2}$. Also $F(b)-F(a)=\int_{a}^{b}\sqrt{1+t^2}dt\,, a,b\in R$, therefore

$$\frac{d}{dx}\int_{x}^{x^2}\sqrt{1+t^2}dt=\frac{d}{dx}\left[\int_{x}^{1}\sqrt{1+t^2}dt+\int_{1}^{x^2}\sqrt{1+t^2}dt\right]$$
$$=-\sqrt{1+x^2}+2x\sqrt{1+x^4}$$

9. (8%) Find $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\frac{\sqrt{1+tan\,(t)}}{cos^2(t)}+t^3sin^2(t))dt$.

Ans:

Note that $t^3sin^2(t)$ is an odd function, so we only need to deal with the first term.

Let $u=1+\ tan\,(t)\ \ \rightarrow du=sec^2(t)dt$

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\frac{\sqrt{1+tan\,(t)}}{cos^2(t)}+t^3sin^2(t))dt=\int_{0}^{2}\sqrt{u}du\ =\frac{4\sqrt{2}}{3}$$