1. Find the following limit. (If the limit does not exist or has an infinite limit, you should point it out. In addition, also remember the definition of definite integral). (20%)

(a) $\lim_{n\to\infty}(\frac{1}{\sqrt{n^2+1^2}}+\frac{1}{\sqrt{n^2+2^2}}\dots+\frac{1}{\sqrt{n^2+n^2}})$

(b) $\lim_{x\to 0}\frac{\int_0^x(1+sin2t)^{\frac{1}{t}}dt}{x}$

(c) $\lim_{x\to\infty}\frac{e^{x^2}}{1-x^3}$

(d) $\lim_{x\to 0^+}\cot x\,(e^x-1)$

**Ans:**

(a) $\lim_{n\to\infty}(\frac{1}{\sqrt{n^2+1^2}}+\frac{1}{\sqrt{n^2+2^2}}\dots+\frac{1}{\sqrt{n^2+n^2}}) = \lim_{n\to\infty}\left(\frac{1}{\sqrt{1+(\frac{1}{n})^2}}+\frac{1}{\sqrt{1+(\frac{2}{n})^2}}+\cdots+\right.$

$\left.\frac{1}{\sqrt{1+(\frac{n}{n})^2}}\right)\frac{1}{n} = \lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{1}{\sqrt{1+(\frac{i}{n})^2}}\right)\frac{1}{n} = \int_0^1\frac{1}{\sqrt{1+x^2}}dx = \int_0^{\frac{\pi}{4}}\frac{sec^2\theta}{\sec\theta}d\theta\ (Let\ \ x =$

$tan\theta, dx = sec^2\theta d\theta) = \int_0^{\frac{\pi}{4}}\sec\theta\, d\theta = \ln|\sec\theta+\tan\theta|]_0^{\frac{\pi}{4}} = \ln|\sqrt{2}+1|$

(b) $\lim_{x\to 0}\frac{\int_0^x(1+sin2t)^{\frac{1}{t}}dt}{x} = \lim_{x\to 0}\frac{(1+sin2x)^{\frac{1}{x}}}{1}$ (L' Hôpital' s rule and fundamental theorem of calculus)

$$y = \lim_{x\to 0}(1+sin2x)^{\frac{1}{x}}$$

$\ln y = \ln\lim_{x\to 0}(1+sin2x)^{\frac{1}{x}} = \lim_{x\to 0}\ln(1+sin2x)^{\frac{1}{x}} = \lim_{x\to 0}\frac{1}{x}\ln(1+sin2x) =$

$\lim_{x\to 0}\frac{2cos2x}{1+sin2x}$ (L' Hôpital' s rule)$=2$

Therefore, $y = e^2$

(c) $\lim_{x\to\infty}\frac{e^{x^2}}{1-x^3} = \lim_{x\to\infty}\frac{2xe^{x^2}}{-3x^2}$ (L' Hôpital' s rule)$= \lim_{x\to\infty}\frac{2e^{x^2}}{-3x} = \lim_{x\to\infty}\frac{4xe^{x^2}}{-3}$ (L' Hôpital' s rule)$= -\infty$

(d) $\lim_{x\to 0^+}\cot x\,(e^x-1) = \lim_{x\to 0^+}\frac{(e^x-1)}{tanx} = \lim_{x\to 0^+}\frac{e^x}{sec^2x}$ (L' Hôpital' s rule) $=1$

2. Given $x^2 - \left(f(x)\right)^3 = xf(x), x \geq 0$ and suppose $f(x)$ has an inverse function, what is the value of $(f^{-1})'(x)$ when $x = 2$ (10%)

   **Ans:**

   Let $y = f(x) = 2$, we have $x^2 - 8 = 2x, x \geq 0 \rightarrow x = 4$

   Differentiate both side with respect to $x$, we have $2x - 3\left(f(x)\right)^2 f'(x) = f(x) + xf'(x)$. Substitute $x = 4$, we get $8 - 12f'(4) = 2 + 4f'(4)$

   Therefore, $f'(4) = \frac{3}{8} \rightarrow (f^{-1})'(2) = \frac{1}{f'(4)} = \frac{8}{3}$

3. Use the Mean Value Theorem to prove that $\forall a \geq 0$, we have $\frac{a}{a+1} \leq \ln(a+1) \leq a$. (Hint: use the theorem in the interval $(0, a)$) (10%)

   **Ans:**

When $a = 0$, the equality holds!

When $a > 0$, let $f(x) = \ln(1+x) \quad \rightarrow f'(x) = \frac{1}{1+x}$

According to the mean value theorem, we know that there exist $c \in (0, a)$ such that

$f'(c) = \frac{f(a)-0}{a-0} = \frac{\ln(1+a)}{a}$

Note that since $f'(x) = \frac{1}{1+x}$. Therefore, in this interval for any $c \in (0, a)$, we have

$\frac{1}{1+a} \leq f'(c) \leq \frac{1}{1+0} = 1$ ($f'(x)$ is strictly decreasing due to $f''(x) = \frac{-1}{(1+x)^2} < 0$)

Therefore, $\frac{1}{1+a} \leq \frac{\ln(1+a)}{a} \leq 1 \rightarrow \frac{a}{a+1} \leq \ln(a+1) \leq a$

4. Evaluate the following integrals. (Hint: Try to use change of variables for all the problems) (15%)

   (a) $\int_4^5 \frac{1}{(x-1)\sqrt{x^2-2x}} dx$

   (b) $\int 2^{sinx} cosx \ dx$

   (c) $\int \frac{5}{1+\sqrt{5x}} dx$

   **Ans:**

(a) $\int_4^5 \frac{1}{(x-1)\sqrt{x^2-2x}} dx = \int_4^5 \frac{1}{(x-1)\sqrt{x^2-2x+1-1}} dx = \int_4^5 \frac{1}{(x-1)\sqrt{(x-1)^2-1}} dx =$

$\sec^{-1}|x-1|]_4^5 = sec^{-1}4 - sec^{-1}3$

(b) $\int 2^{sinx} cosx \ \ dx = \int 2^u \, du$ (Let $u = sinx, du = cosxdx$) $= \frac{1}{\ln 2} 2^u + C =$

$\frac{1}{\ln 2} 2^{sinx} + C$

(c) Let $u = 1 + \sqrt{5x} \to du = \frac{5}{2\sqrt{5x}}$dx $\to dx = \frac{2}{5}(u-1)du$

$$\int \frac{5}{1+\sqrt{5x}} dx = \int \frac{5}{u}\frac{2}{5}(u-1)du = 2\int 1 - \frac{1}{u} du = 2(u - \ln|u|) + C$$
$$= 2\left(1 + \sqrt{5x} - \ln(1+\sqrt{5x})\right) + C$$

5. Evaluate the following integrals. (15%)
   (a) $\int t \csc t \cot t \, dt$
   (b) $\int cot^3\theta csc^4\theta \, d\theta$
   (c) $\int tan^{-1}\sqrt{x} dx$

   **Ans:**

(a) Let $u = t, dv = \csc t \cot t \, dt \to du = dt, v = -csct$

$$\int t \csc t \cot t \, dt = -tcsct + \int csct \ \ dt = -tcsct - \ln|csct + \cot t| + C$$

(b) $\int cot^3\theta csc^4\theta \, \mathrm{d}\theta = \int cot^2\theta csc^3\theta csc\theta cot\,\theta \mathrm{d}\theta = \int (csc^2\theta -$
$1)csc^3\theta csc\theta cot\,\theta \mathrm{d}\theta$

Let $u = csc\theta, du = -csc\theta cot\theta d\theta$

$$\int (csc^2\theta - 1)csc^3\theta csc\theta cot\,\theta \mathrm{d}\theta = -\int (u^2-1)u^3 du = -\int u^5 - u^3 du$$

$$= -\frac{1}{6}u^6 + \frac{1}{4}u^4 + \mathrm{C} = -\frac{1}{6}(csc\theta)^6 + \frac{1}{4}(csc\theta)^4 + \mathrm{C}$$

(c) Let $y = \sqrt{x} \to dy = \frac{1}{2}\frac{1}{\sqrt{x}}dx = \frac{dx}{2y}$

$$\int tan^{-1}\sqrt{x}dx = \int 2y \ \ tan^{-1}y \, dy$$

Let $u = tan^{-1}y, dv = 2ydy \to du = \frac{1}{1+y^2}dy, v = y^2$

$$\int 2y\ tan^{-1}y\,dy = y^2tan^{-1}y - \int y^2\frac{1}{1+y^2}\mathrm{dy}$$

$$= y^2tan^{-1}y - \int 1 - \frac{1}{1+y^2}\mathrm{dy} = y^2tan^{-1}y - y + \tan^{-1}y + C$$

$$= (x+1)tan^{-1}\sqrt{x} - \sqrt{x} + C$$

6. Evaluate the following integral (If the integral diverges, you should point it out). (15%)

(a) $\int_{-1}^{1}\frac{1}{x^2}dx$

(b) $\int_{1}^{\infty}\frac{1}{\sqrt{x^2-0.1}}dx$

(c) $\int_{0}^{\infty}e^{-x}cosx\,dx$

**Ans:**

(a) $\int_{-1}^{1}\frac{1}{x^2}dx = \int_{0}^{1}\frac{1}{x^2}dx+\int_{-1}^{0}\frac{1}{x^2}dx$

Since $\int_{0}^{1}\frac{1}{x^2}dx = \lim_{b\to 0^+}\int_{b}^{1}x^{-2}dx = \lim_{b\to 0^+}-x^{-1}\Big]_{b}^{1} = \lim_{b\to 0^+}\left(-1+\frac{1}{b}\right) = \infty$ is divergent, therefore $\int_{-1}^{1}\frac{1}{x^2}dx$ is divergent.

(b) Since $\lim_{x\to\infty}\frac{\frac{1}{\sqrt{x^2-0.1}}}{\frac{1}{x}} = 1$ on $[1,\infty)$ and $\int_{1}^{\infty}\frac{1}{x}dx$ is divergent, then by the limit comparison test so does $\int_{1}^{\infty}\frac{1}{\sqrt{x^2-0.1}}dx$

(c) $\int_{0}^{\infty}e^{-x}cosx\,dx = \lim_{b\to\infty}\int_{0}^{b}e^{-x}cosx\,dx$

Let $u = cosx, dv = e^{-x}dx \to du = -sinx, v = -e^{-x}$

$$\int e^{-x}cosxdx = -e^{-x}cosx - \int e^{-x}\,sinxdx$$

Let $u = sinx, dv = e^{-x}dx \to du = cosx, v = -e^{-x}$

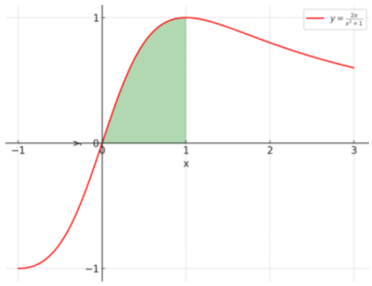
$$\int e^{-x}\,sinxdx = -e^{-x}sinx + \int e^{-x}\,cosxdx$$

$$\int e^{-x}cosxdx = -e^{-x}cosx + e^{-x}sinx - \int e^{-x}\,cosxdx \to \int e^{-x}\,cosxdx$$

$$= \frac{1}{2}(-e^{-x}cosx + e^{-x}sinx)$$

$\lim_{b\to\infty}\int_{0}^{b}e^{-x}cosx\,dx = \lim_{b\to\infty}\frac{1}{2}(-e^{-x}cosx + e^{-x}sinx)\Big]_{0}^{b} = \frac{1}{2}$ which is converge

7. Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \frac{2x}{x^2+1}$ and the $x$-axis $(0 \le x \le 1)$ about the $x$-axis. (10%)




**Ans:**

$$V = \pi \int_0^1 \left(\frac{2x}{x^2+1}\right)^2 dx = 4\pi \int_0^1 \frac{x^2}{(x^2+1)^2}\ dx = 4\pi \int_0^1 \frac{1}{(x^2+1)} - \frac{1}{(x^2+1)^2}\ dx$$

$\int \frac{1}{(x^2+1)} - \frac{1}{(x^2+1)^2}\ dx = \tan^{-1} x - \int \frac{1}{(x^2+1)^2} \mathrm{dx}$ , Let $x = \tan\theta, dx = sec^2\theta d\theta$

$\int \frac{1}{(x^2+1)^2} \mathrm{dx} = \int \frac{sec^2\theta d\theta}{(tan^2\theta+1)^2} = \int \frac{d\theta}{sec^2\theta} = \int cos^2\theta d\theta = \int \frac{1+cos2\theta}{2} d\theta = \frac{1}{2}\theta + \frac{1}{4}sin2\theta +$

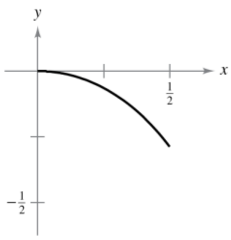
$C = \frac{1}{2}\tan^{-1} x + \frac{1}{4} 2sin\theta cos\theta + C = \frac{1}{2}\tan^{-1} x + \frac{1}{4} 2 \frac{x}{x^2+1} + C = \frac{1}{2}(\tan^{-1} x +$

$\frac{x}{x^2+1}$)+C

$$4\pi \int_0^1 \frac{1}{(x^2+1)} - \frac{1}{(x^2+1)^2}\ dx = 4\pi\left(\tan^{-1} x - \frac{1}{2}\left(\tan^{-1} x + \frac{x}{x^2+1}\right)\right)\Big|_0^1$$

$$= \frac{\pi^2}{2} - \pi$$

8. Find the arc length of the graph of the function $y = \ln(1 - x^2)$ on the interval $0 \le x \le \frac{1}{2}$. (10%)




**Ans:**

$$y = \ln(1 - x^2), y' = \frac{-2x}{(1 - x^2)}$$

Arc length $= \int_0^{\frac{1}{2}} \sqrt{1 + (y')^2}\, dx = \int_0^{\frac{1}{2}} \frac{1+x^2}{1-x^2} dx = \int_0^{\frac{1}{2}} (-1 + \frac{1}{x+1} + \frac{1}{1-x})\, dx =$

$-x + \ln(1 + x) - \ln(1 - x)]_0^{\frac{1}{2}} = \ln 3 - \frac{1}{2}$

9. Assume $f(x)$ is a polynomial whose coefficients are integers, and we know that $\int_1^{\infty} \frac{f(x)}{(x+1)^2(4x^2+1)} dx = \ln \frac{16}{5} + \frac{1}{2}$ (10%)

   (a) Use the limit comparison test for improper integrals, what is the maximum degree of $f(x)$?

   (b) Using partial fraction method, we have $\frac{f(x)}{(x+1)^2(4x^2+1)} = \frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{Cx+D}{4x^2+1}$.

   Find $\int \frac{f(x)}{(x+1)^2(4x^2+1)} dx$

   (c) According to (b) and (c), solve $f(x)$ (Hint: Try to compare the coefficients of the transcendental function)

   **Ans:**

   (a) Since $\int_1^{\infty} \frac{1}{x^p} dx$ is converge if $p > 1$, by the limit comparison test for improper integrals $\lim_{x\to\infty} \frac{\frac{f(x)}{(x+1)^2(4x^2+1)}}{g(x)}$ should have finite $L$, therefore let $g(x) = \frac{1}{x^p}$, $\lim_{x\to\infty} \frac{f(x)x^p}{(x+1)^2(4x^2+1)}$ should have finite $L$. Assume the highest degree of $f(x)$ is $q$, we have $q + p \le 4 \to q \le 4 - p < 3$. The maximum degree of $f(x)$ is 2.

   (b) $\int \frac{f(x)}{(x+1)^2(4x^2+1)} dx = \int \frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{Cx+D}{4x^2+1} dx = \frac{-A}{x+1} + B \ln|x + 1| + \frac{C}{8} \ln|4x^2 + 1| + \frac{D}{2} \tan^{-1} 2x$

   (c) We know that $\int_1^{\infty} \frac{f(x)}{(x+1)^2(4x^2+1)} dx = \lim_{b\to\infty} \int_1^b \frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{Cx+D}{4x^2+1} dx =$

   $\frac{-A}{x+1} + B \ln|x + 1| + \frac{C}{8} \ln|4x^2 + 1| + \frac{D}{2} \tan^{-1} 2x \Big]_1^b$

   We have $D = 0$ (otherwise the results will contains $\pi$)

   Furthermore, since the intergral is convergent, $B = \frac{-C}{4}$ (otherwise the log term will diverge)

$$\rightarrow \lim_{b\to\infty} B\ln\left|\frac{x+1}{(4x^2+1)^{\frac{1}{2}}}\right| + \frac{-A}{x+1}\Bigg]_1^b = B\left(\ln\frac{1}{2} - \ln\frac{2}{\sqrt{5}}\right) + \frac{A}{2}$$

By comparing the terms $\rightarrow B\left(ln\frac{1}{2} - ln\frac{2}{\sqrt{5}}\right) + \frac{A}{2} = ln(\frac{\sqrt{5}}{4})^B + \frac{A}{2} = ln\frac{16}{5} + \frac{1}{2} \rightarrow$

$B = -2, A = 1 \rightarrow C = 8$

$$f(x) = \left(\frac{1}{(x+1)^2} + \frac{-2}{x+1} + \frac{8x}{4x^2+1}\right)(x+1)^2(4x^2+1) = 12x^2 + 6x - 1$$